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The Effect of Meta-Malmquist Index on Portfolio Optimization

Z. Taeb *, SH. Banihashemi †‡

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Abstract

Since the change of conditional value at risk (CVaR) in different confidence levels is very effective in portfolio optimization, the meta-Malmquist index (MMI) is utilized. For this purpose, mean-CVaR models by MMI in the presence of negative data are introduced. Like Markowitz theory in mean-variance framework, we use CVaR as a risk measure and propose our models without considering the skewness and kurtosis of assets return. In our study there are some negative data, so our models is based on Range Directional Measure (RDM) model that can be taken positive and negative data. In this paper efficiencies are obtained in all confidence levels by mean-CVaR models and MMI is calculated on confidence levels as periods in the presence of negative data. This method could help the investors to construct their profitable portfolio by using MMI index. We, also carry out an empirical study within Iran stock exchange market.

Keywords : Portfolio optimization; Efficiency score; Conditional value at risk; Meta Malmquist index; Negative data.

1 Introduction

Portfolio optimization is proven to be a powerful tool for allocation of limited capital to available financial assets through determining maximum return and mitigating the uncertainty (risk). As a result, it has been implemented by various investors to evaluate the portfolio or asset performance and productivity change to select the most proper portfolio. Data Envelopment Analysis (DEA), an efficiency nonparametric analysis tool, was first introduced by Charnes et al. [6] as method for assessing the portfolio performance. His groundbreaking work paved the way for numerous researchers to apply DEA method in various application. For instance, Morey and Morey [19] employed DEA's ability to evaluate mutual funds while Joro and Na [11] presented a DEA-like framework in mean-variance-skewness framework to assess a portfolio performance. Later on, Caves et al. [7] introduced the computation of productivity change by means of efficiency measures which was developed in the context of parametric and non-parametric efficiency measurement by Nishimizu and Page [21] and by Färe et al. [8], respectively. The Färe et al. [8] approach is known as the measure-

*Department of Mathematics, Yadegar Imam Branch, Islamic Azad University, Tehran, Iran.

†Corresponding author. shbanihashemi@atu.ac.ir, Tel:+98(912)4383739.

‡Department of Mathematics, Faculty of Mathematics and Computer Science, Allameh Tabataba'i University, Tehran, Iran.

ment of productivity change through Malmquist indices. Since, in many situations, some inputs and/or outputs and input-output prices are imprecise, Akhbarian [2] propose another method for measuring the overall profit Malmquist productivity index (MPI) when the inputs, outputs, and price vectors vary over intervals. Aghyia et al. [1] employed the Malmquist Productivity Index (MPI) and integrated Data Envelopment Analysis (DEA) for evaluating the function of Decision-Making Units (DMUs) by using the directional distance function with undesirable interval outputs.

Markowitz was the first one who mathematically formulated the portfolio selection process [16, 17]. However, using variance as a risk measure in his theory has been extensively criticized by his peer mathematicians due to its symmetrical measure. Hence, an alternative risk measure called Value-at-Risk (VaR) was proposed by Baumol [5]. Since VaR, as a measure of risk, is not always sub-additive nor convex, Rockafeller and Uryasev [24, 25] defined a more comprehensive risk measure as Conditional Value-at-Risk (CVaR). CVaR, as risk measure, is a coherent risk measure which has translation invariance, positive homogeneity, subadditivity, and monotonicity properties. These unique characteristics has encouraged many researchers such as John and Hafiz [10] and Huang et al. [9] to utilize CVaR as a risk measure for financial problems. Liu [15] developed a method that consider stop profit point and exit time to reduce investment risk and introduce a new risk measure stop point probability cvar (SPP-CVaR). The SPP-CVaR can simultaneously measure the price risk and exit risk. Liang et.al [14] developed a new risk measure exit probability cvar (EP-CVaR) using transition density for the problem of risk assessment when investors implement the stop strategy. The EP-CVaR method can solve the problem of uncertain exit time caused by use of the stop strategy. Kobayashi et al. [13] propose a high-performance algorithm named the bilevel cutting-plane algorithm for exactly solving the cardinality-constrained mean-CVaR portfolio optimization problem for limiting the number of invested assets. In this study, we use Rockafellar and Uryasev [24] technique which estimates the mean-CVaR optimization problem by linear programming problem, based on generated scenarios. Conventional DEA models do not employ negative data for input/outputs, and as a result cannot be used where the data accepts both negative and positive values. While asset returns include both positive and negative values over a period of time. Therefore, to deal with negative rate of return, we use the range directional measure (RDM) model [22]. Portela et al. [22] proposed (RDM), a DEA-like framework based on generic distance function to deal with negative data. He also used RDM efficiency measures to arrive at a Malmquist index, which can reflect productivity change [23]. Also, Mohammadi et al. [20] illustrate how the biennial Malmquist index can be utilized, not only for comparing the performance of a unit in two time periods, but also for comparing the performance of two different units at the same or different time periods in the presence negative data. Kerstens et al. [12] proposed Geometric Representation of the mean-variance-skewness portfolio frontier based upon the shortage function. Banihashemi et al. [4] applied mean-CVaR model in RDM-like framework. By the fact that stock market's return distributions usually exhibit skewness, kurtosis and heavy-tails, Mirsadeghpour et al. [18] considered (multivariate Skewed T) mST and (multivariate Generalized Hyperbolic) mGH distributions inspired by RDM which have impact on input and output for portfolio performance evaluation. In previous works, CVaR is obtained in a confidence level for example 90% or 95% or 99%, and optimal portfolio is selected by obtained CVaR and mean return.

The purpose of this study is to create a better portfolio with new approach by considering the increase in the conditional value at risk in different confidence levels with mean-CVaR models in RDM-like framework. So, portfolio has been optimized by implementing the meta- Malmquist index into mean- CVaR models without considering the skewness and kurtosis of assets return rate. We choose risk measure CVaR as the input and mean return as the only output in mean-CVaR models. CVaR is calculated the at three confidence levels, namely 90%, 95%, and 99%. These confidence levels are referred to as period t in the model. By considering the confidence level as 99% and 95%, efficiencies are obtained by mean-CVaR model. Period frontiers (t and $t+1$) and a meta-frontier (lying above the period t and $t+1$ frontiers) are confidence levels 99%, 95% and 90% at CVaR respectively. We illustrate how the MMI can be used for comparing the performance of an asset in two periods, such as two confidence levels. Since, mean return can take negative values, we utilize mean-CVaR model based on RDM model. In our models, efficiency of under evaluation asset is characterized by projection point and its distance from the efficient frontier. If the under-evaluation asset is not located on the efficient frontier, it is defined as "inefficient asset" for which, the model shows maximal proportionate reduction in CVaR and the same proportional maximization in the mean of return. Efficiencies are obtained from the proposed mean-CVaR models at various confidence levels such as; 90%, 95%, and 99%. Malmquist index is computed for these efficiencies in all confidence levels in the presence of negative data. Also, MMI will be calculated from multiplication efficiency change and technology change for each at time t and $t+1$ based on mean-CVaR models. MMI is calculated based on confidence levels 99% to 95% as periods t to $t+1$ and confidence level 90% is considered as meta frontier. When MMI is greater than 1, productivity of asset j has improved from t to $t+1$. Clearly changes in the risk value (CVaR) at different confidence levels, will affect the MMI.

The rest of the paper is structured as follows. In section 2, we provide an overview of the preliminary concepts of coherent risk measure, VaR and CVaR as risk measures, Malmquist index and meta Malmquist index. Our models have been proposed based on mean-CVaR model and efficiencies have been obtained in all confidence level by using Malmquist productivity index presented in Section 3. In section 4, we present a real world application for Iran Stock Exchange market. Section 5 concludes the paper.

2 Preliminary

The required definitions for the upcoming sections are as follows:

Definition 2.1. *Assume$(\Omega, F, P)$to be the probability space and$I(\Omega, F)$ to be the set of random variables of one dimensional on the space. The function $\rho : I(\Omega, F) \longrightarrow R$ is a coherent risk measure whenever it satisfies following axioms for all $X,Y \in I(\Omega, F)$, X and Y are random variables:*

a) *Monotonicity: If $X \leq Y$, then $\rho(Y) \leq \rho(X)$;*

b) *Subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$;*

c) *Translation Invariance: For all $\alpha \in R, \rho(X+\alpha) = \rho(X) - \alpha$;*

d) *Positive homogeneity: for all $\lambda \geq 0, \rho(\lambda X) = \lambda \rho(X)$.*

Definition 2.2. *The risk measure VaR at confidence level $\alpha \in (0,1)$, is the smallest value Γ such that the probability that the loss exceeds Γ is no larger than $1-\alpha$. The other hand*

$$VaR_\alpha(X) = \inf\{\Gamma \in R | P(X \leq \Gamma) > \alpha\} \quad (2.1)$$

Definition 2.3. *To obtain CVaR as defined in references [24, 25], we define vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ which represents the position of each of n financial instruments in a portfolio.*

On the other hand, the vector of instruments' return is defined as $r = (r^1, r^2, \ldots, r^n)$. The return on a portfolio is the summation of the returns on the individual instruments in the portfolio. As a result, the loss function is equal to the negative value of return on portfolio and is given by:

$$f(\lambda, r) = -(\lambda_1 r^1 + \lambda_2 r^2 + \cdots + \lambda_n r^n) = -\lambda^T r.$$

CVaR is specified as the

$$\begin{aligned} CVaR_\alpha =& E\Big[f(\lambda, r) | f(\lambda, r) \geq VaR_a\Big] \\ =& \frac{1}{1-\alpha} \int_{VaR_\alpha}^{+\infty} x p(x) dx \end{aligned} \quad (2.2)$$

Where E is the expectation operator, and $p(x)$ is the probability density function(PDF) of the loss $f(\lambda, r)$. It is shown that CVaR has an equivalent definition as follows

$$CVaR_\alpha = \min_{\Gamma \in R} F_\alpha(\lambda, \Gamma)$$

where $F(\lambda, \Gamma)$ defined as

$$\begin{aligned} F_\alpha(\lambda, \Gamma) =& \Gamma + \frac{1}{(1-\alpha)Q} \sum_{q=1}^{Q} (f(\lambda, r_q) - \Gamma)^+ \\ =& \Gamma + \frac{1}{(1-\alpha)Q} \sum_{q=1}^{Q} (-\lambda^T r_q - \Gamma)^+ \end{aligned} \tag{2.3}$$

with $(x)^+ = \max\{x, 0\}$ and $r_1, r_2, \ldots, r_Q$ is a sample set of J scenarios of financial instruments log-return and each r^j is a vector in the space R^n, $r_q = (r_q^1, r_q^2, \ldots, r_q^n)$, $(q = 1, 2, \ldots, Q)$.

Definition 2.4. *Malmquist Productivity Index (MPI) is defined with assimilation efficiency changes of each unit and technology changes. MPI can be computed via several functions, such as distance function:*

$$D(X_\circ, Y_\circ) = \inf\{\Theta / (\Theta X_\circ, Y_\circ) \in PPS\}$$

Fare et al. [8] decomposed MPI into two components, using linear inefficiency of technology frontier. Calculation of the MPI requires linear programming problem as introduced below:

$$\begin{aligned} {D_\circ}^t(X_\circ^t, Y_\circ^t) = \quad & \min \quad \Theta \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^t \leq \Theta x_{io}^t, i = 1, \ldots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj}^t \geq y_{ro}^t, r = 1, \ldots, s \\ & \lambda_j \geq 0, \ j = 1, \ldots, n \end{aligned} \tag{2.4}$$

x_{io}^t and y_{ro}^t are the i-th input and the r-th output of DMU_o at time t respectively; where $\circ \in j = 1, 2, \ldots, n$. Alternatively, CCR problem is calculated from $D^{t+1}(X_\circ^{t+1} Y_\circ^{t+1})$ at time $t+1$ instead of t and is the technical efficiency for $DMU_\circ$ at time $t+1$. The value of $D^t(X_\circ^{t+1}, Y_\circ^{t+1})$, for $DMU_\circ$, is the distance of $DMU_\circ$ at $t+1$ from the frontier of time t.

$MPI(M_o)$ will be calculated from multiplication efficiency change and technology change for each input orientedat time t and $t+1$:

$$12 \ M_\circ = \Big[\frac{D_\circ^t(X_\circ^{t+1}, Y_\circ^{t+1})}{D_\circ^t(X_\circ^t, Y_\circ^t)} \cdot \frac{D_\circ^{t+1}(X_\circ^{t+1}, Y_\circ^{t+1})}{D_\circ^{t+1}(X_\circ^t, Y_\circ^t)}\Big]$$

This value defines geometric convex compotation, because it specified the smallest decrease of efficiencies and any small change in each efficiency effects in MPI. Three conditions are available:

1. $M_\circ > 1$, Increase productivity and observe progress.
2. $M_\circ < 1$, Decrease productivity and observe regress.
3. $M_\circ = 1$, No change in productivity at time $t+1$ in comparison to t.

Definition 2.5. *In the conventional DEA models, each* $\boldsymbol{DMU}_j (j = 1, \ldots, n)$ *is specified by a pair of non-negative input and output vector*$(x_{ij}^t, y_{rj}^t) \in \boldsymbol{R}_+^{(m+s)}$*, in which inputs* $x_{ij}^t (i = 1, \ldots, m)$ *are utilized to produce outputs,* $y_{rj}^t (r = 1, \ldots, s)$*. These models cannot be used for the cases where DMUs include both negative and positive inputs and/or outputs. Portela et al. (2004) considered a DEA model which can be applied in cases where input/output data take positive and negative values.*

Ideal point (I) within the attendance of negative data is:

$$\begin{aligned} I = \Big(& \max_j \{y_{rj}^t : r = 1, \ldots, s\}, \\ & \min_j \{x_{ij}^t : i = 1, \ldots, m\} \Big) \end{aligned}$$

and the goal is to project each under evaluation's asset to this ideal point. These ranges assume implicitly the existence of an ideal point with maximum outputs and minimum inputs observed in period t. The model for DMU_o in time period t

is as follows:

$$DR^t(x_o^t, y_o^t, R_{x_o^t}, R_{y_o^t}) = \beta_o^* = \max\Big\{\beta_o \mid$$
$$\sum_{j=1}^{n} \lambda_j x_{ij}^t \leq x_{io}^t - \beta_o R_{x_{io}^t} \ i = 1, \dots, m,$$
$$\text{o R˙y˙ro\^{}t} \quad r=1,\dots,s, \sum_{j=1}^{n} \lambda_j y_{rj}^t \geq y_{ro}^t + \beta$$
$$\sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0 \ j = 1, \dots, n\Big\}$$

Where, directions can be defined as following:

$$R_{t_{x_{ro}}} = x_{io}^t - \min\{x_{ij}^t\} \qquad i = 1, \dots, m$$
$$R_{t_{y_{ro}}} = \max\{y_{rj}^t\} - y_{ro}^t \qquad r = 1, \dots, s$$

The optimum solution to model (2.5) provides an inefficiency measure equal to $DR^t(x_o^t, y_o^t, R_{x_o^t}, R_{y_o^t}) = \beta_o^*$. This measure represents the proportion of the range vectors $R_{y^t_{ro}}, R_{x^t_{ro}}$ which the outputs and inputs of unit "o" should be increased and decreased respectively. As a result the inefficiency measure will reach the frontier. When $\beta_k^* = 0$ the unit is on the frontier.

$$RMD(x_o^t, y_o^t, R_{x_o^t}, R_{y_o^t})$$
$$= 1 - DR^t(x_o^t, y_o^t, R_{x_o^t}, R_{y_o^t}) = 1 - \beta_o^*$$

is efficiency score of DMU_o. Other models that use negative data are modified slacks-based measure model (MSBM), Emrouznejad (2010) and semi-oriented radial measure (SORM), Sharp et al. (2006).

Definition 2.6. *In order to illustrate the meta-Malmquist index let us take a set of assets which we assume use one input (risk) to secure one output (expected return). In Fig. 1 [23] we plot assets and consider two period frontiers (t and $t+1$) and a meta-frontier (lying above the period t and $t+1$ frontiers). Asset F in period t has a RDM efficiency of $\frac{IF'}{IF}$ when it is assessed in relation to the period t frontier. We can also assess the efficiency of asset F in relation to the meta-frontier, which we refer to as metaefficiency. The meta-efficiency of asset F is given by $\frac{IF''}{IF}$, and it can be decomposed into two components: The within-period-efficiency $\frac{IF'}{IF}$ and a technological gap ($\frac{IF''}{IF'}$) That is, $\frac{IF''}{IF} = \frac{IF'}{IF} * \frac{IF''}{IF'}$.*

The technological gap (TG) measures the distance between the period t frontier and the meta-frontier.

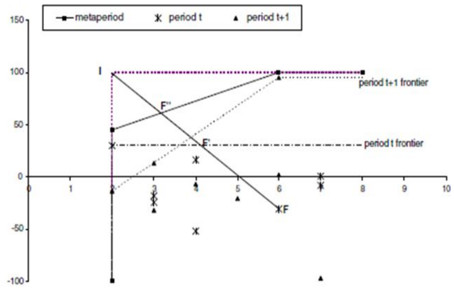

Figure 1: Illustration of the meta-Malmquist Index

In the present context, one advantage of using meta-frontiers is that we can handle VRS technologies which become necessary in the presence of negative data. Using metafrontiers under VRS makes it possible to compute the index for all units. To using the Meta- Malmquist Index, the variable return to scale is observed at first period of t technology [23].

The Meta VRS Malmquist index is defined by:

$$MM_j^{t,t+1} = \frac{RDM^{mf}(x_j^{t+1}, y_j^{t+1}, R^{mf}_{x_j^{t+1}}, R^{mf}_{y_j^{t+1}})}{RDM^{mf}(x_j^t, y_j^t, R^{mf}_{x_j^t}, R^{mf}_{y_j^t})} \tag{2.5}$$

As we know, expected return can take negative value and hence we should apply negative data models. Based on Figure 1, we define meta-malmquist index for portfolio that has negative expected return and has used portela et al. [16] in presence negative data. We have:

$$RDM^{mf}(x_j^t, y_j^t, R^{mf}_{x_j^t}, R^{mf}_{y_j^t}) = RDM^t(x_j^t, y_j^t, R^{mf}_{x_j^t}, R^{mf}_{y_j^t}) \times TG_j^t \tag{2.6}$$

Where TG_j^t is retrieved residually as:

$$TG_j^t = \frac{RDM^{mf}(x_j^t, y_j^t, R^{mf}_{x_j^t}, R^{mf}_{y_j^t})}{RDM^t(x_j^t, y_j^t, R^{mf}_{x_j^t}, R^{mf}_{y_j^t})} \tag{2.7}$$

Using the above definitions, we can define a meta-malmquist index as:

$$MM_j^{t,t+1} = \frac{RDM^{mf}(x_j^{t+1}, y_j^{t+1}, R^{mf}_{x_j^{t+1}}, R^{mf}_{y_j^{t+1}})}{RDM^{mf}(x_j^t, y_j^t, R^{mf}_{x_j^t}, R^{mf}_{y_j^t})} = \frac{RDM^{t+1}(x_j^{t+1}, y_j^{t+1}, R^{mf}_{x_j^{t+1}}, R^{mf}_{y_j^{t+1}})}{RDM^{t}(x_j^t, y_j^t, R^{mf}_{x_j^t}, R^{mf}_{y_j^t})} \times \frac{TG_j^{t+1}}{TG_j^t}. \quad (2.8)$$

Where

$$\frac{TG_j^{t+1}}{TG_j^t} = \frac{\frac{RDM^{mf}(x_j^{t+1}, y_j^{t+1}, R^{mf}_{x_j^{t+1}}, R^{mf}_{y_j^{t+1}})}{RDM^{t+1}(x_j^{t+1}, y_j^{t+1}, R^{mf}_{x_j^{t+1}}, R^{mf}_{y_j^{t+1}})}}{\frac{RDM^{mf}(x_j^t, y_j^t, R^{mf}_{x_j^t}, R^{mf}_{y_j^t})}{RDM^{t}(x_j^t, y_j^t, R^{mf}_{x_j^t}, R^{mf}_{y_j^t})}}. \quad (2.9)$$

Wher $MM_j^{t,t+1}$ greater than 1, productivity of unit j has improved from t to $t+1$.

3 Proposed models in Mean-CVaR framework and calculating Meta Malmquist Index (MMI)

Based on the RDM model provided by Portela et al. [22], we propose the Mean-CVaR model. First, assume there are n financial assets and a portfolio is going to be selected from n financial assets. Return of each asset in period t is defined as $r_1^t, \ldots, r_n^t$. The vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ represents the policy of investing in different proportions of assets in a portfolio. Let

$$g = (R_{tCVaR_o}, R_{t\mu_o}) = \begin{pmatrix} (\max_j(\mu_j^t : j = 1, \ldots, n) - \mu_o^t) = R_{t\mu_o} \\ ([CVaR_o^t - \min(CVaR_j^t : j = 1, \ldots, n)]) = R_{tCVaR_o} \end{pmatrix}$$

be a vector that shows direction in which β as an objective function is going to be maximized, Regarding the negative return values.

Consider a vector with specified direction $g = (R_{tCVaR_o}, R_{t\mu_o})$ and and an under evaluation $(CVaR_o^t, \mu_o^t)$ that μ_o^t is the mean return and $CVaR_o^t$ is the risk measure of t in period t. Now, we introduce the following linear programming model as mean-CVaR model in the direction of g as follows:

$$\begin{aligned} \max \quad & \beta \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j E(r_j^t) \geq \mu_o^t + \beta R_{t\mu_o} \\ & \sum_{j=1}^n \lambda_j CVaR_j^t \leq CVaR_o^t - \beta R_{tCOVaR_o}, \\ & \sum_{j=1}^n \lambda_j = 1 \quad \lambda_j \geq 0 \; j = 1, \ldots, n \end{aligned} \quad (3.10)$$

The first constraint is output and is defined by expected return and second constraint is input and is defined by CVaR as a risk measure that was explained in previous section. Mechanism of the mean-CVaR model is just like the RDM model. When amount of β for under evaluation asset equals to zero, it will be understood that this asset is efficient and mean-CVaR point is part of the weakly efficient frontier. In the other word, $1-\beta$ is amount of the efficiency. The mean-CVaR model seeks simultaneously to increase mean of return and to decrease risk in the direction of the vector g. The use of this model guarantees that a projected mean-CVaR point is part of the weakly efficient subset [2]. We calculate CVaR at three confidence levels namely 90%, 95%, 99%, which is introduced as period t in the model. By considering 99% and 95%, efficiencies are obtained by mean-CVaR model. Meta malmquist index is computed for these efficiencies in the presence of negative data. In our work, we requires the solution of the following linear programming mean -CVaR models (3.11) and (3.12) and linear programming models (3.13) and (3.14) with three confidence levels:

$$\begin{aligned} & DR_o^t(x_o^t, y_o^t, R^t_{R_{x_o}}, R^t_{R_{y_o}}) = DR_o^{99\%}(CVaR_o^{99\%}, Return_o^{99\%}, R^{99\%}_{R_{\mu_o}}, R^{99\%}_{R_{CVaR_o}}) = \max \beta \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j E(r_j^{99\%}) \geq \mu_o^{99\%} + \beta R^{99\%}_{R_{\mu_o}} \\ & \sum_{j=1}^n \lambda_j CVaR_j^{99\%} \leq CVaR_o{}^{99\%} - \beta R^{99\%}_{CVaR_{R_o}}, \\ & \sum_{j=1}^n \lambda_j = 1 \; \lambda_j \geq 0, \; j = 1, \ldots, n \end{aligned} \quad (3.11)$$

$$
\begin{aligned}
& DR_o^{t+1}(x_o^{t+1}, y_o^{t+1}, R_{R_{x_o}}^{t+1}, R_{R_{y_o}}^{t+1}) = DR_o^{95\%}(CVaR_o^{95\%}, \\
& Return_o^{95\%}, R_{R_{\mu_o}}^{95\%}, R_{R_{CVaR_o}}^{95\%}) = \max \beta \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j E(r_j^{95\%}) \geq \mu_o^{95\%} + \beta R_{R_{\mu_o}}^{95\%} \\
& \sum_{j=1}^{n} \lambda_j CVaR_j^{95\%} \leq CVaR_o^{95\%} - \beta R_{CVaR_{R_o}}^{95\%}, \\
& \sum_{j=1}^{n} \lambda_j = 1 \ \lambda_j \geq 0, \ j = 1, \ldots, n
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
& DR_o^{mf}(x_o^{t+1}, y_o^{t+1}, R_{R_{x_o}}^{mf}, R_{R_{y_o}}^{mf}) = DR_o^{90\%}(CVaR_o^{95\%}, \\
& Return_o^{95\%}, R_{R_{\mu_o}}^{90\%}, R_{R_{\mu_o}}^{90\%}) = \max \beta \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j E(r_j^{90\%}) \geq \mu_o^{95\%} + \beta R_{R_{\mu_o}}^{90\%} \\
& \sum_{j=1}^{n} \lambda_j CVaR_j^{90\%} \leq CVaR_o^{95\%} - \beta R_{CVaR_{R_o}}^{90\%}, \\
& \sum_{j=1}^{n} \lambda_j = 1 \ \lambda_j \geq 0, \ j = 1, \ldots, n
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
& DR_o^{mf}(x_o^{t+1}, y_o^{t+1}, R_{R_{x_o}}^{mf}, R_{R_{y_o}}^{mf}) = DR_o^{90\%}(CVaR_o^{99\%}, \\
& Return_o^{99\%}, R_{R_{\mu_o}}^{90\%}, R_{R_{\mu_o}}^{90\%}) = \max \beta \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j E(r_j^{90\%}) \geq \mu_o^{99\%} + \beta R_{R_{\mu_o}}^{90\%} \\
& \sum_{j=1}^{n} \lambda_j CVaR_j^{90\%} \leq CVaR_o^{99\%} - \beta R_{CVaR_{R_o}}^{90\%}, \\
& \sum_{j=1}^{n} \lambda_j = 1 \ \lambda_j \geq 0, \ j = 1, \ldots, n
\end{aligned} \tag{3.14}
$$

Where, $DR_o^{99\%}(CVaR_o^{99\%}, Return_o^{99\%}, R_{R_{\mu_o}}^{99\%}, R_{R_{CVaR_o}}^{99\%})$ and $DR_o^{95\%}(CVaR_o^{95\%}, Return_o^{95\%}, R_{R_{\mu_o}}^{95\%}, R_{R_{CVaR_o}}^{95\%})$ measure the efficiencies of $DMU_o(o \in \{1, \ldots, n\})$ in 99% confidence level and 95% confidence level, respectively. $DR_o^{90\%}(CVaR_o^{95\%}, Return_o^{95\%}, R_{R_{\mu_o}}^{90\%}, R_{R_{CVaR_o}}^{90\%})$ measures its optimistic efficiency in period 95% confidence level using the production technology of meta frontier (90%confidence level).

$DR_o^{90\%}(CVaR_o^{99\%}, Return_o^{99\%}, R_{R_{\mu_o}}^{90\%}, R_{R_{CVaR_o}}^{90\%})$ measures the optimistic efficiency of DMU_o in period 99% confidence level using the production technology of period meta frontier (90% confidence level). In this paper we are devoted to evaluate portfolio productivity and portfolio optimization by implementing the meta-Malmquist index into mean-conditional value at risk (CVaR) model. The MMI value is obtained by applying Conditional Value-at-risk (CVaR) as a risk measure into the proposed models without considering the skewness and kurtosis of assets return rate. Models are based on Range Directional Measure (RDM) model which can take data with both positive and negative values. Efficiencies are obtained from the proposed mean-CVaR models at various confidence levels, such as, 90%, 95%, 99%. We calculate MMI based on $MM_j^{t,t+1}$, and at 99% to 95% confidence levels representing periods t to $t+1$. Morever, we consider 90% confidence level to be meta frontier. When $MM_j^{99\%,95\%}$ is greater than 1, productivity of unit j has progress from 99% to 95%. Also, MMI will be calculated from multiplication efficiency change and technology change for each DMU_o at time t and $t+1$ based on models (3.11), (3.12), (3.13) and (3.14) with correlation to model (2.9). If we want to provide a general idea about the performance in these three confidence levels and make fairly portfolio, we use the MMI. Clearly changes in the risk value (CVaR) at different confidence levels, will affect the meta-Malquist Index. This method helps the investors to construct a profitable portfolio based on each asset, using meta Malmquist.

4 Application in Iranian stock companies

The dataset was randomly collected from the stock's price of the 15 Iranian stock companies, from 25/04/2015 till 25/04/2016. The dataset has been obtained from http://www.tsetmc.ir/. our approach is illustrated using these data set contains price of 15 companies that is specified by symbol of stock companies in tables. Moreover, trading days are recorded according to the market calendar, with all weekends and holidays removed from dataset.

Efficiency of each asset is going to be evaluated at three confidence levels and meta malmquist is calculated for these assets. The software MATLAB was used to calculate conditional value at risk and expected return. In Table 1, we present the results of CVaR and expected return of a sample data set for the stock companies, respectively. These data will be used as the input and the out-

Table 1: Input and output consist of expected return and conditional value at risk as risk measures and inefficiency of the stock companies by using Mean-CVaR mode

Number of asset	Symbol of stock company	Expected Return	Conditional Value at Risk (CVaR)			β with mean-CVaR model		
			%90	%95	%99	%90	%95	%99
1	AZAB	0.0026	0.0392	0.0430	0.0476	0.48	0.40	0.02
2	CONT	0.0085	0.0361	0.0452	0.0513	0.00	0.00	0.00
3	DJBR	0.0013	0.0231	0.0348	0.0901	0.25	0.31	0.92
4	DSIN	0.0023	0.0195	0.0328	0.0941	0.03	0.18	0.93
5	IPAR	0.0019	0.0265	0.0396	0.0681	0.85	0.37	0.23
6	KHAZ	0.0017	0.0471	0.0516	0.0653	0.59	0.55	0.83
7	KRTI	-0.0003	0.0586	0.0802	0.1945	0.69	0.74	0.98
8	NAFT	-0.0006	0.0455	0.0499	0.0574	0.62	0.59	0.76
9	PASH	0.0009	0.0150	0.0245	0.0749	0.00	0.00	0.88
10	RENA	0.0030	0.0433	0.0471	0.0506	0.52	0.45	0.43
11	SHND	-0.0029	0.0755	0.1163	0.3914	0.77	0.83	0.99
12	TRIR	-0.0035	0.0680	0.1062	0.3497	0.76	0.82	0.99
13	TRNS	0.0027	0.0343	0.0422	0.0476	0.40	0.38	0.00
14	PSIR	0.0011	0.0481	0.5991	0.1227	0.61	0.93	0.95
15	GHAT	-0.0023	0.0717	0.1021	0.3059	0.76	0.81	0.99

put of the models.

As mentioned before, we have used Mean-CVaR model to calculate the efficiency of the stock companies. The software GAMS has been applied to measure the relative efficiency of selected stock companies. In this model β shows amount of inefficiency. Therefore, when amount of β for the stock company equals to zero, means that the stock company is efficient. Also, inefficiency scores of these companies have been shown in Table 1 by using mean-CVaR model.

Based on data in Table 1, asset 2 in all levels of CVaR is efficient. However, asset 9 in highest level of CVaR is not efficient. For all assets it can be interpreted, as the confidence level of risk increases, assets get less amount of efficiency and amount of efficiencies are accurate.

By employing the results into the models (3.11) through (3.14), introduced in Section 2, we find the efficiency scores of the assets. We calculate meta-malmquist index based on $MM_j^{t,t+1}$ and for confidence levels 99% to 95% as periods t to $t+1$ in Table 3. Also we consider confidence level 90% as meta frontier. When $MM_j^{99\%,95\%}$ is greater than 1, productivity of unit j has improved from 99% to 95%. In Figure 2, we show 15 Iranian stock companies in three confidence levels. By considering two levels 99% and 95%, efficiencies are obtained by mean-CVaR model.

If we want to provide a general idea about the performance of these three confidence levels, we use the meta-malmquist index. The results in table 3 display that, assets 1, 2, 10 and 13 assure little risk taking at different confidence level and also have high returns. So, these assets are progressing at different confidence levels of meta frontier. As a result, with these assets, a more reliable portfolio can be built and they are suitable for creating portfolio. Also, asset 8 although has low returns, it is favorable due to low risk changes. Assets 7, 11, 12, 14 and 15 assure high risk taking at different confidence level and also have low returns. Therefore, these assets are regressing at different confidence level relative to meta frontier. Assets 3,4,5 and 9 have reasonable returns and risks, but change of their risks, especially at 90% confidence level, is very high and can be used in the next stage of constructing the portfolio.

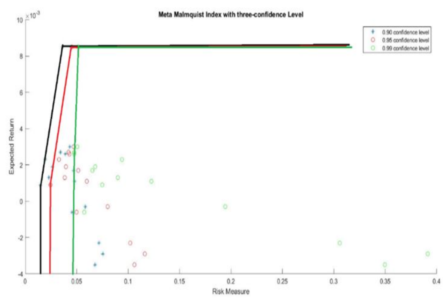

Figure 2: Illustration of the meta-Malmquist Index with real data from Tehran Stock Exchange in three confidence level.

Table 2: Input and output consist of expected return and conditional value at risk as risk measures and inefficiency of the stock companies by using Mean-CVaR mode

Number of asset	Symbol of stock company	$RDM_o^{99\%}$	$RDM_o^{95\%}$	$RDM_o^{90\%}$ ($x_o^{99\%}$, $y_o 99\%$, $R_{R_{x_o}}^{90\%}$,$R_{R_{y_o}}^{90\%}$)	$RDM_o^{95\%}$ ($x_o^{99\%}$,$y_o 99\%$, $R_{R_{x_o}}^{90\%}$,$R_{R_{y_o}}^{90\%}$)
1	AZAB	0.98	0.6	0.31	0.43
2	CONT	1.00	1.00	0.28	0.57
3	DJBR	0.08	0.69	0.001	0.33
4	DSIN	0.07	0.82	0.001	0.36
5	IPAR	0.15	0.63	0.001	0.27
6	KHAZ	0.17	0.45	0.06	0.33
7	KRTI	0.02	0.26	0.002	0.001
8	NAFT	0.24	0.41	0.16	0.3
9	PASH	0.12	1.00	0.001	0.55
10	RENA	0.57	0.55	0.32	0.4
11	SHND	0.01	0.17	0.002	0.001
12	TRIR	0.01	0.18	0.001	0.01
13	TRNS	1	0.62	0.22	0.37
14	PSIR	0.05	0.07	0.003	0.005
15	GHAT	0.01	0.19	0.001	0.001

Table 3: Meta-Malmquist index and components

Number of Asset	Symbol of stock company	Meta-Malmquist Index	Efficiency Change	Technological Change	Productivity Change
1	AZAB	1.3869	0.6122	2.2655	Progress on level 1
2	CONT	2.0357	1.0000	2.0357	Progress on level 1
3	DJBR	Almost Infinite	8.6250	38.2560	Progress on level 2
4	DSIN	Almost Infinite	11.7142	30.9154	Progress on level 2
5	IPAR	Almost Infinite	4.2000	64.9242	Progress on level 2
6	KHAZ	5.4994	2.6470	2.0776	Progress on level 1
7	KRTI	Very Small Number	13.000	Very Small Number	regress
8	NAFT	1.8750	1.7083	1.0976	Progress on level 1
9	PASH	Almost Infinite	8.3333	66.2650	Progress on level 2
10	RENA	1.2498	0.9649	1.2953	Progress on level 1
11	SHND	Very Small Number	17.000	Very Small Number	regress
12	TRIR	Very Small Number	18.000	Very Small Number	regress
13	TRNS	1.6815	0.6200	2.7122	Progress on level 1
14	PSIR	Very Small Number	1.4000	Very Small Number	regress
15	GHAT	Very Small Number	19.000	Very Small Number	regress

5 conclusion

This study considered the evaluation of assets by describing progress and regress productivity by meta malmquist index in the presence negative data. As a risk measure CVaR is used because it is a coherent risk measure. We use mean-CVaR model with negative expected return. In this model expected return is considered as output and CVaR is considered as input. Since expected return may take negative values, conventional DEA models are not appropriate to solve these models. So our models are based on Range Directional Measure (RDM) model. in order to take negative values as outputs or inputs, malmquist index is calculated based on this model. Finally, our model are applied based on data from 15 Companies in Tehran Stock. We calculate meta malmquist index based on $MM_j^{t,t+1}$ and for confidence levels as periods t to t+1. When $MM_j^{t,t+1}$ is greater than 1, productivity of unit j has improved.

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Zohreh Taeb is Assistant professor of Applied Mathematics in operational research at Yadegar Imam Branch, Islamic Azad University. Her research interests are in the areas of Applied Mathematics, Data Envelopment Analysis. She has published research articles in international journals of Mathematics.

Shokoofeh Banihashemi is Assistant professor of Applied Mathematics in operational research at Allameh Tabatabai University. Her research interests are in the areas of Applied Mathematics, Mathematical Finance, Data Envelopment Analysis, Supply Chains. She has published research articles in international journals of Mathematics. She is referee and editor of mathematical and economics journals.