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Research Article

Characterization of Some Distributions by Inequalities on Failure Rate

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Abstract

Characterizing distributions is the process of finding the unique properties that distinguish a particular distribution from others. There are various methods for doing this, such as using moments, maximum entropy, hazard rate, and so on. Nanda (2010) used failure rate and mean residual life functions to characterizing some distributions. In this paper, we extend the results of Nanda (2010). We characterize the Pareto, exponential, and Rayleigh distributions by refining the main results of that paper. We take some inequalities from that paper and propose a new inequality. Then, we raise both sides of the inequalities to some power and show that equality holds if and only if certain distributions are characterized.

Keywords : Statistical distributions; Cauchy schwarz inequality; Hazard rate; Reliability.

1 Introduction

Characterization of distributions through reliability terms has been done by many authors in the literature such as Galambos and Kotz [3], Kotz and Shanbhag [5], Belzunce et al. [2], Nanda et al. [1]. Also, in [4, 6, 7, 8], authors proved some new results on entropy for certain well known distributions and give some characterization results for life distributions via the links between entropy, variance and etc. Life distributions are statistical distributions that describe the time they take for an event to occur, such as the time to failure of a component or the time to death of an organism. The Pareto distribution is often used to model the lifetimes of items that are subject to wear and tear, such as machine components and electronic devices. The exponential distribution is often used to model the lifetimes of items that are subject to random failures, such as light bulbs and radioactive atoms. The Rayleigh distribution is often used to model the lifetimes of items that are subject to fatigue, such as aircraft wings and bridges.

Let X be a nonnegative absolutely continuous random variable with probability density function $f(x)$ and survival function $\overline{F}(x) = P(X \geq x)$. We denote the failure rate of the distribution by $r(x) = \frac{f(x)}{\overline{F}(x)}$. Recently, Nanda [1] proved some interesting results in context of characterization through failure rate and mean residual life function. Nanda [1] showed that for any nonnega-

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tive random variable X, $E(Xr(X)) \geq \frac{2}{1+c^2}$, with equality if and only if X is exponentially distributed, and c is coefficient of variation. Also he showed that for the same rv X, $E(\frac{r(X)}{X}) \geq \frac{2}{\mu^2(1+c^2)}$, the equality holds if and only if X has Rayleigh distribution, with probability density function, $\overline{F}(x) = 2\theta x e^{-\theta x^2} \quad ,x \geq 0 \quad ,\theta > 0$.
We first state the main two results of Nanda [1] and then establish a characterization result for Pareto distribution by the inequality on failure rate. In the rest of the paper, we improve these results by adding some power at the both sides of inequalities to show some new characterization results based on failure rate.

2 Main Results

At the entire of paper we assume that X is finite and absolutely continuous.
First we mention some main results of [1].

Theorem 2.1. *[1] For any nonnegative random variable X, $E(Xr(X)) \geq \frac{2}{1+c^2}$, and the equality holds if and only if X is exponentially distributed.*

Theorem 2.2. *[1] For any nonnegative random variable X,*

$$E(\frac{r(X)}{X}) \geq \frac{2}{\mu^2(1+c^2)},$$

and the equality holds if and only if X has Rayleigh distribution, with probability density function,

$$f(x) = 2\theta x e^{-\theta x^2} \quad ,x \geq 0 \quad ,\theta > 0.$$

Theorem 2.3. *For any nonnegative random variable $X \geq c$,*

$$E(X^3 r(X)) \geq \frac{2}{u^{-1} - k^2}.$$

The equality holds if and only if X has Pareto distribution. Also, $u = \mu^2 + \sigma^2$, and $k = \frac{c}{u}$ and $E(X^2)$ is finite.

Proof. By the Cauchy-Schwartz inequality, we have,

$$[\int_c^\infty x\overline{F}(x)\,dx][\int_c^\infty \frac{x^3 f^2(x)}{\overline{F}(x)}\,dx] \geq (\mu^2+\sigma^2)^2. \tag{2.1}$$

Using

$$\int_c^\infty x\overline{F}(x)\,dx = \frac{\mu^2+\sigma^2}{2} - \frac{c^2}{2}$$

and

$$\int_c^\infty \frac{x^3 f^2(x)}{\overline{F}(x)}\,dx = E(X^3 r(X))$$

the inequality (1) reduces to

$$E(X^3 r(X)) \geq \frac{2}{u^{-1} - k^2},$$

so equality holds if and only if we have some constant $A \geq 2$ such that, for any $X \geq c$,

$$A\sqrt{x\overline{F}(x)} = \sqrt{\frac{x^3 f^2(x)}{\overline{F}(x)}}.$$

It is easy to show that, $r(X) = \frac{A}{x}$, which holds if and only if X has Pareto distribution, that is, $\overline{F}(x) = (\frac{x}{c})^{-A} \qquad x \geq c.$ □

Now question is that, if we add some power to both sides of above inequalities what will be the changes in the above results? For instance, if we take some constant such as B, $E[(\frac{r(X)}{X})]^3 \geq B$, when does equality hold? In the rest of the paper we answer to this question.

Theorem 2.4. *For any nonnegative random variable X,*

$$E[(\frac{r(X)}{X})]^{2^{t+1}-1} \geq \frac{2^{2^{t+1}-1}}{u^{2^{t+1}-1}}, \qquad \forall t \geq 0,$$

and the equality holds if and only if X has Rayleigh distribution, with probability density function, $\overline{F}(x) = 2\theta x e^{-\theta x^2} \quad ,x \geq 0 \quad ,\theta > 0$. where $u = \mu^2 + \sigma^2$.

Proof. We prove by induction on the t. The case $t = 0$ is already established in Nanda [1]. Now assume that Theorem holds for any $1 \leq t \leq n - 1$. □

$$E[(\frac{r(X)}{X})]^{2^n-1} \geq \frac{2^{2^n-1}}{u^{2^n-1}}. \tag{2.2}$$

We prove theorem for $t = n$.
By the Cauchy-Schwartz inequality, we get,
$[\int_0^\infty x\overline{F}(x)\,dx][\int_0^\infty \frac{f^{2^{n+1}}(x)}{x^{2^{n+1}-1}\overline{F}^{2^{n+1}-1}(x)}\,dx] \geq$

$$[E(\frac{r^{2^n-1}(X)}{X^{2^n-1}})]^2. \tag{2.3}$$

Table 1: Simulated Data

Distribution	Parameter	Sample Size	Difference
4*Exponential	2*2.5	10000	0.039
		1000000	0.0012
	2*4.6	10000	0.0017
		1000000	0
4*Rayleigh	2*2	10000	0.0252
		1000000	0.0236
	2*5	10000	0.0024
		1000000	0.0037

Using (2), where $t = n - 1$ and the equalities,

$$\int_0^\infty x\overline{F}(x)\,dx = \frac{\mu^2 + \sigma^2}{2} = \frac{u}{2}$$

and

$$\int_0^\infty \frac{f^{2^{n+1}}(x)}{x^{2^{n+1}-1}\overline{F}^{2^{n+1}-1}(x)}\,dx = E[(\frac{r(X)}{X})]^{2^{n+1}-1}$$

the inequality (3) reduces to

$$E[(\frac{r(X)}{X})]^{2^{n+1}-1} \geq \frac{2^{2^{n+1}-1}}{u^{2^{n+1}-1}},$$

so the equality holds if and only if we have some constant E such that for any $X \geq 0$,

$$\sqrt{x\overline{F}(x)} = E^{\frac{1}{2n}}\sqrt{\frac{f^{2^{n+1}}(x)}{x^{2^{n+1}-1}\overline{F}^{2^{n+1}-1}(x)}}$$

with some calculation we have, $r(X) = \frac{X}{E}$, which holds if and only if X has Rayleigh distribution.

Theorem 2.5. *For any nonnegative random variable X, and $\forall t \geq 0$*

$$E[Xr^{2^{t+1}-1}(X)] \geq \frac{2^{2^{t+1}-1}}{u^{2^t-1}(1+c^2)^{2^t}},$$

the equality holds if and only if X is exponentially distributed.

Proof. The proof is similar to the theorem 2.4. □

Corollary 2.1. *In the theorem 2.5, set t=0, we will have Theorem 2.3 of [1].*

Theorem 2.6. *For any nonnegative random variable* $X \geq c$, and $\forall t \geq 0$

$$E[X^{2^{t+1}+1}r^{2^{t+1}-1}(X)] \geq \frac{2^{2^{t+1}-1}}{u^{2^t-1}(u^{-1}-k^2)^{2^t}},$$

the equality holds if and only if X has Pareto distribution.

Proof. The proof is similar to the theorem 2.4. □

Corollary 2.2. *In the theorem 2.6, set t=0, we will have Theorem 2.3 of [1].*

3 Simulation Study and Conclusions

In this section, we present some results of simulated and real data and also draw some conclusions relevant to the research. For this purpose, we generate data from exponential and Rayleigh distributions with different values of parameters and then evaluate some results obtained in this paper. Table 1 shows the number of data points (10000 and 1000000) generated under eachdistribution and the differences between both sides of the equalitiesproved by the theorems.The difference are close to zero, which indicates that the sample size affects the accuracy of the results. This suggests that one can use this method for goodness of fit tests. To test this idea, we use two real data sets about the lifespan of two electronic equipment in an organizationin Brazil. These two equipments are notebooks and PCs without monitors. We assume that the lifespan follow exponential distributions and apply the results of the theorems to test this hypothesis. Using MATLAB software, we obtain the values of 0.7809 for 33 notebook lifespan data and 0.9032 for 113 PC without monitor lifespan data. These values are the differences between the left and right sides of the equalities in the theorems. As a future work, one can use these kinds of theorems to perform goodness of fit tests.

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