Available online at http://sanad.iau.ir/journal/ijim/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 16, No. 3, 2024 Article ID IJIM-1629, 9 pages
Research Article

Science and Research Branch (IAU)

Gradual improvement of a CCR inefficient DMU using a finite sequence of the intermediate targets

A.R. Fakharzadeh Jahromi[1], H. Rostamzadeh, M. Oroji
Department of Mathematics, Shiraz University of Technology, Shiraz, Iran.

Submission Date: 2022/10/21, Revised Date: 2023/07/04, Date of Acceptance: 2024/05/18

Abstract
An important action to improve the performance of inefficient decision-making units (DMUs) is Finding an efficient target. This target determines the amount of changes in inputs and outputs to achieve efficiency. Although the usual models in data envelopment analysis (DEA) always consider this improvement and present a target as an efficient DMU, but in fact, for some DMUs achieving that target in one step is difficult and even sometimes impossible. For this reason, finding an intermediate target is an important fact in DEA. Accordingly, instead of improving in one step, this work achieves efficiency in several steps; therefore, the improvement is obtained gradually. In this regard, first the efficiency of DMUs is evaluated using the CCR model, and then a sequence of intermediate targets is provided for each inefficient DMU, such that moving in this direction reduces the inefficiency of these DMUs. Also, an example is presented to illustrate the proposed method and its application.

Keywords: Data envelopment analysis, Efficiency, Gradual improvement, Intermediate targets, Returns to scale.

[1] Corresponding author: Email: a_fakharzadeh@sutech.ac.ir

1. Introduction and backgrounds

The purpose of DEA is not only to calculate the efficiency score of DMUs, but also, it improves the performance of inefficient units; in this regard, an important issue is how this improvement is done. By using appropriate models are determined the necessary target on the efficient frontier for the inefficient DMU, so that this DMU with reaching to the target becomes efficient. Despite the determination of the efficient improved target for inefficient DMUs, due to the limitations of the organization (budget, time, necessary infrastructure, etc), it is not always possible for the organization to achieve the improved efficient target in one step. In this paper, based on the latest studies and considering the existing limitations in organization, we are going to present a multi-step method in DEA for propelling an inefficient DMU to an efficient one.

In the second decade after the emergence of the DEA, in 1999, Cooper et al. [1] proposed the RAM model to determine efficient targets for inefficient DMUs; this model determines the farthest target for inefficient DMUs. In the same year, Frey et al [2] found the nearest target for inefficient DMUs. In 2005, Lezano et al. [3] modified the MIP model and they defined intermediate targets in each stage of their proposed model. Dehnukhalji and Soltani [4] introduced a sequence of intermediate targets for propelling each inefficient DMU to an efficient one; the differences between Dehnukhalji and Soltani's method with Lasano one are: returns to scale will not change during Dehnukhalji and Soltani's method, and the obtained target from Dehnukhalji and Soltani's method in each step is the closest target to the efficient frontier. Moreover, Sharafi et al. [5] improve the inefficient units using the gradient line method.

In this paper, by determining the intermediate targets and based on the CCR model, we propel an inefficient DMU to an efficient one in a multi-stage procedure. Our model is derived from the presented work in [4]. Since the CCR model with constant returns to scale property detects the inefficient DMUs much better in comparison with the BCC model, therefore, our model compared to the presented work in [4] can be more popular and practical. This paper is organized in 4 sections. After presenting an interdiction and some necessary literature reviews, in Section 2 by reminding the related basic concepts the suggested method and model are presented. Then, an applicable example is demonstrated to illustrate the mentioned proposed approach in Section 3, and the final section is devoted to some conclusion remarks.

2. Basic concepts and the proposed approach

Consider *n* DMUs with *m* input and *s* output, such that $X_j = \left(x_{1j}, x_{2j}, \ldots, x_{mj}\right)^t > 0$ and $Y_j = \left(y_{1j}, y_{2j}, \ldots, y_{sj}\right)^t > 0$ are the input and the output vectors of DMU_j for $j = 1, 2, \ldots, n,$ respectively.

Definition 1. The set of all possible activities is called the production possibility set (PPS) and it is defined as follows:

$$PPS = \{(X,\ Y) \in R^{m+s} \mid \sum_{j=1}^{n} \lambda_j X_j \leqslant X,\ \sum_{j=1}^{n} \lambda_j Y_j \geqslant Y,\ \lambda_j \in \Lambda^{CRS},\ \lambda_j \in \Lambda^{VRS}\} \quad (1)$$

and we denote, $\Lambda^{CRS} = \{\lambda_j \in R^n \mid \lambda_j \geqslant 0\}$ and $\Lambda^{CRS} = \{\lambda_j \in R^n \mid 1\lambda = 1,\ \lambda_j \geqslant 0\}$.

The returns to scale of a unit in *PPS* depend on the rate of change in the outputs to change in inputs. *CRS* and *VRS* are considered for the states of the constant return to scale and the variable return to scale, respectively. Table 1 shows the CCR and BCC models in the forms of the envelopment and multiplier for evaluating DMU_o ($o \in \{1, 2, \ldots, n\}$). In these models, the optimal value of the objective function θ_O is called the DMU_o efficiency score. DMU_o is efficient if $\theta_O = 1$, otherwise it is called inefficient.

Table 1: CCR and BCC models

envelopment model (2)	Multiplier model (3)
$\theta_o = \text{Min}\,\theta$ $\text{S.to}: \sum_{j=1}^{n} \lambda_j x_{ij} \leqslant \theta X_o;$ $\sum_{j=1}^{n} \lambda_j y_{rj} \geqslant Y_o;$ $\lambda_j \in \Lambda,\ j = 1, 2, ..., n.$	$\theta_o = \text{Max}\,UY_o + u_0$ $\text{S.to}: VX_o = 1;$ $UY_j - VX_j + u_0 \leqslant 0,\ j = 1, 2, ..., n;$ $U \geqslant 0;$ $V \geqslant 0;$ $u_0 \in \Gamma,$

where $\Gamma^{CRS} = \{0\}$ and $\Gamma^{VRS} = \mathbb{R}$.

Definition 2. The radial image of DMU_o in the *CCR* model is as $(\hat{X}_O, \hat{Y}_O) = (\theta_O X_O, Y_O)$.

Theorem 1. Suppose (U^*, V^*, u_0^*) is an optimal solution of model (3) in the evaluation of DMU_o, then $H_O = \{(X, Y) \mid U^*Y - V^*X + u_0^* = 0\}$ is a supporting hyperplane of the production possibility set.

Proof: Refer to [6].

A subset of *PPS* that their radial image belongs to the corresponding support hyperplane of DMU_o (hyperplane H_O) is denoted by P_{Ho}, therefore

$$P_{HO} = \{(X, Y) \in PPS \mid (\theta_O X, Y) \in H_O\}. \qquad (4)$$

We define:

$$\{i_1, i_2, \ldots, i_q\} \equiv \{i \mid v_i^* = 0\}, \quad \{r_1, r_2, \ldots, r_p\} \equiv \{r \mid u_r^* = 0\}, \qquad (5)$$

where v_i^* and u_r^* are the optimal weights of i_{th} input and r_{th} output in model (3) respectively; and also, we define:

$$K_{H_O}=\{(X_j,Y_j)\,|\,j\in\{1,2,\dots,n\},(X_j,Y_j)\in H_O\},\ Pos(A)=\left\{\sum_{j=1}^{n}\lambda_j a_j\,|\,\lambda_j\ge 0,\ j=1,2,\dots,n\right\}. \quad (6)$$

With these symbols, Nasrabadi et al. [7] proved that P_{Ho} is equal to the following set and also, they proved Theorem 2 by regarding:

$$P_{Ho}=\left\{\left(\alpha X+\sum_{k=1}^{q}d_k e_{i_k},\ \beta Y-\sum_{l=1}^{p}d'_l e'_{rl}\right)\Bigg|\ \begin{array}{c}\alpha\ge 1,\ 0<\beta\le 1,\ (X,Y)\in Pos(K_{H_O}),\\ (d,d')=(d_1,\dots,d_q,d'_1,\dots,d'_p)\ge 0,(d'_1,\dots,d'_p)\le\beta(y_{r_1o},\dots,y_{r_po})\end{array}\right\}, \quad (7)$$

where e_{ik} is the standard unit vector in $\mathbb{R}^m$ that its i_k-th component equal to 1 and e'_{rl} is the same vector in $\mathbb{R}^s$ that its i_l-th component equal to 1.

Theorem 2. Suppose (U^*,V^*,u_0^*) is an optimal solution of model (3) in the evaluation of DMU_o; in this case

1 .If $u_0^*=0$, then each $(X,Y)\in P_{H_O}$ has constant returns to scale.

2 .If $u_0^*>0$, then each $(X,Y)\in P_{H_O}$ has non-decreasing returns to scale.

3. If $u_0^*<0$, then each $(X,Y)\in P_{H_O}$ has non-increasing returns to scale.

Proof: See [7].

Theorem 2 shows that if DMU_o belongs to the class of returns to constant scale, then all the points of the set P_{H_O} also belong to this class. Therefore, the return to scale property of each unit in P_{H_O} is similar to DMU_o.

A basic point in determining intermediate targets and improving the performance of inefficient DMUs is the resemblance criterion. In this paper, we define the similarity criterion based on the return to scale property and the efficiency score. In other words, we move an inefficient DMU toward the efficiency frontier via a sequence of targets that all have the same return to scale as the target DMU and their efficiency scores are increasing as well. Also, to reach the efficient frontier in fewer steps, in each step we minimize the distance of the evaluated unit from the efficiency frontier. For finding a sequence of the targets similar to DMU_o, we need a subset of *PPS* including DMUs with the efficiency score greater than or equal to the DMU_o efficiency score, and we limit our search to the subscription of this subset and P_{H_O}.

Theorem 3. Suppose $(X,Y)\in\bar{H}_O=\{(X,Y)\in PPS\,|\,U^*Y-\theta_O V^*X+u_0^*\ge 0\}$ and $\bar{\theta}$ is the efficiency score of $\left(X, Y\right)$; then $\bar{\theta} \geq \theta_O$.

Proof: Refer to [4].

To start our mentioned procedure, at first, we solve the *CCR* model for all DMUs to determine the inefficient DMUs and their corresponding supporting hyperplanes. Then, to find a target similar to DMU_o, we use the following model.

$$\begin{aligned} & \text{Min}\ dist((X,Y),(\bar{X},\bar{Y})) \\ & S.to: (X,Y) \in \underline{P_{H_o}}; \qquad (8) \\ & \quad (\underline{X},\underline{Y}) \in \overline{H}_o; \\ & \quad (\bar{X},\bar{Y}) \in H_o, \end{aligned}$$

where $\left(X, Y\right)$ belongs to the allowed area for changing the inputs and outputs of DMU_o and $\left(\bar{X}, \bar{Y}\right)$ is an arbitrary point on the corresponding supporting hyperplane of DMU_o. But by applying the P_{H_O} set in model (8), nonlinear constraints are created. To get rid of this problem, we present the following lemma which is necessary to prove the next theorem 4.

Lemma 1. The P_{H_O} set is equal to the following set:

$$\bar{P}_{H_o} = \left\{ (X,Y) \mid \begin{array}{c} X = \bar{X} + \sum_{k=1}^{q} d_k e_{i_k}\ ,\ Y = \bar{Y} - \sum_{l=1}^{p} d_l^{'} e_{rl}^{'}\ ,\ \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \sum_{j \in K_{H_O}} \lambda_j \begin{pmatrix} \alpha_j X_j \\ \beta_j Y_j \end{pmatrix}, \alpha_j \geq 1, \\ 0 < \beta_j \leq 1\ ,\ \lambda_j \geq 0\ ,\ \left(d, d^{'}\right) = \left(d_1, \ldots, d_q, d_1^{'}, \ldots, d_p^{'}\right) \geq 0\ ,\ \left(d_1^{'}, \ldots, d_p^{'}\right) \leq \beta\left(y_{r_1 O}, \ldots, y_{r_p O}\right) \end{array} \right\}. \quad (9)$$

Proof: If $\alpha_j = \alpha$ and $\beta_j = \beta$ for all j, then $\bar{P}_{H_O} \subseteq P_{H_O}$. Conversely, if we put $\alpha = \dfrac{\sum_{j \in K_{H_O}} \lambda_j \alpha_j X_j}{\sum_{j \in K_{H_O}} \lambda_j X_j}$ and $\beta = \dfrac{\sum_{j \in K_{H_O}} \lambda_j \beta_j Y_j}{\sum_{j \in K_{H_O}} \lambda_j Y_j}$, then we can conclude that $P_{H_O} \subseteq \bar{P}_{H_O}$. □

Theorem 4. The $\bar{P}_{H_O}$ set is equal to the following set:

$$\hat{P}_{H_O} = \left\{ \left(X, Y\right) \mid X \geq \sum_{j \in K_{H_O}} \lambda_j X_j\ ,\ Y \leq \sum_{j \in K_{H_O}} \lambda_j Y_j\ ,\ \lambda_j \geq 0 \right\}. \quad (10)$$

Proof: It is clear that $\bar{P}_{H_O} \subseteq \hat{P}_{H_O}$. Suppose $\left(X, Y\right) \in \hat{P}_{H_O}$ then $X \geq \sum_{j \in K_{H_O}} \lambda_j X_j$ and $Y \leq \sum_{j \in K_{H_O}} \lambda_j Y_j$, therefore, there exist $S, S^{'} \geq 0$ such that

$$X = \sum_{j \in K_{H_O}} \lambda_j X_j + S = \sum_{j \in K_{H_O}} \lambda_j X_j + \sum_{j \in K_{H_O}} \eta_j \lambda_j X_j = \sum_{j \in K_{H_O}} \left(1+\eta_j\right) \lambda_j X_j = \sum_{j \in K_{H_O}} \alpha_j \lambda_j X_j = \bar{X},$$

$$Y = \sum_{j \in K_{H_O}} \lambda_j Y_j - S' = \sum_{j \in K_{H_O}} \lambda_j Y_j - \sum_{j \in K_{H_O}} \eta'_j \lambda_j Y_j = \sum_{j \in K_{H_O}} \left(1-\eta'_j\right) \lambda_j Y_j = \sum_{j \in K_{H_O}} \beta_j \lambda_j Y_j = \bar{Y}$$

which $\eta_j \geq 0,\ \alpha_j = 1+\eta_j,\ 0 \leq \eta'_j < 1$ and $\beta_j = 1-\eta'_j$ for all $j \in K_{H_O}$, therefore $\left(X, Y\right) \in \bar{P}_{H_O}$. □

According to lemma 1, the sets P_{H_O} and $\hat{P}_{H_O}$ are equal, and according to theorem 4, the sets of $\bar{P}_{H_O}$ and $\hat{P}_{H_O}$ are also equal. Therefore, the set $\hat{P}_{H_O}$ can be replaced by the set P_{H_O}, and therefore, model (8) could be rewritten as model (11):

$$\begin{aligned}
&\text{Min } 1\frac{|X-\bar{X}|}{X_o} + 1\frac{|Y-\bar{Y}|}{Y_o} \\
&\text{S.to}: \sum_{j \in K_{H_o}} \lambda_j^t X_j \leqslant X; \\
&\qquad \sum_{j \in K_{H_o}} \lambda_j^t Y_j \geqslant Y; \\
&\qquad U_o^* Y - \theta^{*t-1} V_o^* X + u_0^* \geqslant 0; \\
&\qquad U_o^* \bar{Y} - V_o^* \bar{X} + u_0^* = 0; \\
&\qquad L_I^t \leqslant \frac{X - X^{t-1}}{X^{t-1}} \leqslant 0; \\
&\qquad 0 \leqslant \frac{Y - Y^{t-1}}{Y^{t-1}} \leqslant U_o^t; \\
&\qquad \lambda_j^t \geqslant 0,\ \forall j \in K_{H_o},
\end{aligned} \tag{11}$$

where all the variables are positive, *o* is the index of the evaluated DMU and *t* is the index of the intermediate target in step *t*. X^t and Y^t are the input and the output vectors of the intermediate target in step *t* respectively and θ^{*^t} is the optimal value of the CCR model in the evaluation of $\left(X^t, Y^t\right)$; also, L_I^t s are the lower bounds of the allow able changes of the inputs and U_O^t s are the upper bounds of the allow able changes of the outputs in step *t*. Note that these lower and upper bounds are parameters that should be determined by the decision-maker based on the ability of the DMU_o to change the inputs and outputs; these bounds can be changed at each stage as well. The objective function of model (11) is a nonlinear function, which can be linearized by changing the variables $\left|X-\bar{X}\right| = a+b$ and $\left|Y-\bar{Y}\right| = c+d$, where $a, b, c, d \geq 0$.

Theorem 5. Model (11) is feasible and has a finite optimal solution.

Proof: Since $\left(X^{t-1}, Y^{t-1}\right) \in P_{H_O}$, then $\left(X^{t}, Y^{t}\right) \in P_{H_O}$, therefore $P_{H_O} \neq \varnothing$ in each step. As a result, there are coefficients λ_j^t which leads to the feasibility of model (11) and other variables are chosen as follows:

$$\left(X, Y\right) = \left(X^{t-1}, Y^{t-1}\right), \qquad \left(\bar{X}, \bar{Y}\right) = \left(\theta^{*^{t-1}} X^{t-1}, Y^{t-1}\right). \tag{12}$$

Also, the existence of a zero-lower bound for the objective function guarantees that the optimal value of the objective function is finite. □

Now, to determine a sequence of targets for the inefficient DMU_o, we present the following algorithm:

Step 1: Put t=0;

Step 2: Put $Y^t = Y_O$, $X^t = X_O$;

Step 3: Evaluate $\left(X_O, Y_O\right)$ by model (3) and determine the values of $\left(U_O^*, V_O^*, u_0^*\right)$;

Step 4: Evaluate $\left(X^t, Y^t\right)$ by model (3) and obtain the value of θ^{*^t};

Step 5: If $\theta^{*^t} = 1$, stop and otherwise go to step 6;

Step 6: Put $t+1 \to t$ and get the values $\left(X^*, Y^*\right)$ from model (11);

Step 7: Put $\left(X^t, Y^t\right) = \left(X^*, Y^*\right)$ and go to step 4.

According to $\theta^{*^t} \leq 1$ and increasing of θ^{*^t}, the algorithm ends after the finite number of iterations.

3. Numerical example

Consider an organization including 6 DMUs with one input and one output, whose data are taken from [4] and are reported in Table 2. As seen in this table, the BCC efficiency scores of DMUs A, B and E are equal to one, as a result, these DMUs are efficient based on the BCC model. But according to the CCR model, only DMU B is efficient and DMUs A, C, D, E and F are inefficient. As expected, the CCR model detects inefficiencies, much better.

Table 2. Data and efficiency scores of BCC and CCR.

DMU	X	Y	θ^*_{BCC}	θ^*_{CCR}
A	10	10	1.00	0.50
B	18	36	1.00	1.00
C	30	10	0.33	0.17
D	35	40	0.69	0.57
E	55	60	1.00	0.55
F	60	20	0.22	0.17

Suppose that the decision maker has allowed that the maximum of the input reduction be 20% and the output increase be 30%. The intermediate targets with the CCR efficiency scores of these targets for the inefficient DMUs are presented in Table 3. As an example, DMU C with activity vector (10, 30) has the CCR efficiency score 0.17. The decision-maker must reduce about 83% of its the input in one step that unit C achieves efficiency, while the capacity reduction of the input is maximum 20%. Therefore, it is impossible that the decision-maker reaches to this target in one step. By applying the mentioned algorithm on DMU C, the first intermediate target is obtained with the activity vector (13, 24). Therefore, it is enough that in the first step the input decreases to 6 and the output increases to 3. Thus, with this improvement, the efficiency score will be 0.27. With 4 repetitions of the algorithm, the final efficiency for DMU C is obtained with the activity vector (24.50, 12.29). The results of this example are reported in Table 3. Note that in running the algorithm for each inefficient DMU, the returns to scale do not change, and also the efficiency scores are increasing as expected and the final efficiency score is equal to one.

Table 3. Intermediate targets and their CCR efficiency scores.

DMU		Step0	Step1	Step2	Step3	Step4
A	X	10	8	6.50	---	---
	Y	10	13	13	---	---
	θ^*_{CCR}	0.50	0.81	1.00	---	---
C	X	30	24	19.20	15.36	12.29
	Y	10	13	16.90	21.97	24.50
	θ^*_{CCR}	0.17	0.27	0.44	0.72	1.00
D	X	35	28	26	---	---
	Y	40	52	52	---	---
	θ^*_{CCR}	0.57	0.93	1.00	---	---
E	X	55	44	39	---	---
	Y	60	78	78	---	---
	θ^*_{CCR}	0.55	0.89	1.00	---	---
F	X	60	48	38.40	30.72	24.58
	Y	20	26	33.80	43.94	49.15
	θ^*_{CCR}	0.17	0.27	0.44	0.72	1.00

4. Conclusion

In this paper, an algorithm is presented to generate a sequence of intermediate targets for each inefficient DMU based on CCR model, such that the return to scale of all the targets in this sequence is the same with the inefficient DMU. Also, the efficiency scores of the targets in this sequence are increasing and the final target is efficient. In other words, the introduced targets in the sequence of intermediate are completely similar to the inefficient DMU, and the efficiency scores gradually are improved to get to the efficiency. In each step, according to the decision-maker's opinion and based on management constraints, the lower and upper bounds for changing the inputs and outputs of the inefficient DMU are determined. Based on these bounds, a sequence of targets similar to the inefficient DMU and as close as possible to the efficiency frontier is obtained.

References

[1] Cooper, W.W., Park, K.S. and Pastor, J.T. (1999), “RAM: A range adjusted measure of inefficiency for use with additive models and relations to other models and measures in DEA”, Journal of Productivity Analysis, Vol. 11 No. 1, pp. 5-42. https://doi.org/10.1023/A:1007701304281

[2] Frei, F.X. and Harker, P.T. (1999), “Projections onto efficient frontiers: Theoretical and computational extensions to DEA”, Journal of Productivity Analysis, Vol. 11 No. 3, pp. 275-300. https://doi.org/10.1023/A:1007746205433

[3] Lozano, S. and Villa, G. (2005), “Determining a sequence of targets in DEA”, Journal of the Operational Research targets in DEA, Journal of the Operational Research Society, Vol. 56 No.12, pp.1439-1447. https://doi.org/10.1057/palgrave.jors.2601964

[4] Dehnokhalaji, A. and Soltani, N. (2018), “Gradual efficiency improvement through a sequence of targets”, Journal of the Operational Research Society, Vol. 70 No. 12, pp. 2143-2152. https://doi.org/10.1080/01605682.2018.1529723

[5] Sharafi, H. (2021). “Gradual Improvement of Benchmarking in Data Envelopment Analysis Using Gradient Line Method”, International Journal of Data Development Analysis, Vol. 9 No.3, pp.31-46. http://ijdea.srbiau.ac.ir

[6] Cooper, W.W., Seiford, L.M. and Tone, K. (2006), “Data envelopment analysis: a comprehensive text with models, applications, references and DEA-solver software”, New York, NY:Springer. https://doi.org/10.1007/978-0-387-45283-8

[7] Nasrabadi, N., Dehnokhalaji, A. and Soleimani-Damaneh, M. (2014), “Characterizing a subset of the PPS maintaining the reference hyperplane of the radial projection point”, Journal of the Operational Research Society, Vol. 65 No.12, pp.1876-1885. https://doi.org/10.1057/jors.2012.170