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Research Article

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Fuzzy Approach to Solve a Global Mixed Integer Multi-Objective FractionalL Signomial Geometric Programming Problem

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Abstract

This study proposes a method for solving mixed integer multi-objective fractional signomial geometric programming (MIMOFSGP) problems.In this paper we firstly convert a multi-objective fractional signomial programming into a non-fractional problem by a new convenient reformulation strategy. Then the fuzzy programming technique is applied to find the optimal compromise solution and a convex relaxation is used to reach a global solution. A mixed integer compromise optimal solution of the convex programming problem can finally be found by use of nonlinear branch and bound algorithm. Then on using the Spacial branch and bound algorithm, we obtain a solution that has the shortest distance from the solution of the original problem. Finally two examples are included to demonstrate the correctness and efficiency of the proposed approach.

Keywords : Multi-objective programming; Geometric programming; Fractional programming; Mixed integer programming; Spacial branch and bound algorithm.

1 Introduction

FRactional geometric programming problems (FGP) is used to solve a class of geometric programming problems to minimize a fractional objective function under certain constraints. A few methods have been applied in the recent past to convert a fractional signomial objective function into a non-fractional signomial objective function to find the optimal solution by use of some common mathematical programming techniques. The mixed integer fractional signomial geometric programming problems(MIFSGP) in which the objective function appears as a quotient of two signomial functions subject to certain constraints with integer and continuous decision variables is an important part of geometric programming problems in the wide scope of engineering design, management and finance problems. For instance, Ray and Saini (2001) [18], Arora (1989) [2] addressed a few meth-

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ods to solve some real (FGP) problems of design engineering. Ching Ter Chang (2002) [10], Jung-Fa Tsai (2007) [19] and Saraj and Bazikar [3] proposed some reformulation techniques to convert geometric fractional functions to geometric non-fractional functions. In practice, some of the (FGP) problems are formulated as multi-objective fractional programming problems (MOFGP). Usually a multi objective optimization problem doesnt have any single optimal solution which optimizes all the objective functions simultaneously. Anyway the decision makers look for the most compromise solution for all objectives. In the recent past a few approaches have been used to find global compromise solutions namely weighting methods, goal programming and fuzzy techniques. Although these approaches are applied in multi objective nonlinear programming [17], there is no absolutely successful extension to find a global lower bound for multi objective fractional geometric programming (MOFGP) problems. In this paper, we first define a new variable for every fractional objective function.This technique reformulates (MOFGP) problem to a non-convex mixed integer multi objective geometric programming (MOGP) problem by adding some constraints. Reformulated non-convex (MOGP) problem poses additional challenges, because it contains non- convex terms in the objective functions or in the constraints. In addition, existence of the integer variables generally makes the feasible region non- convex. So applying convex relaxation will yield the best lower bound for objectives and the yield optimal compromise solution is infeasible for the original (MOFGP) problem.Therefore by using the spatial branch and bound algorithm, we find both a feasible solution for the (MIGP) problems and the tightest global lower bound to the (local) original lower bound for multi objective functions[6]. The rest of the paper is organized as follows: formulation of multi-objective mixed integer fractional geometric programming problems for (MIMOFGP) are discussed in Section 2 and 3, respectively. In section 4, Fuzzy notations and the formulation of a crisp model for the fuzzy geometric programming problem is used. The convexification strategies discussed in Section 5. And also, the spatial branch and bound algorithm [9] is expressed to assess lower bound of convex relaxation of (MIGP). An illustrative example has been incorporated in Section 6. Finally, some conclusions are brought from the obtained results in Section 7.

2 Mixed integer multi-objective signomial geometric fractional programming (MOMISGFP) problems.

a signomial function is a sum of positive or negative signomial terms consisting of a product of power functions,ie.,

$$p(\mathbf{x}) = f(x_1, x_2, \ldots, x_n) = \Sigma_{t=1}^{T} \sigma_t c_t \Pi_{j=1}^{n} x_j^{\alpha_{tj}}$$

Where $\mathbf{x}$ is a vector of real or integer positive variables. T and n are the number of terms and variables respectively in every signomial function. c_t is absolute value of coefficients, and σ_t is sign of coefficients ($+1$ or -1). If all the terms in a signomial function are positive, ie, all coefficients $\sigma_t = +1$, then the function is called posynomial. A multi-objective mixed integer fractional signomial geometric programming problem is expressed in its normal form as follows:
Obtain $x = (x_1, x_2, x_3, \ldots, x_n)^T$, so as to

$$Minimize(f^1(\mathbf{x}), f^2(\mathbf{x}), f^3(\mathbf{x}), \ldots, f^p(\mathbf{x})) \quad (2.1)$$

$$f^k(\mathbf{x}) = \frac{(p^k(\mathbf{x}))}{(q^k(\mathbf{x}))}, q^k(\mathbf{x}) > 0, k \in \{1, 2, \ldots, p\}$$

Subject to

$$g_i(\mathbf{x}) \leq \xi_i, \xi_i = \mp 1, i = 1, 2, \ldots, m$$

Where
$p^k(\mathbf{x}), q^k(\mathbf{x})$ and $g_i(\mathbf{x})$ are signomial functions and

$$\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)^T \in X, X \subseteq R,$$

$\mathbf{x}$ is a vector of real or integer positive variable

$$0 \leq \underline{x}_j \leq x_j \leq \overline{x}_j$$
$$j = 1, 2, \ldots, n$$

It is assumed, the mentioned problem is feasible and has an optimal compromise solution.

3 Strategy of reformulation

In this article a new and convenient technique is presented to convert a multi-objective signomial geometric fractional programming (MOSGFP) problem to a multi-objective signomial geometric programming (MOSGP) problem with p fractional objective functions. Consider the following signomial geometric fractional programming problem:

$$Minimize f^k(\mathbf{x}) = \frac{p^k(\mathbf{x})}{q^k(\mathbf{x})} = \frac{\Sigma_{t=1}^{T_\circ^k} \sigma_{\circ t}^k c_{\circ t}^k \Pi_{j=1}^n x_j^{\alpha_{\circ tj}^k}}{\Sigma_{t=1}^{T_\circ'^k} \sigma_{\circ t}'^k c_{\circ t}'^k \Pi_{j=1}^n x_j^{\alpha_{\circ tj}'^k}} \quad (3.1)$$

$$k \in \{1, 2, \ldots, p\}$$

Subject to

$$g_i(\mathbf{x}) \leq \xi_i, \xi_i = \mp 1, i = 1, 2, \ldots, m$$

Where

$$g_i(\mathbf{x}) = \Sigma_{t=1}^{T_i} \sigma_{it}\ c_{it} \Pi_{j=1}^n x_j^{\alpha_{itj}}$$

$$q^k(\mathbf{x}) > 0, \ for\ \mathbf{x} \in X\ and$$
$$\forall \mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)^T \in X \subseteq R,$$

$$0 \leq \underline{x}_j \leq x_j \leq \overline{x}_j,$$

$$T_\circ^k = Number\ of\ terms\ in\ p^k(\mathbf{x}),$$
$$T_\circ'^k = number\ of\ terms\ in\ q^k(\mathbf{x}),$$

$$c_{\circ t}, c'_{\circ t} \in R^+$$

$$0 \leq \underline{x}_j \leq x_j \leq \overline{x}_j\ ,\ j = 1, 2, \ldots, n$$

Suppose x_{n+k} is a positive variable, let $q^k(\mathbf{x}) = x_{n+k}$ then we have:

$$f^k(\mathbf{x}) = x_{n+k}^{-1} * p^k(\mathbf{x})\ for\ k \in \{1, 2, \ldots, p\},$$

To find lower and upper bounds of every x_{n+k} for every $k \in \{1, 2, \ldots, p\}$ we should solve two following subproblems:

$$L_{n+k} := \min q^k(\mathbf{x})\ ,\ k \in \{1, 2, \ldots, p\}$$

Subject to

$$g_i(\mathbf{x}) \leq \xi_i\ ,\ \xi_i = \mp 1\ ,\ i = 1, 2, \ldots, m$$
$$for\ \mathbf{x} \in X$$

And

$$U_{n+k} := \max q^k(\mathbf{x})\ ,\ k \in \{1, 2, \ldots, p\}$$

Subject to

$$g_i(\mathbf{x}) \leq \xi_i\ ,\ \xi_i = \mp 1\ ,\ i = 1, 2, \ldots, m$$
$$for\ \mathbf{x} \in X$$

So, the problem 3.2 leads to the following signomial geometric programming problem which contains a new equality constraint, $x_{n+k}^{-1} * q^k(\mathbf{x}) = 1$

$$Minimize\ \ x_{n+k}^{-1} * p^k(\mathbf{x}) \quad (3.2)$$

Subject to

$$x_{n+k}^{-1} * q^k(\mathbf{x}) = 1 \quad (3.3)$$

$$g_i(\mathbf{x}) \leq \xi_i\ ,\ \xi_i = \mp 1\ ,\ i = 1, 2, \ldots, m$$

$$L_{n+k} \leq x_{n+k} \leq U_{n+k}\ ,\ k \in \{1, 2, \ldots, p\}\ ,\ \mathbf{x} > 0$$

We can replace the equation (3.3) by two non-equation as following:

$$x_{n+k}^{-1} * q^k(\mathbf{x}) \leq 1 \quad (3.4a)$$
$$-x_{n+k}^{-1} * q^k(\mathbf{x}) \leq -1 \quad (3.4b)$$

Therefore the original MOSGFP problem is reformulated to:

$$Minimize\ (x_{n+1}^{-1} * p^1(\mathbf{x}),$$
$$x_{n+2}^{-1} * p^2(\mathbf{x}), \ldots, x_{n+P}^{-1} * p^P(\mathbf{x})) \quad (3.5)$$

Subject to

$$x_{n+k}^{-1} * q^k(\mathbf{x}) \leq 1$$

$$-x_{n+k}^{-1} * q^k(\mathbf{x}) \leq -1\ ,\ k = 1, 2, \ldots, P$$

$$g_i(\mathbf{x}) \leq \xi_i\ ,\ \xi_i = \mp 1\ ,\ i = 1, 2, \ldots, m$$

$$L_{n+k} \leq x_{n+k} \leq U_{n+k}, k \in \{1, 2, \ldots, P\}, \mathbf{x} > 0$$

Where

$$p^k(\mathbf{x}) = \Sigma_{t=1}^{T_\circ^k}\ \sigma_{\circ t}^k\ c_{\circ t}^k \Pi_{j=1}^n x_j^{\alpha_{\circ tj}^k},$$

$$q^k(\mathbf{x}) = \Sigma_{t=1}^{T_\circ'^k}\ \sigma_{\circ t}'^k\ c_{\circ t}'^k \Pi_{j=1}^n x_j^{\alpha_{\circ tj}'^k}$$

$$g_i(\mathbf{x}) = \Sigma_{t=1}^{T_i}\ \sigma_{it}\ c_{it} \Pi_{j=1}^n\ x_j^{\alpha_{itj}}$$

$$q^k(\mathbf{x}) > 0,$$
$$\forall \mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)^T \in X \subseteq \mathcal{R},$$
$$0 \le \underline{x}_j \le x_j \le \overline{x}_j,$$
$$j = 1, 2, \ldots, n$$

Proposition 3.1. *if $0 < q^k(\mathbf{x}^*) = x^*_{n+k}$ where $\mathbf{x}^*$ is an optimal solution of (3.2) therefore it is an optimal solution of (3.1).*

$$\forall\ \mathbf{x} \ne \mathbf{x}^*,$$
$$(x^*_{n+k})^{-1} * p^k(x^*) \le (x_{n+k})^{-1} * p^k(x) \quad (3.6)$$

Proof. Suppose $\mathbf{x}^* = (x^*_1, x^*_2, \ldots, x^*_n, x^*_{n+k})$ is an optimal solution of (3.2), consequently

$$(x^*_{n+k})^{-1} * q^k(\mathbf{x}^*) = 1$$

$$g_i(\mathbf{x}^*) \le \xi_i\ ,\ \xi_i = \mp 1\ ,\ i = 1, 2, \ldots, m$$

Thus $\mathbf{x}^*$ is satisfied in constraint of (3.1).
Also by replacing
$q^k(\mathbf{x}^*) = x^*_{n+k}, q^k(\mathbf{x}) = x_{n+k}$
in the objective function (3.2) we have :

$$x^{*-1}_{n+k} * \Sigma_{t=1}^{T^k_\circ}\ \sigma^k_{\circ t}\ c^k_{\circ t} \Pi_{j=1}^n\ x_j^{*\alpha^k_{\circ tj}} \le$$
$$x^{-1}_{n+k} * \Sigma_{t=1}^{T^k_\circ}\ \sigma^k_{\circ t}\ c^k_{\circ t} \Pi_{j=1}^n\ x_j^{\alpha^k_{\circ tj}}$$

$$\frac{\Sigma_{t=1}^{T^k_\circ}\ \sigma^k_{\circ t}\ c^k_{\circ t} \Pi_{j=1}^n\ x_j^{\alpha^k_{\circ tj}}}{\Sigma_{t=1}^{T'^k_\circ}\ \sigma'^k_{\circ t}\ c'^k_{\circ t} \Pi_{j=1}^n\ x_j^{*\alpha'^k_{\circ tj}}} \le$$
$$\frac{\Sigma_{j \in J_k}\ c_j \Pi_{j \in I_k}\ x_i^{\alpha_{ji}}}{\Sigma_{j \in J'_k}\ c'_j \Pi_{j \in I_k}\ x_i^{\alpha_{ji}}} \quad (3.7)$$

Hence, $\mathbf{x}^*$ is an optimal solution of (3.1). □

Then it is possible for every fractional objective function $f^k(\mathbf{x}) = \frac{p^k(\mathbf{x})}{q^k(\mathbf{x})}$ in any MOSGFP problem using p new positive variables, x_{n+k} where $k = 1, \ldots, P$ and letting $q^k(\mathbf{x}) = x_{n+k}$, therefore the MOGFP problem (2.1) , leads to the following multi objective geometric programming problem which is reformulated as $f^k(\mathbf{x}) = x^{-1}_{n+k} * p^k(\mathbf{x})$ for $k \in \{1, 2, \ldots, P\}$ and contains of P new equal constraints:

$$Minimize\ (x^{-1}_{n+1} * p^1(\mathbf{x}),$$
$$x^{-1}_{n+2} * p^2(\mathbf{x}), \ldots, x^{-1}_{n+P} * p^P(\mathbf{x})) \quad (3.8)$$

Subject to

$$x^{-1}_{n+k} * q^k(\mathbf{x}) \le 1$$
$$-x^{-1}_{n+k} * q^k(\mathbf{x}) \le -1\ ,\ k = 1, 2, \ldots, P$$
$$g_i(\mathbf{x}) \le \xi_i\ ,\ \xi_i = \mp 1\ ,\ i = 1, 2, \ldots, m$$
$$L_{n+k} \le x_{n+k} \le U_{n+k}\ ,\ k \in \{1, 2, \ldots, P\}\ ,\ \mathbf{x} > 0$$

Where

$$p^k(\mathbf{x}) = \Sigma_{t=1}^{T^k_\circ}\ \sigma^k_{\circ t}\ c^k_{\circ t} \Pi_{j=1}^n\ x_j^{\alpha^k_{\circ tj}},$$
$$q^k(\mathbf{x}) = \Sigma_{t=1}^{T'^k_\circ}\ \sigma'^k_{\circ t}\ c'^k_{\circ t} \Pi_{j=1}^n\ x_j^{\alpha'^k_{\circ tj}}$$
$$g_i(\mathbf{x}) = \Sigma_{t=1}^{T_i}\ \sigma_{it}\ c_{it} \Pi_{j=1}^n\ x_j^{\alpha_{itj}}$$

$$q^k(\mathbf{x}), x_{n+k} > 0$$
$$\forall \mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)^T \in X \subseteq R,$$
$$0 \le \underline{x}_j \le x_j \le \overline{x}_j\ ,\ j = 1, 2, \ldots, n$$

4 Convexification strategies

A convex relaxation is an efficient tool to reach a global solution in non-convex (MIGP) problems. It is used to enlarge the feasible set of (3.2) and compute a lower bound on the optimal solution of MIGP problem.Since the nonlinear functions in constraints and objective of (3.2) are signomial functions, obviously each signomial function is convex if all the terms are convex [14]. Convexity requirements for signomial terms are provided with the following Theorem, [See 15].

Theorem 4.1. *(Maranas and Floudas 1995). A positive signomial term*
$f(\mathbf{x}) = c\Pi_{i=1}^n\ x_i^{\alpha_i}$ is convex if one of the following conditions hold:

(i) $\alpha_i \le 0 (for\ i = 1, \ldots, n).$

(ii) $\exists\ k \ne i\ ,\ \alpha_k > 0\ ,\ \alpha_i \le 0,\ \Sigma_{i=1}^n\ \alpha_i \ge 1.$

And a negative signomial term
$f(\mathbf{x}) = c\Pi_{i=1}^n\ x_i^{\alpha_i}$ is convex if $\alpha_i > 0$, (for $i = 1, \ldots, n$) and $\Sigma_{i=1}^n\ \alpha_i \le 1$
$\forall c < 0$, Then $\alpha_i \ge 0$, (for $i = 1, \ldots, n$) and $\Sigma_{i=1}^n\ \alpha_i \le 1$.
From theorem 1 it can be deduced that it is possible to convexify positive and negative signomial terms by using power transformations, so this fact

is applied in this paper to convexify non-convex signomial functions.
After utilizing power transformations to convexify a signomial term, the power transformation functions require to be approximated by piecewise linear functions. This study uses a common piecewise linear approximation, special order set type 2 (SOS2) (Beale and Tomlin) [12].
The fuzzy technique programming generates the compromise solution of a multi-objective optimization problem.It is applicable to the multi-objective signomial programming problems that presented in the past section. It is assumed that the offered problem in (3.8) is feasible and it has an optimal compromise solution.

5 Fuzzy programming to reformulate multi-objective signomial geometric programming problems

Fuzzy set theory ,which is a generalization of classical set theory to understand the ambiguity and uncertainty in the complication of the problems, was first presented by zadeh in 1965 [5]. Specifically a relatively practical introduction of fuzzy set theory into conventional multi objective linear programming problems was first introduced by Zimmermann in 1978 [20]. It has been successfully applied so far to solve different types of multi-objective decision making problems in the presence of fuzziness.
A fuzzy set is associated with its membership function which is defined from its elements into the interval $[0, 1]$. The following procedure from [17] is applied in solving a MOMISGP by a signomial geometric programming method with a linear membership function to reach an optimal compromise solution.
Define a fuzzy membership function $\mu_k(x)$ for the k-th objective function $x_{n+k}^{-1} * p^k(\mathbf{x})$ as

$$\mu_k(x)=\begin{cases} 1 & ,if\ x_{n+k}^{-1} * p^k(x) \le L_k \\ \frac{U_k - x_{n+k}^{-1} \cdot p^k(x)}{U_k - L_k} & ,if\ L_k \le x_{n+k}^{-1} * p^k(x) \le U_k \\ 0 & ,if\ x_{n+k}^{-1} * p^k(x) \ge U_k \end{cases} \quad (5.1)$$

Where

$$L_k \neq U_k\ ,\ k = 1, 2, \ldots, P$$

Then, maximize the membership function $\mu_k(x)$, $k = 1, 2, \ldots, P$ subject to the constraints (3.5) and then use max-min operator [7] to find a crisp model.
Considering a dummy variable T and formulating a crisp model for the fuzzy geometric programming problem as:

$$Max\ T\ i.e.\ Min : T^{-1} \quad (5.2)$$

Subject to

$$\frac{U_k - x_{n+k}^{-1} * p^k(x)}{U_k - L_k} \ge T$$

$$x_{n+k}^{-1} * q^k(\mathbf{x}) \le 1$$

$$-x_{n+k}^{-1} * q^k(\mathbf{x}) \le -1$$

$$g_i(\mathbf{x}) \le \xi_i\ ,\ \xi_i = \mp 1\ ,\ i = 1, 2, \ldots, m$$

$$L_{n+k} \le x_{n+k} \le U_{n+k}\ ,\ k \in \{1, 2, \ldots, P\}$$

Where

$$\begin{aligned} & p^k(\mathbf{x}) = \Sigma_{t=1}^{T_\circ^k}\ \sigma_{\circ t}^k\ c_{\circ t}^k \Pi_{j=1}^n\ x_j^{\alpha_{\circ tj}^k}, \\ & q^k(\mathbf{x}) = \Sigma_{t=1}^{T_\circ'^k}\ \sigma_{\circ t}'^k\ c_{\circ t}'^k \Pi_{j=1}^n\ x_j^{\alpha_{\circ tj}'^k} \\ & g_i(\mathbf{x}) = \Sigma_{t=1}^{T_i}\ \sigma_{it}\ c_{it} \Pi_{j=1}^n\ x_j^{\alpha_{itj}} \\ & \mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)^T > 0 \end{aligned}$$

$$T \ge 0$$

Further the inequalities in (5.1) can be reformulation as:

$$x_{n+k}^{-1} * p^k(\mathbf{x}) + (U_k - L_k)T \le U_k\ ,\ k = 1, 2, \ldots, P$$

Here, we can solve the crisp mixed integer geometric programming problem defined in(3.5)by using geometric programming algorithm to find $\mathbf{x}^*$ and estimate all the p number of signomial objective functions in (3.5) and consequently (3.1) at the optimal solution $\mathbf{x}^*$.

Example 5.1.

$$\begin{aligned} & Min\ f_1(\mathbf{x}) = \frac{x_1 x_2^{0.5} x_3^{1.2}}{2x_2x_3 + x_2^{2.1} + x_3^{1.3}} \\ & Min\ f_2(\mathbf{x}) = \frac{-2x_2^{-1}x_3^{-1} + 2x_1x_3}{-x_1^3 x_3^{-1.2} + x_3^2} \end{aligned} \quad (6.1)$$

$$\begin{aligned} s.t \quad & 2x_1 + 3x_2 + x_3 \ge 8 & (1) \\ & 3x_1 + x_2 \ge 5 & (2) \\ & x_2 + 2x_3 \ge 10 & (3) \\ & 2x_1 + x_3 \le 4 & (4) \\ & 0.2 \le x_1 \le 5, 1 \le x_2 \le 5, 1 \le x_3 \le 5 & (5) \end{aligned}$$

x_1 is a real variable and x_2, x_3 are integers

STEP1:

Replacing the denominators by $x_4\&x_5$ in the objective functions as follows:

$$2x_2x_3 + x_2^{2.1}x_3^{1.3} = x_4$$
$$-x_1^3x_3^{-1.2} + x_3^2 = x_5$$

Which further can be written as:

$$2x_2x_3x_4^{-1} + x_2^{2.1}x_3^{1.3}x_4^{-1} = 1$$
$$-x_1^3x_3^{-1.2}x_5^{-1} + x_3^2x_5^{-1} = 1$$

Which are converted as follows.

$$2x_2x_3x_4^{-1} + x_2^{2.1}x_3^{1.3}x_4^{-1} \leq 1 \quad (6)$$
$$-2x_2x_3x_4^{-1} - x_2^{2.1}x_3^{1.3}x_4^{-1} \leq -1 \quad (7)$$
$$-x_1^3x_3^{-1.2}x_5^{-1} + x_3^2x_5^{-1} \leq 1 \quad (8)$$
$$x_1^3x_3^{-1.2}x_5^{-1} - x_3^2x_5^{-1} \leq -1 \quad (9)$$

STEP2:

To find upper and lower bounds of this new variables, its needed to solve the following subproblems:

$$Min \quad l_4 : 2x_2x_3 + x_2^{2.1}x_3^{1.3} \quad (I)$$

s.t set of constraints (1) - (5)

$$Max \quad u_4 : 2x_2x_3 + x_2^{2.1}x_3^{1.3} \quad (II)$$

s.t set of constraints (1)-(5)

$$Min \quad l_5 : -x_1^3x_3^{-1.2} + x_3^2 \quad (III)$$

s.t set of constraints (1)-(5)

$$Max \quad u_5 : -x_1^3x_3^{-1.2} + x_3^2 \quad (IV)$$

s.t set of constraints (1) - (5)
The solutions of this subproblems are as follows:

$$l_4 = 93\ ,\ u_4 = 191.25\ ,\ l_5 = 6.1099\ ,\ u_5 = 12.9588$$

Therefore, the original problem is rewrite as following:

$$Min \quad f_1 = x_1x_2^{0.5}x_3^{1.2}x_4^{-1} \quad (6.2)$$

$$Min \quad f_2 = -2x_2^{-1}x_3^{-1}x_5^{-1} + 2x_1x_3x_5^{-1}$$

s.t set of constraints (1)-(9)

$$93 \leq x_4 \leq 191.25\ ,\ 6.1099 \leq x_5 \leq 12.9617 \quad (10)$$

STEP3:

Now, the multi-objective problem is reduced to single objective problem by fuzzy programming technique as (5.3),so by obtaining upper bounds U_i and lower bound L_i for two objectives ($i = 1, 2$), then $0.010876211 \leq f_1 \leq 0.04139956$ and $0.013 \leq f_2 \leq 0.58$
So we have:

$$Max\ T \quad (6.3)$$

$$s.t$$
$$x_1x_2^{0.5}x_3^{1.2}x_4^{-1} + 0.3052335T \leq 0.04139956$$
$$-2x_2^{-1}x_3^{-1}x_5^{-1} + 2x_1x_3x_5^{-1} + 0.567T \leq 0.58$$

s.t set of constraints (1) - (10)
Where x_1, x_4, x_5 are real and x_2, x_3 are integer variables.
Now, base on the branch and bound algorithm by solving this single programming problem by relaxing x_2, x_3, the optimal compromised solution is obtained as: $T = 0.844123$,

$$x_1 = 0.2, x_2 = 4.4, x_3 = 3.6, x_4 = 150.3780$$
$$and\ x_5 = 12.9617$$

Then we have:

$$f_1(x) = 0.0297521 \quad and \quad f_2(x) = 0.1013820$$

To reach the optimal integer solution and the integer compromised solution is achieved:

Table1: branching on integer variable

Case	$1:\ x_3 \geq \mathbf{4}$	$2:\ x_3 \leq \mathbf{3}$
T	infeasible	0.81385061
x_1	0.2	0.2
x_2	5	5
x_3	4	3
$f_1(x)$	–	0.010960338
$f_2(x)$	–	0.11854671

In this example in case 1, the problem is infeasible, so by comparing the solutions, it is clear that the optimal compromised integer solution for (6.3) and accordingly (6.1) is case 2 will be achieved.

6 Bound assessment algorithm

The best known method to find exact or at least ε-approximate solutions to non-convex mixed integer geometric programming problems is the spatial Branch-and-Bound algorithm, which rests on computing lower bounds to the value of the objective function to be minimized on each region. These lower bounds are often computed by solving convex

relaxations of the original program .The (SBB) algorithm works in a common application of tight lower bounds computed through a convex relaxation. In order to upgrade quickly evaluated quality of our proposed method to achieve lower bounds of the MOMIFGP problem (2.1), we implemented (using AMPL [13]) a simplified partial SBB algorithm [1],[16] which finds generally ε-approximate solutions for an arbitrarily small positive ε. The SBB algorithm works by recursively partitioning the search space along the coordinate direction that contributes most to the gap between lower and upper bounds on the optimal objective function value computed in each subproblem. For a nonlinear minimization problem, the lower bound is usually computed by constructing and solving a convex relaxation, and the upper bound can simply be a local optimum found by a (local) NLP solver.
At each branching step, registers the most reassuring node and discards the others,.As our purpose is a MOMIGFP problem that has both integer variables and convexified constraints, branching may be required on both integer and continuous decision variables.Since this algorithm works on minimization programming so we define a new variable S such that $S = T^{-1}$ in (5.3).

Propose Algorithm 1: The partial depth-first SBB algorithm

Input P as a non-convex MIGP problem and an iteration limit
Let count $\leftarrow 0$
Solve $\mathbf{P}$ locally to find x^* with objective function value S^*(incumbent)
Construct a convex relaxation $\mathbf{R}$ of $\mathbf{P}$
Solve $\mathbf{R}$ to find an optimum $\bar{x}$ with function value $\bar{S}$ (lower bound)
Choose the branching variable i with branching point $\bar{x}_i$
Let end $\leftarrow$ false

While end do
Let $\mathbf{P}_0$ be defined as $\mathbf{P}$ with the added constraint $x_i^L \leq x_i \leq \bar{x}_i$
Let $\mathbf{P}_1$ be defined as $\mathbf{P}$ with the added constraint $\bar{x}_i \leq x_i \leq x_i^U$
For $k \in \{0,1\}$, let $\mathbf{R_k}$ be the convex relaxation of $\mathbf{P}_k$
For $k \in \{0,1\}$, let $\bar{x}_k$ be the optimum of $\mathbf{R}_k$ with value $\bar{S}_k$
Let $l = \text{argmin } (\bar{S}_0, \bar{S}_1)$ (best lower bound to S^*)
if $\bar{S}_l < S^*$ then
Let **end** $\leftarrow$ true (node cannot improve)
else if $\bar{S}_l > \bar{S}$ then
Let $\bar{S} \leftarrow \bar{S}$ (overall bound improvement)
end
if x^* is infeasible in $\mathbf{P}_l$ then
Solve $\mathbf{P}_l$ locally to find $\hat{x}$ with value $\hat{S}$
if $\hat{S} < S^*$ then
Let $S^* \leftarrow \hat{S}$ and $x^* \rightarrow \hat{x}$ (improve the non-convex problem solution)
end
if $|S^* - \bar{S}| < \varepsilon$ or **end if** $>$ limit then
Let **end** $\leftarrow$ true (global optimum)
end
if $\bar{S}_l = \infty$ then
Let **end** $\leftarrow$ true (infeasible node)
end if
Let $\mathbf{P} \leftarrow \mathbf{P}_l$ and $\bar{x} \leftarrow \bar{x}_l$
Update the branching variable i and branching point $\bar{x}_i$
Increase count
end while

7 Numerical example

The test results are summarized in Table 2.
In this section, the proposed approach is tested on an example. Consider the following mixed integer multi-objective fractional geometric programming problem:

Example 7.1. *Find $X = (x_1, x_2)$ so as to,*

$$Min : f_1(x) = \frac{20x_1^{-1}x_2^3 + 60x_1^{-1}x_2^{-1}}{(\frac{1}{3})x_1x_2 + (\frac{1}{3})x_2},$$

$$Min : f_2(x) = \frac{50x_1^{-1}x_2^{-1} + 60x_1^2x_2^{-2}}{(\frac{1}{4})x_1x_2 + (\frac{1}{4})x_1},$$

Subject to

$$1 \leq x_1, x_2 \leq 5,$$
$$x_1 \in R, x_2 \in Z$$

Ideal solutions:

$$Min : f_1(x) = 6.9282032, x_1 = 5,$$
$$x_2 = 1.3160740, x_3 = 2.632148,$$
$$Max : f_1(x) = 753.59999, x_1 = 1, x_2 = 5,$$
$$x_3 = 3.3333333,$$
$$Min : f_2(x) = 4.8657616, x_1 = 2.0274006,$$
$$x_2 = 5, x_4 = 3.0411010,$$
$$Max : f_2(x) = 604, x_1 = 5, x_2 = 1,$$
$$x_3 = 2.4999981,$$

Using the ideal solution, we can obtained the lower bound L_i and upper bound U_i of the objective functions $f_i, i = 1, 2$

$$L_1 = 6.9282032 < f_1 < 753.5999999 = U_1;$$
$$and \quad L_2 = 4.8657616 < f_2 < 604$$

Step1:

Solving problem (**P**)

Using the fuzzy programming steps of ε-constraint method in [17] which causes the programming problem to be reformulated as:

$$\begin{aligned}
&Max: T\\
&20x_1^{-1}x_2^3x_3^{-1}+60x_1^{-1}x_2^{-1}x_3^{-1}\\
&+T(66.28606-20.670048)\leq 66.28606,\\
&50x_1^{-1}x_2^{-2}x_4^{-1}+60x_1^2x_2^{-2}x_4^{-1}\\
&+T(442.924839-36.123848)\leq 442.924839,\\
&(\frac{1}{4})x_1x_2x_4^{-1}+(\frac{1}{4})x_1x_4^{-1}=1,\\
&(\frac{1}{3})x_1x_2^2x_3^{-1}+(\frac{1}{3})x_2x_3^{-1}=1,\\
&1\leq x_1,x_2\leq 5,\\
&x_2\in Z, x_1,x_3,x_4\in R
\end{aligned}$$

The optimal conciliation solution is achieved as:

$$\begin{aligned}
&T^{-1*}=S^*=1.0337966, x_1^*=5, x_2^*=4,\\
&x_3^*=7.9999999\\
&f_1(x)=32.375\\
&f_2(x)=15.4
\end{aligned}$$

Step2:

Relaxing Problem(**R**)

The mentioned problem has 5 non-convex terms, to obtain a global solution, the convex relaxation which is used is power transformation [21] and piecewise linear approximation applied to underestimate convexified terms. So the non-convex primary programming problem is converted to:

$$\begin{aligned}
&Max: T\\
&50x_1^{-1}x_2^{-1}x_4^{-1}+60y_1^{-2}x_2^{-2}x_4^{-1}\\
&+T(442.924839-36.123848)\leq 442.924839,\\
&20x_1^{-1}x_2^3x_3^{-1}+60x_1^{-1}x_2^{-1}x_3^{-1}\\
&+T(66.28606-20.670048)\leq 66.28606,\\
&(\frac{1}{4})y_1^{-1}y_2^{-1}x_4^{-1}+(\frac{1}{4})y_1^{-1}x_4^{-1}=1,\\
&(\frac{1}{3})y_1^{-1}y_2^{-1}x_3^{-1}+(\frac{1}{3})y_2^{-1}x_3^{-1}=1,
\end{aligned}$$

Where $y_i=x_i^{-1}$,

$$\begin{aligned}
&1\leq x_1,x_2\leq 5,\\
&x_2\in Z, x_1,x_3,x_4\in R
\end{aligned}$$

The optimal solution is obtained as:

$$\begin{aligned}
&\bar{S}=1.0303940, \bar{x}_1=4.44, \bar{x}_2=3,\\
&\bar{x}_3=4.2051309, \bar{x}_4=3.2051313\\
&f_1(x)=29.9933677, f_2(x)=42.17546515
\end{aligned}$$

Table2: the achieved optimal conciliation solutions:

Table2: non-inferior solution for multi-objective fractional geometric programming problem

Problem	**Non–convex(P)**	**Convex(R)**
S	$S^*=1.033796$	$\bar{S}=1.030394$
$\mathbf{x_1}$	5	4.44
$\mathbf{x_2}$	4	3
$\mathbf{x_3}$	7.999999	4.2051309
$\mathbf{x_4}$	6.2499999	3.205131
$f_1(x)$	32.375	29.9933677
$f_2(x)$	15.4	26.7754

Therefore the difference between objectives of two problem is:

$$|S^*-\bar{S}|=1.033796-1.030394=0.34026*10^{-2}$$

Suppose the maximum of difference between objectives S^* and $\bar{S}$ is accepted for $\varepsilon=0.1*10^{-4}$ it means, we should find a tighter relaxation to implementing of the proposed algorithm:

Step3

Choose $\bar{x}_2$ as the branching point,
Define $\mathbf{P}_0$ as $\mathbf{P}$ with the added constraint $1\leq x_2\leq 3$ and let $\mathbf{R}_0$ be the convex relaxation of $\mathbf{P}_0$,
Define $\mathbf{P}_1$ as $\mathbf{P}$ with the added constraint $3\leq x_2\leq 5$, now let $\mathbf{R}_1$ be the convex relaxation of $\mathbf{P}_1$,
The optimal solutions of two relaxed problems $\mathbf{R}_0$ and $\mathbf{R}_1$ are obtained:

Table3: comparison of relaxed problems with different region

Problem	$\mathbf{R}_0$	$\mathbf{R}_1$
$\bar{S}$	1.03039412107	1.03039412223
x_1	4.44	4.44
x_2	3	3
x_3	3.205130	4.20513195
x_4	3.205130	3.20523239
$f_1(x)$	29.993387	29.993387
$f_2(x)$	42.1754509	42.1754496
$\|S^*-\bar{S}\|$	$0.34571879*10^{-2}$	$0.34025432*10^{-2}$

Hence, $l=1$, and $S^*<\bar{S}_1<\bar{S}$ then node and lower bound can improve, so $\bar{S}_1\to\bar{S}$

Step4:

Since x^* is feasible in $\mathbf{P_1}$, so no any change in x^* and S^*.

Step5:

$$|\bar{S} - S^*| = 0.34025432 * 10^{-2} > 0.1 * 10^{-4}$$

Here, in the next iteration of the algorithm, another variable is chosen to branch.

Repeat step 3:

Define $\mathbf{P}_0$ as $\mathbf{P}$ with the added constraint $1 \leq x_1 < 4.44$, and let $\mathbf{R}_0$ be the convex relaxation of $\mathbf{P}_0$,
Define $\mathbf{P}_1$ as $\mathbf{P}$ with the added constraint $4.44 \leq x_2 \leq 5$,
Now let $\mathbf{R}_1$ be the convex relaxation of $\mathbf{P}_1$,
The optimal solutions of two relaxed problems $\mathbf{R_0}$ and $\mathbf{R}_1$ are obtained:

Table4: Comparisonofrelaxedproblemswithdifferent region

Problem	problem$\mathbf{R}_0$	problem$\mathbf{R}_1$
$\bar{S}$	1.0322634	1.0337966
x_1	4.1156168	5
x_2	3	4
x_3	4.3525059	7.9999948
x_4	3.3525067	6.2499986
$f_1(x)$	31.2618066	32.37496278
$f_2(x)$	34.8908291	15.4
$\|S^* - \bar{S}\|$	0.15910^{-2}	$0.613 * 10^{-6}$

Hence, $l = 1$, and $S^* < \bar{S}_1 < \bar{S}$, then node and lower bound can improve, so $\bar{S}_1 \rightarrow \bar{S}$

Step4:

Since x^* is feasible in $\mathbf{P}_1$ then x^* and S^* wouldnt change.

Step5:

$$|\bar{S} - S^*| = 0.613 * 10^{-6} < 0.1 * 10^{-4}$$

By performance of the mentioned algorithm, we have so far found a lower bound for our convex relaxed problem in such a way that, the optimal solution of the our non convex problem is feasible over there and more over the obtained objective functions values of the convex problem is somehow near to the those of primary non convex problem till some acceptable extend. Here Table5 shows the achieved optimal conciliation solutions:

Table5: Non-inferior solution for mixed integer
multi objective fractional geometric programming problem

Problem	Non–convex(P)	Convex(R)
S	1.0337966653957	1.0337966044722
$\mathbf{x_1}$	1	1
$\mathbf{x_2}$	5	5
$\mathbf{x_3}$	4	4
$\mathbf{x_4}$	7.99999999	7.99999448
$f_1(x)$	32.37496278	32.37400000
$f_2(x)$	15.4	15.4

The result of this algorithm is the nearest optimal solution to the optimal solution of a primitive programming problem.
Obviously, the difference between every objective value in non-convex multi-objective primary problem and relaxed convexified problems is shown in table 6.

Table6: Comparing convex objectives and non-convex objectives on the first and last regions

$f_1^{\mathbf{P}}(x) - f_1^{\mathbf{R}_0}(x)$	$f_1^{\mathbf{P}}(x) - f_1^{\mathbf{R}}(x)$
0.0009627	2.3806323
$f_2^{p}(x) - f_2^{\mathbf{R}_0}(x)$	$f_2^{p}(x) - f_2^{\mathbf{R}}(x)$
0	27.109069

In this comparison, it is clear that, $\mathbf{R}_0$ achieves a closer lower bound to lower bound of $\mathbf{p}$.

8 Conclusion

In this paper a mixed integer multi-objective fractional geometric programming problem (MIMOFGP) is discussed and it is demonstrated efficiently by replacing a new additional variable for (MOFGP) problems. Since non-convex geometric programming needs to be converted to convex relaxation to reach a solution globally, a power transformation and piecewise linear approximation is applied to convexify and underestimate the primary nonconvex problem. To provide flexibility and for utilizing more effective relaxation strategies; an algorithm based on spatial branch and bound method is used. Since this algorithm works on an optimal solution, we applied a fuzzy strategy to reach an optimal solution for a multi-objective geometric programming problem. The results showed that the implementation of this algorithm on the multi-objective programming problems can make the lower bound of a relaxed problem closer to the lower bound of a non-convex multi-objective programming problem. This work can be proposed as an effective and essential step of convex relaxation for every mixed integer multi-objective programming problem.

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