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Prime Filters and Zariski Topology on Equality Algebras

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Abstract

In this paper, we present some characterizations of prime and maximal filters. Moreover, we introduce $\cap$-irreducible filters of an equality algebra, investigate some results about them and relations between maximal, prime, $\vee$-irreducible and $\cap$-irreducible filters in equality algebra. Also, we introduce spectrum of an equality algebra and prove that the spectrum endowed with Zariski topology is a compact T_0 topological space and maximal spectrum (as a subspace of that) is a compact T_1 topological space.

Keywords : Equality algebra; Prime filter; Maximal filter; $\vee$-irreducible filter; $\cap$-irreducible filter; Zariski topology.

1 Introduction

A More general algebraic structure in logic without contractions is residuated lattices [23] and Ono [19] considered them as an algebraic structure of substructural logics. Among logical algebras, residuated lattices have received the most attention due to their interesting properties and including two important sub-classes: BL-algebras and MV-algebras. Fuzzy type theory was developed as a higher order fuzzy logic. Novák and De Beats generalized residuated lattices and proposed EQ-algebra [18]. Recently, Ganji Saffar defined the concepts of fuzzy n-fold obstinate (pre)filter and maximal fuzzy (pre)filter of EQ-algebras and discussed the properties of them [8]. Because by replacing the product operation with a lesser or equal operation, we get an EQ-algebra again, Jenei [11] introduced a new algebra, called equality algebra. Since equality algebra can be a good alternative to possible algebraic semantics for fuzzy type theory, the study of equality algebra is very valuable.
In [5], it was proved that equality algebras and BCK-meet semilattices (under distributivity condition) correspond to each other. Because different filters have natural expressions as diverse sets of provable formulas, filter theory has a significant impact on the study of logical algebras. For this, in [1], Borzooei et al. introduced some types of filters in equality algebras. For more recent studies about equality algebras, you can see [2, 6, 10, 17, 20].
Algebra studies the property of operations and algorithmic computations of a space, while topology provide a framework to understanding the geometric properties of it.

In recent years, the study of topological con-

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cepts on logical algebraic structures has received much attention. To read more about the commonalities between topology and logical algebras, you can refer to sources such as: In [9], G. Georgescu et al. gave topological characterizations to the lifting property for Boolean elements and several properties related to it. In [14], every MV-algebra was equipped with a filter topology which that became a topological MV-algebra. In [7], Foruzesh et al. introduced the inverse topology on the set of all minimal prime ideals of an MV-algebra (namely $Min(A)$ and showed that $Min(A)$, with the inverse topology is a compact space, Hausdorff, T_0-space and T_1-space. (for more details of these topological algebras, see [4, 13, 16, 22, 24, 25].

In this paper, we characterize maximal and prime filters. Furthermore, we bring forward $\cap$-irreducible filters of an equality algebra and investigate basic properties of them. In [21], three kinds of prime filters of residuated lattices are defined and their properties are investigated. Similarly, we give interesting results about the relation between prime, $\vee$-irreducible and $\cap$-irreducible filters of an equality algebra.

The paper is organized as: In Section 2, we gather the basic notions and results on topology and equality algebras, used in the squeal. In Section 3, we study maximal, prime, $\vee$-irreducible and $\cap$-irreducible filters. Then we get some interesting results about them and investigate relation between them. In Section 4, we introduce Zariski topology on equality algebra and show that $\mathcal{S}pec()$ equipped with Zariski topology is a compact T_0-space. Moreover, maximal spectrum of an equality algebra as a subspace of that is a compact T_1-space.

2 Preliminaries

In this section, we have compiled some basic concepts about topology and equality algebra that will be used in later sections. We only remind some definitions and results. More concepts about topology can be found in [15].

Recall that we say (A, τ) is a *topological space*, where τ is a family of subsets of the set A satisfying: (i) $A, \emptyset \in \tau$, (ii) the intersection of any finite members of τ is in it, and (iii) the any union of members of τ is in it. Any member of τ is called an *open subset* of A, and $A \setminus U$, is a *closed set* which is the complement of an open set U. For $X \mathcal{S} A$, *closure* of X is the intersection of all closed sets containing X and denoted by $Cl(X)$. Also,

$$Cl(x) = \bigcap \{V \mathcal{S} A | \ V \text{ is closed and } x \in V\}.$$

A subfamily $\{U_\alpha\}_{\alpha \in I}$ of τ is called a *base* of τ if for every $x \in U \in \tau$ there exists an $\alpha \in I$ such that $x \in U_\alpha \mathcal{S} U$. A collection $\{U_\alpha\}_{\alpha \in I}$ of subsets of A is said to be an *open covering* if its elements are open subsets of A and the union of elements of it is equal to A. The set $X \mathcal{S} A$ is called *compact* if every open covering of X contains a finite sub-collection that also covers X. A topological space (A, τ) is *compact space* if each open covering of A is reducible to a finite one. Suppose the topological space (A, τ), then

T_0: for any $x, y \in A$ and $x \neq y$, there exists an open set in A that contains x or y, but not both of them.

T_1: for all $x, y \in A$ and $x \neq y$, there exist open sets U_1 and U_2 in A such that $x \in U_1$ and $y \in U_2$ but $y \notin U_1$ and $x \notin U_2$.

T_2: for all $x, y \in A$ and $x \neq y$, there exist two distinct open sets U_1 and U_2 in A such that $x \in U_1$, $y \in U_2$ and $U_1 \cap U_2 = \emptyset$.

A topological space satisfying T_i is called T_i-*space*, for each $i = 0, 1, 2$. A T_2-space is also called a *Hausdorff* space. A topological space (A, τ) is said to be *disconnected* if it is the union of two disjoint non-empty open sets. Otherwise, A is said to be *connected*. A subset of a topological space is said to be connected if it is connected under its subspace topology.

Definition 2.1. [11] Algebraic structure $(; \wedge, \sim, 1)$ of type $(2, 2, 0)$ is called an *equality algebra*, if it satisfies the following conditions, for all $, , \in$,

$(E1)$ $(, \wedge, 1)$ is a commutative idempotent integral monoid,

$(E2)$ $\sim = \sim$,

$(E3)$ $\sim = 1$,

$(E4)$ $\sim 1 =$,

$(E5)$ $\leq \leq$ implies $\sim \leq \sim$ and $\sim \leq \sim$,

$(E6)$ $\sim\leq (\wedge) \sim (\wedge)$,
$(E7)$ $\sim\leq (\sim) \sim (\sim)$.
The operation $\wedge$ is called *meet* and $\sim$ is an *equality* operation. On the equality algebra, we write $\leq$ if and only if $\wedge =$. Then the relation " $\leq$ " is a partial order on . Also, we define the operation "$\twoheadrightarrow$" on as: $\twoheadrightarrow = \sim (\wedge)$.

Note: Equality algebra $(; \wedge, \sim, 1)$ is denoted by unless otherwise state.

If there exists an element $0 \in$ such that $0 \leq$, for every $\in$, then is called *bounded*. In a bounded equality algebra , we define the negation operation "$^-$ " by: $^- = \twoheadrightarrow 0 = \sim 0$, for all $\in$. If 1 is the unique upper bound of the set $\{\twoheadrightarrow, \twoheadrightarrow\}$, for every $, \in$, then is called *prelinear*. A *lattice equality algebra* is an equality algebra which is a lattice.

Proposition 2.1. [11, 26] *The following conditions hold, for any $, , \in$:*
(i) $\twoheadrightarrow = 1$ *if and only if* $\leq$,
(ii) $1 \twoheadrightarrow =$, $\twoheadrightarrow 1 = 1$, *and* $\twoheadrightarrow = 1$,
(iii) $\leq \twoheadrightarrow$,
(iv) $\leq (\twoheadrightarrow) \twoheadrightarrow$,
(v) $\twoheadrightarrow (\twoheadrightarrow) = \twoheadrightarrow (\twoheadrightarrow)$,
(vi) $\leq$ *implies* $\twoheadrightarrow \leq \twoheadrightarrow$ *and* $\twoheadrightarrow \leq \twoheadrightarrow$,
(vii) $\twoheadrightarrow = \twoheadrightarrow (\wedge)$,
$(viii)$ *If is a lattice, then* $= (\vee)$.

Theorem 2.1. [26] If is prelinear, then it is a distributive lattice.

Definition 2.2. [12] Let be a non-empty subset of . Then is called a *filter* of , if for all $, \in$, we have
(i) $\in$ and $\leq$ imply $\in$,
(ii) $\in$ and $\sim \in$ imply $\in$.

Proposition 2.2. [5, 12] *Let $\emptyset \neq \mathcal{S}$. Then is a filter of if and only if, for all $, \in$, $1 \in$, and if $\in$ and $\twoheadrightarrow \in$, then $\in$.*

Clearly, $1 \in$, for any filter of . A filter of is called a *proper filter* of if $\neq$. Clearly, a filter of bounded equality algebra is proper if and only if it is not containing 0. The set of all filters of is denoted by $\mathcal{F}()$. In addition, is called *simple* if $\mathcal{F}() = \{\{1\}, \}$. Let $\in \mathcal{F}()$. Define the relation θ on by

$$(,) \in \theta \text{ if and only if } \{\twoheadrightarrow, \twoheadrightarrow\}\mathcal{S}.$$

Theorem 2.2. [12] *Let $\theta, \psi \in \mathcal{C}on()$ and $\in \mathcal{F}()$. Then*
(i) $\theta \in \mathcal{C}on()$,
(ii) $[1]_\theta \in \mathcal{F}()$, *where* $[1]_\theta = \{| \ (1) \in \theta\}$,
(iii) *if* $[1]_\theta = [1]_\psi$, *then* $\theta = \psi$,
(iv) $\theta_{[1]_\theta} = \theta$ *and* $[1]_\theta =$.

Let $= \{[]| \in\}$ where $[] = \{\in | \ (,) \in \theta\}$. Then define the binary relation $\leq$ on by:

$$[] \leq [] \quad \text{if and only if} \quad \twoheadrightarrow \in,$$

which is an order relation on . For any $, \in$, define

$$[] \sim [] = [\sim] \quad \text{and} \quad [] \wedge [] = [\wedge].$$

Then $(, \sim, \wedge, 1\ )$ is called a *quotient equality algebra* and denoted by , where $1 = [1]_\theta =$.

Definition 2.3. [17] Let $\emptyset \neq \subseteq$. The smallest filter of containing is called *the generated filter by in* which is denoted by $\langle\rangle$. Indeed, $\langle\rangle = \bigcap\limits_{S \in F()}$.

Proposition 2.3. [17] *Let $\emptyset \neq \subseteq$. Then* $\langle\rangle = \{\in| \ 1 \twoheadrightarrow (2 \twoheadrightarrow (... \twoheadrightarrow (n \twoheadrightarrow)...)) = 1$, *for some* $n \in \mathbb{N}$ *and* $1, ..., n \in\}$.
In particular, for any element , we have $\langle = \{\in|^n = 1$, *for some* $n \in \mathbb{N}\}$, *where* $\twoheadrightarrow^0 =$ *and* $\twoheadrightarrow^n = \twoheadrightarrow (\twoheadrightarrow^{n-1})$. *If* $\in \mathcal{F}()$ *and* $\backslash$, *then* $\langle \cup \{\rangle = \{\in|^n \in$, *for some* $n \in \mathbb{N}\}$. *If* $, \in \mathcal{F}()$, *then* $\langle \cup \rangle = \{\in| \ g \twoheadrightarrow \in$, *for some* $g \in\} = \{\in| \ f \twoheadrightarrow \in$, *for some* $f \in\}$.

Remark 2.1. *The algebraic structure* $(\mathcal{F}(), \wedge, \vee, \{1\},)$ *is a bounded complete lattice, where, for every* $, \in \mathcal{F}()$,

$$\wedge = \cap, \qquad \vee = \langle \cup \rangle.$$

Theorem 2.3. [17] *Let be lattice, $, \mathcal{S}, \in \mathcal{F}()$ and $\subsetneq$. Then*
(i) *if* $\mathcal{S}$, *then* $\langle\rangle\mathcal{S}\langle\rangle$,
(ii) $\langle \cup \{\rangle \cap \langle \cup \{\}\rangle = \langle \cup \{\underline{}\}\rangle$.
(iii) $\langle \underline{}\rangle = \langle \cap \langle \underline{}\rangle$.

3 Results on prime and maximal filters

In some logical algebras such as BL-algebras and MV-algebras, prime filter defined as $\vee$-irreducible filter and easily proved that it is an $\cap$-irreducible filter, too. In addition, maximal spectrum is always a subset of prime spectrum. But in some others, such as residuated lattices, prime, $\vee$-irreducible and $\cap$-irreducible filters are different (and are equivalent under certain conditions), and it is only with "De Morgan's condition" that a maximal filter becomes a prime filter.
In this section, we want to examine these three types of prime filters and maximal filters on and then find the relationship between them so that we can make the right decision about which type of filter to consider as the spectrum of an equality algebra.

Definition 3.1. [1, 17] Suppose that $\in \mathcal{F}()$ is a proper filter. Then
(i) is a *maximal filter* of , if it is not included in any other proper filter of . The set of all maximal filters of is denoted by $\mathcal{M}ax()$.
(ii) is a *prime filter* of , if $\twoheadrightarrow\in$ or $\twoheadrightarrow\in$, for all $,\in$. The set of all prime filters of is denoted by $\mathcal{P}rime()$.
(iii) Suppose is a lattice equality algebra. Then we say is a $\vee-$irreducible filter of , if $\vee \in$ implies $\in$ or $\in$, for all $,\in$.

Proposition 3.1. *For any proper filter of , there exists a maximal filter of that contains .*

Proof. Zorn's Lemma states that a partially ordered set containing upper bounds for every chain (that is, every totally ordered subset) necessarily contains at least one maximal element. Consider

$$\Sigma = \{\in \mathcal{F}()| \neq, \mathcal{S}\}.$$

Since $\in \Sigma$, then $\Sigma \neq \emptyset$. Let $\{_i\}_{i\in I}$ be a chain in partially ordered set $(\Sigma, \mathcal{S})$, where $_i\mathcal{S}_{i+1}$ for any $i \in I$. Put $= \bigcup_{i\in I} {}_i$. Then it is easy to see that $\in \mathcal{F}()$ and $\mathcal{S} \bigcup_{i\in I} {}_i =$. If $=$, then $1 \in \bigcup_{i\in I} {}_i$ and so there exists $i_0 \in I$ such that $1 \in_{i_0}$. Thus $_{i_0} =$ which is a contradiction. Hence, $\neq$ and so $\in \Sigma$. By the simple way, we can see that is a maximal element of Σ. Hence, using Zorn's Lemma, there exists a maximal element $\in \Sigma$ which is a maximal filter of and $\mathcal{S}$. □

Remark 3.1. *Suppose is a proper filter of . Then each filter of quotient equality algebra has the form , where $\in \mathcal{F}()$ and $\mathcal{S}$. Indeed,*

$$\mathcal{F}(\ ) = \{\ |\ \mathcal{S} \in \mathcal{F}()\}.$$

Theorem 3.1. *Let $\in \mathcal{F}()$ be proper and $\in \backslash$. Then the following statements are equivalent:*
(i) $\in \mathcal{M}ax()$,
(ii) $\langle \cup \{\} \rangle =$,
(iii) *is a simple equality algebra.*

Proof. $(i) \Rightarrow (ii)$ Let $\in \mathcal{M}ax()$ and $\in \backslash$. Then by Proposition 2.3, $\subsetneq \langle \cup \{\} \rangle \mathcal{S}$. Since $\in \mathcal{M}ax()$, we have $\langle \cup \{\} \rangle =$.
$(ii) \Rightarrow (i)$ Let $\mathcal{S}\mathcal{S}$ and $\neq$. Then there is $\in$ such that $\notin$ and so by (ii), $\langle \cup \{\} \rangle =$. Moreover, since $\cup \{\} \mathcal{S}$, we have $= \langle \cup \{\} \rangle \mathcal{S}$. Hence, $=$ and $\in \mathcal{M}ax()$.
$(i) \Rightarrow (iii)$ Let $\in \mathcal{M}ax()$ and be a proper filter of . Then by Remark 3.1, we get $\mathcal{S}\mathcal{S}$ where, $\neq$, we get $\neq$. Hence, since $\in \mathcal{M}ax()$, we have $=$ and so $= = 1 =$. Therefore, $\mathcal{F}(\ ) = \{, \}$ and is a simple equality algebra.
$(iii) \Rightarrow (i)$ Consider is simple and $\mathcal{S}\mathcal{S}$. If $\neq$, then $1 = \neq \in \mathcal{F}(\ )$. Thus $=$ and so $=$. Therefore, $\in \mathcal{M}ax()$. □

Proposition 3.2. *Let be bounded. Then $\in \mathcal{M}ax()$ if and only if for all $\in \backslash$, there is $n \in \mathbb{N}$ such that $\twoheadrightarrow^n 0 \in$.*

Proof. Suppose $\in \mathcal{M}ax()$ and $\in \backslash$. Then by Theorem 3.1, $\langle \cup \{\} \rangle =$. Thus $0 \in \langle \cup \{\} \rangle$. Hence, by Proposition 2.3, there exists $n \in \mathbb{N}$ such that $\twoheadrightarrow^n 0 \in$.
Conversely, let for each $\in \backslash$, as a consequence $\twoheadrightarrow^n 0 \in$, for some $n \in \mathbb{N}$. Then by Proposition 2.3, $0 \in \langle \cup \{\} \rangle$ and so $\langle \cup \{\} \rangle =$. Therefore, by Theorem 3.1, $\in \mathcal{M}ax()$. □

Theorem 3.2. *Suppose $\in \mathcal{F}()$. Then $\in \mathcal{P}rime()$ if and only if is a chain.*

Proof. Let $\in \mathcal{P}rime()$ and $[], [] \in /$. Since is prime, we have $\twoheadrightarrow\in$ or $\twoheadrightarrow\in$. So $[] \leq []$ or $[] \leq []$. Therefore, is a chain. Conversely, let be a chain and $, \in$. Since $[] \leq []$ or $[] \leq []$, we get $\twoheadrightarrow\in$ or $\twoheadrightarrow\in$ and so $\in \mathcal{P}rime()$. □

Corollary 3.1. *Each maximal filter of is a prime filter of . Indeed, $\mathcal{M}ax()\mathcal{S}\mathcal{P}rime()$.*

Proof. Let $\in \mathcal{M}ax()$. Then by Theorem 3.1, is simple. Hence is a chain and by Theorem 3.2, we get $\in \mathcal{P}rime()$. □

Theorem 3.3. *If $\in \mathcal{P}rime()$ and $\mathcal{A} = \{\in \mathcal{F}()| \ \mathcal{S}$ and is proper$\}$, then $(\mathcal{A}, \mathcal{S})$ is linearly ordered.*

Proof. Let $, \in \mathcal{A}$ such that $\nsubseteq$ and $\nsubseteq$. Thus there are $a \in \setminus$ and $b \in \setminus$. Since is prime, we have $a \twoheadrightarrow b \in$ or $b \twoheadrightarrow a \in$. If $a \twoheadrightarrow b \in \mathcal{S}$, then since $a \in$ and $\in \mathcal{F}()$, we obtain $b \in$, which is a contradiction. Similarly, if $b \twoheadrightarrow a \in \mathcal{S}$, then $a \in$, which is a contradiction, too. Thus $\mathcal{S}$ or $\mathcal{S}$. Therefore, $\mathcal{A}$ is a chain. □

Corollary 3.2. *For any $\in \mathcal{P}rime()$, there is a unique maximal filter of that contains .*

Proof. By Proposition 3.1, there exists at least one maximal filter of that contains . By the contrary, let $_{1},_{2} \in \mathcal{M}ax()$ such that $\mathcal{S}_1$ and $\mathcal{S}_2$. Then by Theorem 3.3, $_{1},_{2} \in \mathcal{A}$ and they are comparable. Hence, $_{1}\mathcal{S}_2$ or $_{2}\mathcal{S}_1$ which are contradictions with $_{1},_{2}$ are maximal filters of . Therefore, maximal filter of that contains is unique. □

Theorem 3.4. *Let $, \in \mathcal{F}()$ and $\mathcal{S}$. Then*
(i) if $\in \mathcal{P}rime()$, then $\in \mathcal{P}rime()$, too.
(ii) If $\{1\} \in \mathcal{P}rime()$, then $\in \mathcal{P}rime()$, for any $\in \mathcal{F}()$.
(iii) If $\{1\} \in \mathcal{P}rime()$, then is chain.

Proof. (*i*) It is straightforward.
(*ii*) Since any filter of contains $\{1\}$, by (i) the proof is easy.
(*iii*) Suppose $\{1\} \in \mathcal{P}rime()$ and $, \in$. Then we have $\twoheadrightarrow= 1$ or $\twoheadrightarrow= 1$. Hence $\leq$ or $\leq$. Therefore, is chain. □

Theorem 3.5. *Let be a lattice. If $\in \mathcal{P}rime()$, then is a $\vee$-irreducible filter of .*

Proof. Suppose $\in \mathcal{P}rime()$ and $\underline{\in}$. By definition of prime filter, we get $\underline{\in}$ or $\underline{\twoheadrightarrow}$. Let $\underline{\in}$. Also, by Proposition 2.1(viii), $\underline{=}(\underline{)}$.*Since* $\underline{,}$ $\underline{\in}$*and* $\in \mathcal{F}()$, then $\underline{\in}$. Similarly, if $\underline{,}$ then . Hence, is a $\vee$-irreducible filter of . □

Theorem 3.6. *Let be prelinear. If is a $\vee$-irreducible filter of , then $\in \mathcal{P}rime()$.*

Proof. Consider is prelinear. Then by Theorem 2.1, is a lattice and for every $, \in$, $() \vee () = 1 \in$. Thus by definition of $\vee$-irreducible, we get $\in$ or $\in$. Hence, $\in \mathcal{P}rime()$ is a prime filter of . □

Next, we get that under which conditions $\{1\}$ is a $\vee$-irreducible filter of . For this, we define subdirectly irreducible equality algebra.

Remark 3.2. There is a one-one corresponding between $\mathcal{C}on()$ and $\mathcal{F}()$. It is enough to define the map $\phi : \mathcal{C}on() \longrightarrow \mathcal{F}()$ as $\theta \longmapsto [1]_\theta$. Then by Theorem 2.2(iii), we get ϕ is well-defined and one-one. If $\in \mathcal{F}()$, then $\theta \in \mathcal{C}on()$ and by Theorem 2.2(iv), we have $\phi(\theta) = [1]_\theta =$. So ϕ is onto.

Definition 3.2. [3] An algebraic structure A is called subdirectly irreducible if and only if A is trivial or there is a minimum congruence in $\mathcal{C}on(A) \setminus \{\Delta\}$, where $\Delta = \{(,) \in \times| \in\}$.

Lemma 3.1. *Let $, \in \mathcal{F}()$ and $\theta, \psi \in \mathcal{C}on()$. Then*
(i) if $\mathcal{S}$, then $\theta\mathcal{S}\theta$,
(ii) if $\theta\mathcal{S}\psi$, then $[1]_\theta\mathcal{S}[1]_\psi$,
(iii) $\theta = \Delta$ if and only if $[1]_\theta = \{1\}$,
(iv) $= \{1\}$ if and only if $\theta = \Delta$.

Proof. Proofs of (*i*) and (*ii*) are straightforward.
(*iii*) If $\theta = \Delta$, then $[1]_\theta = [1]_\Delta = \{x \in | \ (x, 1) \in \Delta\} = \{1\}$. Conversely, if $[1]_\theta = \{1\}$, then by Theorem 2.2(iv), we obtain $\theta = \theta_{[1]_\theta} = \theta_{\{1\}}$, where

$$\begin{aligned} \theta_{\{1\}} &= \{(x, y) \in \times| \ xy = 1 = yx\} \\ &= \{(x, y) \in \times| \ x = y\} = \Delta. \end{aligned}$$

(*iv*) Let $= \{1\}$. Then $\theta = \theta_{\{1\}} = \Delta$. Conversely, if $\theta = \Delta$, then by Theorem 2.2(iv), we get $= [1]_\theta = [1]_\Delta = \{1\}$. □

Lemma 3.2. *Let $\theta \in \mathcal{C}on()$. Then θ is a minimum element of $\mathcal{C}on() \setminus \{\Delta\}$ if and only if $[1]_\theta$ is a minimum element of $(\mathcal{F}() \setminus \{1\}; \mathcal{S})$.*

Proof. Let θ be a minimum element of $Con()\setminus\{\Delta\}$ and $\in \mathcal{F}()\setminus\{1\}$ be arbitrary. Then by Lemma 3.1(iv), we have $\theta \in Con()\setminus\{\Delta\}$ and so $\theta\mathcal{S}\theta$. Hence, by Lemma 3.1(ii) and Theorem 2.2(iv), we get $[1]_\theta\mathcal{S}[1]_\theta =$. Therefore, $[1]_\theta$ is minimum in $\mathcal{F}()\setminus\{1\}$.

Conversely, let $[1]_\theta$ be minimum in $\mathcal{F}()\setminus\{1\}$ and $\psi \in Con()\setminus\{\Delta\}$ be arbitrary. Then by Lemma 3.1(iii), $[1]_\psi \in \mathcal{F}()\setminus\{1\}$ and so $[1]_\theta\mathcal{S}[1]_\psi$. Hence, by Lemma 3.1(i) and Theorem 2.2 (iv) we get

$$\theta = \theta_{[1]_\theta}\mathcal{S}\theta_{[1]_\psi} = \psi.$$

Therefore, θ is a minimum element of $Con()\setminus\{\Delta\}$. □

Theorem 3.7. *Equality algebra is subdirectly irreducible if and only if there exists $\in \mathcal{F}()\setminus\{1\}$ such that for any $1 \neq\in$ such that $\mathcal{S}\langle\rangle$.*

Proof. Consider is subdirectly irreducible. Then there is a minimum congruence as $\theta \in Con()\setminus\Delta$. Suppose $:= [1]_\theta$. From Lemma 3.2, we get $[1]_\theta =$ is a minimum element of $\mathcal{F}()\setminus\{1\}$. Thus for any $1 \neq\in$, we have $\mathcal{S}\langle\rangle$.
Conversely, let $\neq \{1\}$ be a filter of such that $\mathcal{S}\langle\rangle$, for all $1 \neq\in$. It is easy to see that is a minimum element of $\mathcal{F}()\setminus\{1\}$. Now, take $\theta := \theta$. By Theorem 2.2(iv), we have $[1]_\theta =$ and by Lemma 3.2 we get $\theta = \theta$ is a minimum congruence relation in $Con()\setminus\Delta$. Therefore, is a subdirectly irreducible equality algebra. □

Theorem 3.8. *If lattice equality algebra is subdirectly irreducible, then $\{1\}$ is a $\vee$-irreducible filter of .*

Proof. Suppose is subdirectly irreducible with θ as the minimum element of $Con()\setminus\Delta$ and $\{1\}$ is not a $\vee$-irreducible filter of . Thus there exist $\Subset$ such that $\underline{=}1$ and $1 \neq \underline{\ }$*ByLemma*3.2, $[1]^{\cdot}\theta$ is the minimum filter of $\mathcal{F}()\setminus\{1\}$ and since $\langle\neq \{1\} \neq \langle\underline{\rangle}$, we get $[1]_\theta\mathcal{S}\langle\cap\langle\underline{\rangle}$. So by Theorem 2.3(iii), $[1]_\theta\mathcal{S}\langle\underline{\rangle} = \langle 1\rangle = \{1\}$. Hence $[1]_\theta = \{1\}$, which is a contradiction. Therefore, $\{1\}$ is a $\vee$-irreducible filter of . □

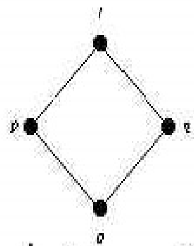
Corollary 3.3. *If is a simple lattice equality algebra, then $\{1\}$ is $\vee$-irreducible filter of .*

Proof. Let be a simple lattice equality algebra. Then $\mathcal{F}() = \{\{1\},\ \}$. If $\{1\}$ is not a $\vee$-irreducible filter of , then there are $\Subset$ such that $\underline{=}1$ and $1 \neq \underline{\ }$*Since* $\langle\neq \{1\} \neq \langle\underline{\rangle}$, we get $\langle== \langle\underline{\rangle}$. Thus by Theorem 2.3(iii), we have $\{1\} = \langle\underline{\rangle} = \langle\cap\langle\underline{\rangle} =$, which is a contradiction. Therefore, $\{1\}$ is a $\vee$-irreducible filter of . □

In the following, we introduce new type of filter on equality algebras which is called $\cap$-irreducible. Then we give some properties and investigate relations between maximal, prime, $\vee$-irreducible and $\cap$-irreducible filters of an equality algebra.

Definition 3.3. *Suppose $\in \mathcal{F}()$ is proper. Then is called an* $\cap$-irreducible filter of *if for every proper filters $, \in \mathcal{F}()$, $= \cap$ implies $=$ or $=$.*

Example 3.1. *Let $E = \{0, \underset{,}{}1\}$ be a set by following Hasse diagram. Define the operation $\sim$ on as follows:*



$\sim$	0	p	q	1
0	1	q	p	0
p	q	1	0	p
q	p	0	1	q
1	0	p	q	1

$\rightarrow$	0	p	q	1
0	1	1	1	1
p	q	1	q	1
q	p	p	1	1
1	0	p	q	1

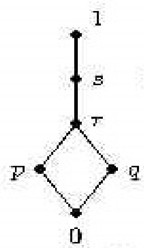
Then $(E, \wedge, \rightarrow, 1)$ is an equality algebra. Clearly, $= \{1\} \in \mathcal{F}()$. Since there are not two proper filters $_1, _2$ of such that $= _1 \cap _2$, we get is an $\cap$-irreducible filter. Also, $= \{1\}$ is not $\cap$-irreducible. Since $=_1 \cap_2$, where $_1 = \{1\}$ and $_2 = \{\underset{,}{}1\}$ but $_1 \neq\neq_2$.

Theorem 3.9. (*i*) *Any prime filter of is an* $\cap$-*irreducible filter of .*
(*ii*) *If is a lattice, then* $\cap$*-irreducible and* $\vee$*-irreducible filters of are coincide.*

Proof. (i) Let $\mathcal{LF} \in \mathcal{P}rime(\mathcal{LE})$, $\mathcal{LF} = \mathcal{LF}_1 \cap \mathcal{LF}_2$ and $\mathcal{LF}_2 \neq \mathcal{LF} \neq_1$. Thus there exist $\in \mathcal{LF}_1 \setminus \mathcal{LF}$ and $\in \mathcal{LF}_2 \setminus \mathcal{LF}$. From $\mathcal{LF}$ is prime, we obtain $\in \mathcal{LF}$ or $\in \mathcal{LF}$. If $\in \mathcal{LFS}_1$, then since $\in \mathcal{LF}_1$ and $\mathcal{LF}_1 \in \mathcal{F}(\mathcal{LE})$, we get $\in \mathcal{LF}_1$. Thus $\in \mathcal{LF}_1 \cap \mathcal{LF}_2 = \mathcal{LF}$, which is a contradiction. Similarly, whereas $\in \mathcal{LF}$, we get $\in \mathcal{LF}_1 \cap \mathcal{LF}_2 = \mathcal{LF}$, which is a contradiction. Afterwards, $\mathcal{LF} = \mathcal{LF}_1$ or $\mathcal{LF} = \mathcal{LF}_2$. Hence, $\mathcal{LF}$ is an $\cap$-irreducible filter of .
(ii) Suppose $\mathcal{LF}$ is an $\cap$-irreducible filter of $\mathcal{LE}$ and $, \in \mathcal{LE}$. If $\vee \in \mathcal{LF}$, then by Theorem 2.3(ii) $\mathcal{LF} = \langle \mathcal{LF} \cup \{\} \rangle \cap \langle \mathcal{LF} \cup \{\} \rangle$. Since is $\cap$-irreducible, we get $= \langle \cup \{\} \rangle$ or $= \langle \cup \{\} \rangle$ and so $\in \mathcal{LF}$ or $\in \mathcal{LF}$. Hence $\mathcal{LF}$ is a $\vee$-irreducible filter of $\mathcal{LE}$.
Conversely, let $\mathcal{LF}$ be a $\vee$-irreducible filter of $\mathcal{LE}$ and $\mathcal{LF} = \mathcal{LG}\cap$. If $\neq$ and $\neq$, then there exist $\in \setminus$ and $\in \mathcal{LH} \setminus \mathcal{LF}$. Since $, \leq \vee$ and $, \in \mathcal{F}()$, we obtain $\vee \in \cap =$. From is a $\vee$-irreducible filter of $\mathcal{LE}$, we conclude $\in \mathcal{LF}$ or $\in \mathcal{LF}$, which is a contradiction. Therefore, $\mathcal{LF} = \mathcal{LG}$ or $\mathcal{LF} = \mathcal{LH}$ and so $\mathcal{LF}$ is an $\cap$-irreducible filter of $\mathcal{LE}$. □

The following example shows that the converse of Theorem 3.9(i) is not true in general.

Example 3.2. *Consider* $= \{0, , , , 1\}$ *with the following Hasse diagram. Define the operation* $\sim$ *on as follows:*



$\sim$	0	p	q	r	s	1
0	1	s	s	s	r	0
p	s	1	r	s	r	p
q	s	r	1	s	r	q
r	s	s	s	1	s	r
s	r	r	r	s	1	s
1	0	p	q	r	s	1

$\rightarrow$	0	p	q	r	s	1
0	1	1	1	1	1	1
p	s	1	s	1	1	1
q	s	s	1	1	1	1
r	s	s	s	1	1	1
s	r	r	r	s	1	1
1	0	p	q	r	s	1

Then $(\mathcal{LE}, \wedge, \sim, 1)$ is an equality algebra and $\mathcal{F}(\mathcal{LE}) = \{\{1\}, \mathcal{LE}\}$. Clearly, $= \{1\}$ is an $\cap$-irreducible filter of $\mathcal{LE}$. But $\mathcal{LF}$ is not a prime filter of $\mathcal{LE}$, because $=\notin \mathcal{LF}$ and $\notin \mathcal{LF}$.

Theorem 3.10. *Suppose* $\mathcal{LE}$ *is prelinear and* $\mathcal{LF} \in \mathcal{F}(\mathcal{LE})$. *If* $\mathcal{LF}$ *is an* $\cap$*-irreducible filter of* $\mathcal{LE}$*, then it is a prime filter, too.*

Proof. Since $\mathcal{LE}$ is prelinear, by Theorem 2.1, we obtain $\mathcal{LE}$ is a lattice. Hence by Theorems 3.9(ii) and 3.6, the proof is clear . □

Theorem 3.11. (i) *Any maximal filter of* $\mathcal{LE}$ *is an* $\cap$*-irreducible filter of* $\mathcal{LE}$*.*
(ii) *Any maximal filter of a lattice equality algebra is a* $\vee$*-irreducible.*

Proof. (i) By Corollary 3.1, we have $\mathcal{M}ax(\mathcal{LE})\mathcal{SP}rime(\mathcal{LE})$. Thus by Theorem 3.9(i), the proof is clear.
(ii) By (i) and Theorem 3.9(ii), the proof is complete. □

Remark 3.3. When $\mathcal{LE}$ is a lattice, $\cap$-irreducible and $\vee$-irreducible filters are identical and any prime filter is a $\vee$-irreducible filter of $\mathcal{LE}$. Whenever, is prelinear, then $\vee$-irreducible and $\cap$-irreducible filters are coincide.

4 Zariski topology on equality algebras

Although Zariski (spectrum) topology has already been defined on algebras such as BL-algebras and MV-algebras, here we want to examine this topology on equality algebras, which is more comprehensive than the previous ones. In the previous section, we examined the types of prime filters so that we can make the right choice for the spectrum set and introduce the spectrum topology on equal algebra. According to Remark 3.3, it seems that it is better to equate the spectrum with the set of all $\cap$-irreducible filters. But because we need a lattice structure to prove the topological properties in Propositions 4.1(vii), 4.3(vii) and so Theorem 4.2, we have to equate the lattice condition to equality algebra and consider the set of all $\vee$-irreducible filters as its spectrum.

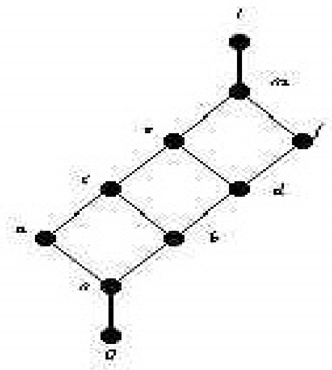
In this section, we introduce Zariski topology on equality algebra and show that *spectrum* of an equality algebra (the set of all $\vee$-irreducible filters of $\mathcal{LE}$, which is denoted by $\mathcal{S}pec(\mathcal{LE})$) with Zariski topology is a compact T_0-space. Moreover, maximal spectrum of an equality algebra as a subspace of spectrum is a compact T_1-space. Also, we prove that under which conditions (maximal) spectrum will be a Hausdorff space.

Note: From now on, let $\mathcal{LE}$ be a lattice equality algebra unless otherwise state.

Theorem 4.1. [17] Let $\in \mathcal{F}()$. Then for each , there is $\in \mathcal{S}pec()$ such that $\mathcal{S}$ and .

Definition 4.1. *Let $\mathcal{LA}\mathcal{S}\mathcal{LE}$. Then the set of all $\vee$-irreducible filters of $\mathcal{LE}$ containing $\mathcal{LA}$ is denoted by $V(\mathcal{LA}) = \{\mathcal{LP} \in \mathcal{S}pec(\mathcal{LE})|\ \mathcal{S}\mathcal{LP}\}$. For any $\mathcal{L}E$, we denote $V(\{\})$ by $V($ for short and $V(= \{\mathcal{LP} \in \mathcal{S}pec(\mathcal{LE})|\ \mathcal{LP}\}$.*

Example 4.1. *Let $\mathcal{L}E = \{0, n, a, b, c, d, e, f, m, 1\}$ with the following Hasse diagram.*



Define the operation as fig **??**.

Then $(\mathcal{L}E, \sim, \wedge, 1)$ is an equality algebra and $\mathcal{S}pec(\mathcal{LE}) = \{\underbrace{\{1\}}_{\mathcal{LP}_1}, \underbrace{\{f, m, 1\}}_{\mathcal{LP}_2}, \underbrace{\{a, c, e, m, 1\}}_{\mathcal{LP}_3}\}$. If $\mathcal{LA} = \{e, m\}$, then $V(\mathcal{LA}) = \{\mathcal{LP}_3\}$. Also, $V(b) = \emptyset$ and $V(m) = \{\mathcal{LP}_2, \mathcal{LP}_3\}$.

$\sim$	0	n	a	b	c	d	e	f	m	1
0	1	m	f	e	d	c	b	a	n	0
n	m	1	f	e	d	c	b	a	n	n
a	f	f	1	d	e	b	c	n	a	a
b	e	e	d	1	f	e	d	c	b	b
c	d	d	e	f	1	d	e	b	c	c
d	c	c	b	e	d	1	f	e	d	d
e	b	b	c	d	e	f	1	d	e	e
f	a	a	n	c	b	e	d	1	f	f
m	n	n	a	b	c	d	e	f	1	m
1	0	n	a	b	c	d	e	f	m	1

$\to$	0	n	a	b	c	d	e	f	m	1
0	1	1	1	1	1	1	1	1	1	1
n	m	1	1	1	1	1	1	1	1	1
a	f	f	1	f	1	f	1	f	1	1
b	e	e	e	1	1	1	1	1	1	1
c	d	d	e	f	1	f	1	f	1	1
d	c	c	c	e	e	1	1	1	1	1
e	b	b	c	d	e	f	1	f	1	1
f	a	a	a	c	c	e	e	1	1	1
m	n	n	a	b	c	d	e	f	1	1
1	0	n	a	b	c	d	e	f	m	1

Figure 1

Proposition 4.1. *Let $\mathcal{LA}, \mathcal{LB}\mathcal{S}E$ and $\mathcal{LF}, \mathcal{LG} \in \mathcal{F}(\mathcal{LE})$. Then*

(i) $V(\mathcal{LA}) = \mathcal{S}pec(\mathcal{LE})$ if and only if $\mathcal{LA} = \emptyset$ or $\mathcal{LA} = \{1\}$;

(ii) if $\mathcal{LA}\mathcal{S}\mathcal{LB}$, then $V(\mathcal{LB})\mathcal{S}V(\mathcal{LA})$;

(iii) if $\mathcal{LE}$ is bounded, then $V(0) = \emptyset$;

(iv) $V(\mathcal{LA}) = \emptyset$ if and only if $\langle \mathcal{LA} \rangle = E$. Particularly, $V(\mathcal{L}E) = \emptyset$.

(v) $V(\mathcal{LA}) = V(\langle \mathcal{LA} \rangle)$;

(vi) $V(\bigcup_{i \in \Delta} \mathcal{LA}_i) = \bigcap_{i \in \Delta} V(\mathcal{LA}_i)$;

(vii) $V(\langle \mathcal{LA} \rangle \cap \langle \mathcal{LB} \rangle) = V(\mathcal{LA}) \cup V(\mathcal{LB})$;

(viii) $V(\mathcal{LA}) = V(\mathcal{LB})$ if and only if $\langle \mathcal{LA} \rangle = \langle \mathcal{LB} \rangle$;

(ix) $V(\mathcal{LF}) = V()$ if and only if $\mathcal{LF} = \mathcal{LG}$.

(x) if $\mathcal{LA}$, then $V(\mathcal{LA})\mathcal{S}V($.

Proof. (i) Let $V(\mathcal{LA}) = \mathcal{S}pec(\mathcal{LE})$, $\mathcal{LA} \neq \emptyset$ and $\mathcal{LA} \neq \{1\}$. Then there exists $1 \neq \mathcal{LA}$. By Theorem 4.1(i), there is $\mathcal{LP} \in \mathcal{S}pec(\mathcal{LE})$ such that $\mathcal{LP}$. Afterwards, $\mathcal{LA} \nsubseteq \mathcal{LP}$ and so $\mathcal{LP} \notin V(\mathcal{LA}) = \mathcal{S}pec(\mathcal{LE})$ which is a contradiction. Hence $\mathcal{LA} = \emptyset$ or $\mathcal{LA} = \{1\}$. Conversely, $V(\{1\}) = \{\mathcal{LP} \in \mathcal{S}pec(\mathcal{LE})|\ \{1\}\mathcal{S}\mathcal{LP}\} = \mathcal{S}pec(\mathcal{LE})$ and it is clear that $V(\emptyset) = \mathcal{S}pec(\mathcal{LE})$.

(ii) The proof is straightforward;

(iii) Let $\mathcal{LE}$ be bounded. We know $\mathcal{LF} \in \mathcal{F}(\mathcal{LE})$ is proper if and only if $0 \notin \mathcal{LF}$. So $V(\{0\}) = \{\mathcal{LP} \in \mathcal{S}pec(\mathcal{LE})|\ \{0\}\mathcal{S}\mathcal{LP}\} = \emptyset$.

(iv) Let $V(\mathcal{LA}) = \emptyset$ and $\langle \mathcal{LA} \rangle \neq \mathcal{L}E$. From Proposition 3.1, Corollary 3.1 and Theorem 3.5, respectively, we get there is $\mathcal{LP} \in \mathcal{S}pec(\mathcal{LE})$ such that $\mathcal{LA}\mathcal{S}\langle \mathcal{LA} \rangle \mathcal{S}\mathcal{LP}$. So $\mathcal{LP} \in V(\mathcal{LA}) = \emptyset$, which is a contradiction. Hence, $\langle \mathcal{LA} \rangle$ is not proper and $\langle \mathcal{LA} \rangle = E$. Conversely, if $\langle \mathcal{LA} \rangle = E$, then there is no proper filter of $\mathcal{LE}$ containing $E = \langle \mathcal{LA} \rangle$. So by (ii) we obtain that $V(\mathcal{LA}) = V(\langle \mathcal{LA} \rangle) = \emptyset$.

(v) Let $\mathcal{LP} \in V(\mathcal{LA})$. Then $\mathcal{LA}\mathcal{S}\mathcal{LP}$ and since $\langle \mathcal{LA} \rangle$ is the smallest filter of $\mathcal{LE}$ containing $\mathcal{LA}$, we have $\langle \mathcal{LA} \rangle \mathcal{S}\mathcal{LP}$. Thus $\mathcal{LP} \in V(\langle \mathcal{LA} \rangle)$ and so

$V(\mathcal{L}A)\mathcal{S}V(\langle\mathcal{L}A\rangle)$. Since $\mathcal{L}A\mathcal{S}\langle\mathcal{L}A\rangle$, by (ii), the converse holds. Therefore, $V(\mathcal{L}A) = V(\langle\mathcal{L}A\rangle)$.
(vi) Since for each $i \in \Delta$, $\mathcal{L}A_i\mathcal{S} \bigcup_{i\in\Delta} \mathcal{L}A_i$, by (ii) we get $V(\bigcup_{i\in\Delta} \mathcal{L}A_i)\mathcal{S}V(\mathcal{L}A_i)$. Hence, $V(\bigcup_{i\in\Delta} \mathcal{L}A_i)\mathcal{S} \bigcap_{i\in\Delta} V(\mathcal{L}A_i)$. Conversely, if $\mathcal{L}P \in \bigcap_{i\in\Delta} V(\mathcal{L}A_i)$, then for any $i \in \Delta$, $\mathcal{L}P \in V(\mathcal{L}A_i)$ and so $\mathcal{L}A_i\mathcal{S}\mathcal{L}P$. Thus $\bigcup_{i\in\Delta} \mathcal{L}A_i\mathcal{S}\mathcal{L}P$ and so $\mathcal{L}P \in V(\bigcup_{i\in\Delta} \mathcal{L}A_i)$.
(vii) We know $\langle\mathcal{L}A\rangle \cap \langle\mathcal{L}B\rangle\mathcal{S}\langle\mathcal{L}A\rangle, \langle\mathcal{L}B\rangle$. From (ii) and (v) we obtain $V(\mathcal{L}A)\cup V(\mathcal{L}B)\mathcal{S}V(\langle\mathcal{L}A\rangle\cap\langle\mathcal{L}B\rangle)$. Conversely, let $\mathcal{L}P \in V(\langle\mathcal{L}A\rangle \cap \langle\mathcal{L}B\rangle)$, $\mathcal{L}P \notin V(\mathcal{L}A)$ and $\mathcal{L}P \notin V(\mathcal{L}B)$. Then $\langle\mathcal{L}A\rangle \cap \langle\mathcal{L}B\rangle\mathcal{S}\mathcal{L}P$, $\mathcal{L}A \nsubseteq \mathcal{L}P$ and $\mathcal{L}B \nsubseteq \mathcal{L}P$. Thus there are $\in \mathcal{L}A$ and $\in \mathcal{L}B$ such that $, \notin \mathcal{L}P$. Since $, \leq \vee$ and $\langle\mathcal{L}A\rangle, \langle\rangle$ are filters of , we get $\vee \in \langle\mathcal{L}A\rangle\cap\langle\rangle\mathcal{S}$. Thus $\vee \in$ and $, \notin \mathcal{L}P$, which is a contradiction with $\in \mathcal{S}pec()$. Hence $\in V(\mathcal{L}A)$ or $\in V()$ and so $V(\langle\rangle \cap \langle\mathcal{L}B\rangle)\mathcal{S}V(\mathcal{L}A) \cup V(\mathcal{L}B)$. Therefore, $V(\langle\mathcal{L}A\rangle \cap \langle\mathcal{L}B\rangle) = V(\mathcal{L}A) \cup V(\mathcal{L}B)$.
$(viii)$ By (v), we have

$$\begin{aligned} V(\mathcal{L}A) &= V(\langle\mathcal{L}A\rangle) = \{\mathcal{L}P \in \mathcal{S}pec(\mathcal{L}\mathcal{E})|\ \langle\mathcal{L}A\rangle\mathcal{S}\mathcal{L}P\} \\ &= \{\mathcal{L}P \in \mathcal{S}pec(\mathcal{L}\mathcal{E})|\ \langle\mathcal{L}B\rangle\mathcal{S}\mathcal{L}P\} \\ &= V(\langle\mathcal{L}B\rangle) = V(\mathcal{L}B). \end{aligned}$$

(ix) By (viii), the proof is clear.
(x) For any $\mathcal{L}A$, by (vi), we get
$V(\mathcal{L}A) = V(\bigcup_{LA} \{) = \bigcap_{LA} V(\mathcal{S}V($. □

Proposition 4.2. *Suppose $\in\mathcal{L}E$. Then*
(i) $V(= \mathcal{S}pec(\mathcal{L}\mathcal{E})$ if and only if $\bar{1}$;
(ii) $V(= \emptyset$ if and only if $\langle = \mathcal{L}E$;
(iii) $V(= V()$ if and only if $\langle = \langle\rangle$;
(iv) if then $V(SV()$;
(v) $V() = V(\cap V()$;
(vi) $V() = V(\cup V()$.

Proof. By Proposition 4.1(i), (iv) and (viii), respectively, it is easy to see that (i), (ii) and (iii) hold.
(iv) Let *and* LP$\in$ $V($. Then and since $\mathcal{L}P \in \mathcal{F}(\mathcal{L}\mathcal{E})$, we get $\in\mathcal{L}P$. So $\mathcal{L}P \in V()$. Therefore, $V(\mathcal{S}V()$.
(v) Let $\mathcal{L}P \in V()$. Then $\in\mathcal{L}P$ and since $\leq$*weget* $\in$LP .*Thus* LP$\in$ $V(\cap V()$ and so $V()\mathcal{S}V(\cap V()$.
Conversely, let $\mathcal{L}P \in V(\cap V()$. Then $\in\mathcal{L}P$. Hence by Proposition 2.1(iii) and (vii), respectively, we get $()$.
Since $\in\mathcal{L}P$ and $\mathcal{L}P \in \mathcal{F}(\mathcal{L}\mathcal{E})$, we get $\in\mathcal{L}P$ and so $\mathcal{L}P \in V()$. Hence, $V(\cap V()\mathcal{S}V()$ and the proof is complete.
(vi) $\mathcal{L}P \in V()$ if and only if $\in\mathcal{L}P$ if and only if $\mathcal{L}P$ or $\in\mathcal{L}P$ if and only if $\mathcal{L}P \in V(\cup V()$. □

Definition 4.2. *Let $\mathcal{L}A\mathcal{S}\mathcal{L}E$. The complement of $V(\mathcal{L}A)$ in $\mathcal{S}pec(\mathcal{L}\mathcal{E})$ is denoted by $U(\mathcal{L}A)$. Indeed,*

$$U(\mathcal{L}A) = \{\mathcal{L}P \in \mathcal{S}pec(\mathcal{L}\mathcal{E})|\ \mathcal{L}A \nsubseteq \mathcal{L}P\}.$$

For each $\mathcal{L}E$, we denote $U(\{)$ by $U($ for short. Indeed, $U(= \{\mathcal{L}P \in \mathcal{S}pec(\mathcal{L}E)|\ \mathcal{L}P\}$.

Proposition 4.3. *Let $\mathcal{L}A, \mathcal{L}B\mathcal{S}\mathcal{L}E$. Then*
(i) $U(\{1\}) = U(\emptyset) = \emptyset$. If $\mathcal{L}\mathcal{E}$ is bounded, then $U(\{0\}) = \mathcal{S}pec(\mathcal{L}\mathcal{E})$;
(ii) if $\mathcal{L}A\mathcal{S}\mathcal{L}B$, then $U(\mathcal{L}A)\mathcal{S}U(\mathcal{L}B)$;
(iii) $U(\mathcal{L}A) = U(\langle\mathcal{L}A\rangle)$;
(iv) $U(\mathcal{L}A) = \mathcal{S}pec(\mathcal{L}\mathcal{E})$ if and only if $\langle\mathcal{L}A\rangle = \mathcal{L}E$. Particularly, $U(\mathcal{L}E) = \mathcal{S}pec(\mathcal{L}\mathcal{E})$;
(v) $U(\mathcal{L}A) = \emptyset$ if and only if $\mathcal{L}A = \emptyset$ or $\mathcal{L}A = \{1\}$;
(vi) $U(\bigcup_{i\in\Delta} \mathcal{L}A_i) = \bigcup_{i\in\Delta} U(\mathcal{L}A_i)$;
(vii) $U(\langle\mathcal{L}A\rangle \cap \langle\mathcal{L}B\rangle) = U(\mathcal{L}A) \cap U(\mathcal{L}B)$;
$(viii)$ $U(\mathcal{L}A) = U(\mathcal{L}B)$ if and only if $\langle\mathcal{L}A\rangle = \langle\mathcal{L}B\rangle$;
(ix) $U(\mathcal{L}F) = U(\mathcal{L}G)$ if and only if $\mathcal{L}F = \mathcal{L}G$;
(x) if $\mathcal{L}A$, then $U(\mathcal{S}U(\mathcal{L}A)$.

Proof. Proofs of (iii), (iv), (v), $(viii)$, (ix) and (x) are straightforward.
(i) By Proposition 4.1(i) and (iii), $V(\{1\}) = V(\emptyset) = \mathcal{S}pec(\mathcal{L}\mathcal{E})$ and $V(\{0\}) = \emptyset$. So by complement of them the proof is clear.
(ii) Suppose $\mathcal{L}A\mathcal{S}\mathcal{L}B$. From Proposition 4.1(ii), $V(\mathcal{L}B)\mathcal{S}V(\mathcal{L}A)$. So by complement $\mathcal{S}pec(E) \setminus V(\mathcal{L}A)\mathcal{S}\mathcal{S}pec(E)\setminus V(\mathcal{L}B)$. Thus $U(\mathcal{L}A)\mathcal{S}U(\mathcal{L}B)$.

(vi) By Proposition 4.1(vi), we have

$$\begin{aligned} U(\bigcup_{i\in\Delta} \mathcal{L}A_i) &= \mathcal{S}pec(\mathcal{LE}) \setminus V(\bigcup_{i\in\Delta} \mathcal{L}A_i) \\ &= \mathcal{S}pec(\mathcal{LE}) \setminus \bigcap_{i\in\Delta} V(\mathcal{L}A_i) \\ &= \bigcup_{i\in\Delta} [\mathcal{S}pec(\mathcal{LE}) \setminus V(\mathcal{L}A_i)] \\ &= \bigcup_{i\in\Delta} U(\mathcal{L}A_i). \end{aligned}$$

(vii) From Proposition 4.1(vii)

$$\begin{aligned} U(\langle\rangle \cap \langle\rangle) = \mathcal{S}pec() &\setminus [V(\langle\rangle \cap \langle\rangle)] \\ = \mathcal{S}pec() &\setminus [V() \cup V()] \\ = [\mathcal{S}pec() \setminus V()] &\cap [\mathcal{S}pec() \setminus V()] \\ = U() &\cap U(). \end{aligned}$$

□

Proposition 4.4. *Let $\in\mathcal{LE}$. Then*
(i) $U(= \mathcal{S}pec(\mathcal{LE})$ if and only if $\langle = \mathcal{LE}$.
(ii) $U(= \emptyset$ if and only if $\bar{1}$.
(iii) $U(= U(\underline{)}$ if and only if $\langle = \langle\underline{\rangle}$.
(iv) if $\underline{}_{,}$ then $U()\mathcal{S}U($.
(v) $U(\underline{)}=U(\cup U(\underline{)}$.
(vi) $U(\underline{)} = U(\cap U(\underline{)}$.
(vii) if $\mathcal{LE}$ is bounded, then $V(\mathcal{S}U(\hat{-})$.

Proof. The proofs of $(i) - (vi)$ are directly results of Proposition 4.2 $(i) - (vi)$.
(vii) Let $\mathcal{LE}$ be bounded and $\mathcal{LP} \in V($. Then $\mathcal{LP}$. If $\hat{-} \in \mathcal{LP}$, then $\hat{-} = 0 \in \mathcal{LP}$. Since $\mathcal{LP} \in \mathcal{F}(\mathcal{LE})$ we get $0 \in \mathcal{LP}$. Thus $\mathcal{LP} = E$, which is a contradiction and so $\hat{-} \notin \mathcal{LP}$ which implies $\mathcal{LP} \in U(\hat{-})$. Therefore, $V(\mathcal{S}U(\hat{-})$. □

The following example shows that the converse of Proposition 4.4(vii) is not true in general.

Example 4.2. *Suppose $\mathcal{LE}$ is the lattice equality algebra as in Example 4.1. Then $V(m) = \{\mathcal{LP}_2, \mathcal{LP}_3\}$, $U(m^-) = U(n) = \mathcal{S}pec(\mathcal{LE})$ and so $U(m^-) \nsubseteq V(m)$.*

Proposition 4.5. *Let $\mathcal{LE}$ be bounded and $\mathcal{LE}$. If $\hat{-} = 1$, then $U(\hat{-}) = V($.*

Proof. By Proposition 4.4(vii), $V(\mathcal{S}U(\hat{-})$. For the converse, let $\mathcal{LP} \in U(\hat{-})$. Then $\hat{-} \notin \mathcal{LP}$. Since $\hat{-} = 1 \in \mathcal{LP}$ and $\mathcal{LP}$ is a $\vee$-irreducible filter of $\mathcal{LE}$, we get $\mathcal{LP}$. Hence, $\mathcal{LP} \in V($ and so $U(\hat{-})\mathcal{S}V($. Therefore, $U(\hat{-}) = V($. □

Theorem 4.2. *Let $\tau = \{U(\mathcal{L}A)|\ \mathcal{L}A\mathcal{SLE}\}$. Then τ is a topology on $\mathcal{S}pec(\mathcal{LE})$.*

Proof. By Proposition 4.3(i) and (iv), $\emptyset$, $\mathcal{S}pec(\mathcal{LE}) \in \tau$. Also, by Proposition 4.3(vii),

$$\bigcap_{1\leq i\leq n} U(\mathcal{L}A_i) = U(\bigcap_{1\leq i\leq n} \langle\mathcal{L}A_i\rangle) \in \tau.$$

Finally, by Proposition 4.3(vi), an arbitrary union of elements of τ is an element of τ. Hence τ is a topology on $\mathcal{S}pec()$. □

Definition 4.3. *The topology induced by $\tau = \{U(\mathcal{L}A)|\ \mathcal{L}A\mathcal{SLE}\}$ on $\mathcal{S}pec(\mathcal{LE})$ is called the* Zariski topology *and $U(\mathcal{L}A)$ is the open subsets of $\mathcal{S}pec(\mathcal{LE})$ for any $\mathcal{L}A\mathcal{SLE}$.*

Proposition 4.6. *Let $\beta = \{U(\}_{\mathcal{LE}}$. Then β is a basis for Zariski topology $(\mathcal{S}pec(\mathcal{LE}), \tau)$.*

Proof. Let $U(\mathcal{L}A) \in \tau$. From Proposition 4.3(vi), $U(\mathcal{L}A) = U(\bigcup_{LA} \{) = \bigcup_{LA} U($ which is the union of some elements of β. □

Example 4.3. *Suppose $\mathcal{LE}$ is the lattice equality algebra as in Example 4.1. Then*
$U(0) = \mathcal{S}pec(\mathcal{LE}) = U(n) = U(b) = U(d)$,
$U(a) = \{\mathcal{LP}_1, \mathcal{LP}_2\} = U(c) = U(e)$,
$U(f) = \{\mathcal{LP}_1, \mathcal{LP}_3\}$, $U(m) = \{\mathcal{LP}_1\}, U(1) = \emptyset$.

Hence, $\tau =$

$$\{\emptyset, \{\mathcal{LP}_1\}, \{\mathcal{LP}_1, \mathcal{LP}_2\}, \{\mathcal{LP}_1, \mathcal{LP}_3\}, \mathcal{S}pec(\mathcal{LE})\} = \beta$$

.

Proposition 4.7. *Let $\mathcal{LP}, \mathcal{LQ} \in \mathcal{S}pec(\mathcal{LE})$. Then*
(i) $\{\mathcal{LP}\}$ is closed if and only if $\in \mathcal{M}ax(\mathcal{LE})$.
(ii) $Cl(\mathcal{LP}) = V(\mathcal{LP})$.
(iii) $\mathcal{LQ} \in Cl(\mathcal{LP})$ if and only if $\mathcal{LPSLQ}$.

Proof. (i) Consider $\{\mathcal{L}P\}$ is closed in $\mathcal{S}pec(\mathcal{L}\mathcal{E})$. By definition of closed subset, there is a proper subset $\mathcal{L}A\mathcal{S}\mathcal{L}E$ such that $\{\mathcal{L}P\} = V(\mathcal{L}A)$. By Proposition 3.1, there exists a maximal filter $\mathcal{L}M$ of $\mathcal{L}\mathcal{E}$ containing $\mathcal{L}P$. Since $\mathcal{L}P \in V(\mathcal{L}A)$, we have $\mathcal{L}A\mathcal{S}\mathcal{L}P\mathcal{S}\mathcal{L}M$ and so by Theorem 3.11(ii), $\mathcal{L}M \in V(\mathcal{L}A) = \{\mathcal{L}P\}$. Hence $\mathcal{L}P = \mathcal{L}M \in \mathcal{M}ax(\mathcal{L}\mathcal{E})$. Conversely, let $\mathcal{L}P$ be a maximal filter of $\mathcal{L}\mathcal{E}$. Then $V(\mathcal{L}P) = \{\mathcal{L}Q \in \mathcal{S}pec(\mathcal{L}\mathcal{E})|\ \mathcal{L}P\mathcal{S}\mathcal{L}Q \subsetneq \mathcal{L}E\} = \{\mathcal{L}P\}$. Therefore, $\{\mathcal{L}P\}$ is closed in $\mathcal{S}pec(\mathcal{L}\mathcal{E})$.
(ii) By definition of $Cl(\mathcal{L}P)$ and from $V(\mathcal{L}P)$ is a closed subset containing $\mathcal{L}P$, we obtain $Cl(\mathcal{L}P)\mathcal{S}V(\mathcal{L}P)$. Conversely, consider $\mathcal{L}Q \in V(\mathcal{L}P)$ and $\mathcal{L}Q \neq \mathcal{L}P$. We claim that $\mathcal{L}Q$ is in all closed subsets containing $\mathcal{L}P$. For this, let V be an arbitrary closed subset containing $\mathcal{L}P$ such that $V = V(\mathcal{L}A)$ for some non-empty subset $\mathcal{L}A\mathcal{S}\mathcal{L}E$. Since $\mathcal{L}P \in V(\mathcal{L}A)$ and $\mathcal{L}Q \in V(\mathcal{L}P)$ we get $\mathcal{L}A\mathcal{S}\mathcal{L}P$ and $\mathcal{L}P\mathcal{S}\mathcal{L}Q$. Thus $\mathcal{L}A\mathcal{S}\mathcal{L}Q$ and so $\mathcal{L}Q \in V(\mathcal{L}A) = V$. From $\mathcal{L}Q \in \bigcap_{LP\in V} V = Cl(\mathcal{L}P)$ we have $V(\mathcal{L}P)\mathcal{S}Cl(\mathcal{L}P)$. Therefore, $Cl(\mathcal{L}P) = V(\mathcal{L}P)$.
(iii) This is the result of (ii). □

Theorem 4.3. *Let $X\mathcal{S}\mathcal{S}pec(\mathcal{L}\mathcal{E})$. Then $Cl(X) = V(X_0)$, where $X_0 = \bigcap_{LP\in X} \mathcal{L}P$.*

Proof. In Zariski topology, $V(X_0)$ is a closed subset of $\mathcal{S}pec(\mathcal{L}\mathcal{E})$. Since for any $\mathcal{L}P \in X$, $X_0 = \bigcap_{LP\in X} \mathcal{L}P\mathcal{S}\mathcal{L}P$, then $\mathcal{L}P \in V(X_0)$ and so $X\mathcal{S}V(X_0)$. Now, we prove $V(X_0)$ is the smallest closed subset of $\mathcal{S}pec(\mathcal{L}\mathcal{E})$ contains X. Suppose $V(A)$ is an arbitrary closed subset that contains X. Then for any $\mathcal{L}P \in X$, $\mathcal{L}P \in V(A)$ and so $A\mathcal{S}\mathcal{L}P$. Hence $A\mathcal{S} \bigcap_{LP\in X} \mathcal{L}P = X_0$. Thus by Proposition 4.1(v), we get $V(X_0)\mathcal{S}V(A)$. Therefore, $Cl(X) = V(X_0)$. □

Theorem 4.4. *Let . Then*
(i) $U($ is compact in $(\mathcal{S}pec(), \tau)$;
(ii) if is bounded, then $(\mathcal{S}pec(), \tau)$ is a compact topological space.

Proof. (i) From Proposition 4.6, we can suppose that any cover of $U($ is a union of basic open sets of $\mathcal{S}pec(\mathcal{L}\mathcal{E})$. Let $U(= \bigcup_{i\in\Delta} U(i)$. Then by Proposition 4.3(vi), $U(= U(\bigcup_{i\in\Delta}\{i\})$. Thus by Proposition 4.3(viii), we get $\langle = \langle \bigcup_{i\in\Delta}\{i\}\rangle$ and so $\langle \bigcup_{i\in\Delta}\{i\}\rangle$. Hence, there are $i_1, ..., i_n \in \Delta$ such that

$$i_1(...(i_n...) = 1 \quad \text{and} \quad i_j \in \bigcup_{i\in\Delta}\{i\}, \quad \text{where } 1 \leq j \leq n.$$

Without loss of generality, we conclude that there exist $1, ..., n$ such that $\langle \bigcup_{1\leq i\leq n}\{i\}\rangle$. Hence, by Proposition 4.3(x), (iii) and (vi) respectively, we have

$$\begin{aligned} U(\ \mathcal{S} \ U(\langle \bigcup_{1\leq i\leq n} i\rangle) &= U(\bigcup_{1\leq i\leq n} i) \\ &= \bigcup_{1\leq i\leq n} U(i)\mathcal{S} \bigcup_{i\in\Delta} U(i) = U(, \end{aligned}$$

which implies $U(= \bigcup_{1\leq i\leq n} U(i)$. Therefore, $U($ is compact.
(ii) From Proposition 4.3(i), we have $U(0) = \mathcal{S}pec(\mathcal{L}\mathcal{E})$. Then by (i), $\mathcal{S}pec(\mathcal{L}\mathcal{E})$ is compact. □

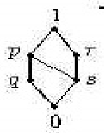
Theorem 4.5. *$(\mathcal{S}pec(\mathcal{L}\mathcal{E}), \tau)$ is a T_0-topological space.*

Proof. Consider $\mathcal{L}P$ and are two distinct elements of $\mathcal{S}pec(\mathcal{L}\mathcal{E})$. From $\mathcal{L}P \neq \mathcal{L}Q$, we get $\mathcal{L}P \not\subsetneq \mathcal{L}Q$ or $\mathcal{L}Q \not\subsetneq \mathcal{L}P$. If $\mathcal{L}P \not\subsetneq \mathcal{L}Q$, then there is $\in \mathcal{L}P$ such that $\notin \mathcal{L}Q$. Thus $\mathcal{L}Q \in U()$ and $\mathcal{L}P \notin U()$. By the similar way, another case can be proved. □

Example 4.4. *Suppose $\mathcal{L}\mathcal{E}$ is the lattice equality algebra as in Example 4.1 and $\mathcal{L}P_1, \mathcal{L}P_3 \in \mathcal{S}pec(\mathcal{L}\mathcal{E})$. Since there is no open subset $U \in \tau$ such that $\mathcal{L}P_3 \in U$ and $\mathcal{L}P_1 \notin U$, then $(\mathcal{S}pec(\mathcal{L}\mathcal{E}), \tau)$ is not a T_1-space. Also it is not a Hausdorff space.*

Definition 4.4. *Suppose $\mathcal{L}\mathcal{E}$ is bounded. Then $B(\mathcal{L}\mathcal{E})$ is the set of all $\in \mathcal{L}E$ such that $\vee^- = 1$ and $\wedge^- = 0$.*

Example 4.5. *Suppose $(E = \{0, , , 1\}, \leq)$ is a lattice with the following Hasse diagram and the operation "$\sim$" is defined on as follows:*



$\sim$	0	p	q	r	s	1
0	1	s	r	q	p	0
p	s	1	p	s	r	p
q	r	p	1	0	s	q
r	q	s	0	1	p	r
s	p	r	s	p	1	s
1	0	p	q	r	s	1

$\rightarrow$	0	p	q	r	s	1
0	1	1	1	1	1	1
p	s	1	p	r	r	1
q	r	1	1	r	r	1
r	q	p	q	1	p	1
s	p	1	p	1	1	1
1	0	p	q	r	s	1

Then $(\mathcal{L}E, \sim, \wedge, 1)$ is a bounded equality algebra and $B(\mathcal{L}\mathcal{E}) = \{0, , , 1\}$.

Lemma 4.1. *Suppose $\mathcal{L}\mathcal{E}$ is bounded, $B(\mathcal{L}\mathcal{E}) = \mathcal{L}E$, $\in \mathcal{L}E$ and $\mathcal{L}P \in \mathcal{S}pec(\mathcal{L}\mathcal{E})$. Then $\in$ if and only if $^- \notin \mathcal{L}P$.*

Proof. Let $\in \mathcal{L}P$ and $^- \in \mathcal{L}P$. Since $\mathcal{L}P \in \mathcal{F}(\mathcal{L}\mathcal{E})$, $0 \in \mathcal{L}P$, which is a contradiction. Therefore, $^- \notin \mathcal{L}P$. For the converse, let $^- \notin \mathcal{L}P$. Since $\vee^- = 1 \in \mathcal{L}P$ and $\mathcal{L}P \in \mathcal{S}pec(\mathcal{L}\mathcal{E})$, then $\in \mathcal{L}P$. □

Theorem 4.6. *Suppose $\mathcal{L}\mathcal{E}$ is bounded. Then*
(i) $B(\mathcal{L}\mathcal{E}) = \mathcal{L}E$ implies $(\mathcal{S}pec(\mathcal{L}\mathcal{E}), \tau)$ is a Hausdorff space.
(iii) If $(\mathcal{S}pec(\mathcal{L}\mathcal{E}), \tau)$ is connected, then $B(\mathcal{L}\mathcal{E}) = \{0, 1\}$.

Proof. (*i*) Let $\mathcal{L}P_1, \mathcal{L}P_2 \in \mathcal{S}pec(\mathcal{L}\mathcal{E})$ and $\mathcal{L}P_1 \neq \mathcal{L}P_2$. Then $\mathcal{L}P_1 \nsubseteq \mathcal{L}P_2$ or $\mathcal{L}P_2 \nsubseteq \mathcal{L}P_1$. Suppose $\mathcal{L}P_1 \nsubseteq \mathcal{L}P_2$. Then there is $\mathcal{L}P_1 \setminus \mathcal{L}P_2$. Since $\mathcal{L}P_2$, we get $\mathcal{L}P_2 \in U($. By Lemma 4.1, $\mathcal{L}P_1$ if and only if $\hat{-} \notin \mathcal{L}P_1$. Thus $\hat{-} \notin \mathcal{L}P_1$ and so $\mathcal{L}P_1 \in U(\hat{-})$. Moreover, by Proposition 4.4(vi), (ii) and since $\mathcal{L}E$ is complemented, we obtain $U(\cap U(\hat{-}) = U(\hat{-}) = U(1) = \emptyset$. Therefore, $(\mathcal{S}pec(\mathcal{L}\mathcal{E}), \tau)$ is a Hausdorff space.
(*ii*) Consider $(\mathcal{S}pec(\mathcal{L}\mathcal{E}), \tau)$ is connected and there is $\in B(\mathcal{L}\mathcal{E})$ such that $\neq 0, 1$. Hence $\vee^- = 1$ and $\wedge^- = 0$. By Proposition 4.4(ii), $U() = \emptyset$ if and only if $= 1$ if and only if $^- = 0$. Since $\neq 0, 1$, we conclude $U() \neq \emptyset \neq U(^-)$. In addition, by Proposition 4.4(v) and (vi),

$$U() \cap U(^-) = U(\vee^-) = U(1) = \emptyset,$$

$$U() \cup U(^-) = U(\wedge^-) = U(0) = \mathcal{S}pec(\mathcal{L}\mathcal{E}).$$

Since $\mathcal{L}\mathcal{E}$ is connected, we get $U() = \emptyset$ or $U(^-) = \emptyset$, which is a contradiction. Therefore, $B(\mathcal{L}\mathcal{E}) = \{0, 1\}$. □

Remark 4.1. By Theorem 3.11(ii), $\mathcal{M}ax(\mathcal{L}\mathcal{E})\mathcal{S}\mathcal{S}pec(\mathcal{L}\mathcal{E})$. Thus we can consider the topology induced by Zariski topology on $\mathcal{M}ax(\mathcal{L}\mathcal{E})$ that is called *maximal spectrum* of $\mathcal{L}\mathcal{E}$. For $\mathcal{L}A\mathcal{S}\mathcal{L}E$ and $\in \mathcal{L}E$, define

$$\begin{aligned} V_M(\mathcal{L}A) &= V(\mathcal{L}A) \cap \mathcal{M}ax(\mathcal{L}\mathcal{E}), \\ V_M() &= V() \cap \mathcal{M}ax(\mathcal{L}\mathcal{E}), \\ U_M(\mathcal{L}A) &= U(\mathcal{L}A) \cap \mathcal{M}ax(\mathcal{L}\mathcal{E}), \\ U_M() &= U() \cap \mathcal{M}ax(\mathcal{L}\mathcal{E}). \end{aligned}$$

Then $\{U_M(\mathcal{L}A) | \mathcal{L}A\mathcal{S}\mathcal{L}E\}$ and $\{U_M() | \in \mathcal{L}E\}$ are the family of open sets and basis for the topology on $\mathcal{M}ax(\mathcal{L}\mathcal{E})$. Also, all the results of Propositions 4.1, 4.2 and 4.3 hold. Therefore, $\mathcal{M}ax(\mathcal{L}\mathcal{E})$ is a compact T_0-space.

Theorem 4.7. *The topological space $(\mathcal{M}ax(\mathcal{L}\mathcal{E}), \tau)$ is a T_1-space.*

Proof. Let $\mathcal{L}M_1, \mathcal{L}M_2$ be two distinct elements of $\mathcal{M}ax(\mathcal{L}\mathcal{E})$. Since any maximal filter is not included in any other proper filter of $\mathcal{L}\mathcal{E}$ and $\mathcal{L}M_1, \mathcal{L}M_2 \in \mathcal{M}ax(\mathcal{L}\mathcal{E})$, we have $\mathcal{L}M_1 \nsubseteq \mathcal{L}M_2$ and $\mathcal{L}M_2 \nsubseteq \mathcal{L}M_1$. Hence, there are $\mathcal{L}M_1 \setminus \mathcal{L}M_2$ and $\in \mathcal{L}M_2 \setminus \mathcal{L}M_1$. Since, $\mathcal{L}M_1$ and $\mathcal{L}M_2$, we get $\mathcal{L}M_1 \notin U($ and $\mathcal{L}M_2 \in U($. Similarly, $\mathcal{L}M_1 \in U(\underline{)}$ and $\mathcal{L}M_2 \notin U(\underline{)}$. Thus $U(\neq U(\underline{)}$ are two open sets that contains one and not containing another one. Therefore, $(\mathcal{M}ax(\mathcal{L}\mathcal{E}), \tau)$ is a T_1-space. □

Theorem 4.8. *The topological space $(\mathcal{S}pec(\mathcal{L}\mathcal{E}), \tau)$ is a T_1-space if and only if $\mathcal{S}pec(\mathcal{L}\mathcal{E}) = \mathcal{M}ax(\mathcal{L}\mathcal{E})$.*

Proof. Let $(\mathcal{S}pec(\mathcal{L}\mathcal{E}), \tau)$ be T_1-space. Since $\mathcal{M}ax(\mathcal{L}\mathcal{E})\mathcal{S}\mathcal{S}pec(\mathcal{L}\mathcal{E})$, it is enough to show that $\mathcal{S}pec(\mathcal{L}\mathcal{E})\mathcal{S}\mathcal{M}ax(\mathcal{L}\mathcal{E})$. For this, let $\mathcal{L}P \in \mathcal{S}pec(\mathcal{L}\mathcal{E})$. Then by Proposition 3.1, there is

$\mathcal{LM} \in \mathcal{M}ax(\mathcal{LE})$ such that $\mathcal{LPSLM}$. If $\mathcal{LP} = \mathcal{LM}$, then $\mathcal{LP} \in \mathcal{M}ax(\mathcal{LE})$. Now, let $\mathcal{LP} \neq \mathcal{LM} \in \mathcal{S}pec(\mathcal{LE})$. Since $(\mathcal{S}pec(\mathcal{LE}), \tau)$ is a T_1-space, then there exists $U \in \tau$ such that $\mathcal{LM} \in U$ and $\mathcal{LP} \notin U$. Also, since $\mathcal{LPSLM}$, we get $\mathcal{LP} \in U$, which is a contradiction and so $\mathcal{LP} = \mathcal{LM}$. Therefore, $\mathcal{S}pec(\mathcal{LE}) = \mathcal{M}ax(\mathcal{LE})$. Conversely, let $\mathcal{S}pec(\mathcal{LE}) = \mathcal{M}ax(\mathcal{LE})$. By Theorem 4.7, $(\mathcal{S}pec(\mathcal{LE}), \tau)$ is a T_1-space. □

Example 4.6. *Consider $\mathcal{LE}$ is the equality algebra as in Example 4.5. Then*

$$\mathcal{S}pec(\mathcal{LE}) = \{\underbrace{\{,1\}}_{\mathcal{LP}},\ \underbrace{\{,1\}}_{\mathcal{LQ}}\} = \mathcal{M}ax(\mathcal{LE}),$$

and $\tau = \{\emptyset,\ \{\mathcal{LP}\},\ \{\mathcal{LQ}\},\ \mathcal{S}pec(\mathcal{LE})\}$. Clearly, $(\mathcal{S}pec(\mathcal{LE}), \tau)$ is a T_1-space.

Theorem 4.9. *If $\mathcal{LE}$ is prelinear, then $(\mathcal{M}ax(\mathcal{LE}), \tau)$ is a Hausdorff space.*

Proof. Suppose $\mathcal{LM}, \mathcal{LN} \in \mathcal{M}ax(\mathcal{LE})$ and $\mathcal{LM} \neq \mathcal{LN}$. Since maximal filters are not included in any other proper filter of $\mathcal{LE}$, then we get $\mathcal{LM} \nsubseteq \mathcal{LN}$ and $\mathcal{LN} \nsubseteq \mathcal{LM}$. Thus there exist $\in \mathcal{LM} \setminus \mathcal{LN}$ and $\in \mathcal{LN} \setminus \mathcal{LM}$. Suppose $a =$ and $b =$. If $a \in \mathcal{LM}$, then since $\in \mathcal{LM}$ and $\mathcal{LM} \in \mathcal{F}(\mathcal{LE})$, we get $\in \mathcal{LM}$, which is a contradiction. Thus $a \notin \mathcal{LM}$ and so $\mathcal{LM} \in U(a)$. Similarly, $\mathcal{LN} \in U(b)$. Also, by Proposition 4.4(vi), (ii) and prelinearity of $\mathcal{LE}$, we have $U(a) \cap U(b) = U(a \vee b) = U(() \vee ()) = U(1) = \emptyset$. Therefore, $\mathcal{M}ax(\mathcal{LE})$ is a Hausdorff space. □

5 Conclusions and future works

In this paper, the notion of ∩-irreducible filter in equality algebras is introduced, and some properties and relations between maximal, prime, ∨-irreducible and ∩-irreducible filters of an equality algebra are investigated. For more generality, the set of all ∨-irreducible filters of an equality algebra is considered as the spectrum of it. Finally, a topology on spectrum (called Zariski topology) of an equality algebra is constructed and showed that the spectrum of an equality algebra with Zariski topology is a compact T_0-space. Moreover, maximal spectrum of equality algebra as a subspace of spectrum is compact T_1-space. Moreover, the conditions that (maximal) spectrum will be a Hausdorff space are studied. For future work, we want to continue study of topology on an equality algebra and construct topological equality algebras and more types of topology on equality algebras will be considered.

References

[1] R. A.Borzooei, F. Zebardast, M. Aaly Kologani, Some type filters in equality algebras, *Categories and General Algebraic Structures with Applications* 7 (2017) 33-55.

[2] R. A. Borzooei, M. Zarean, O. Zahiri, Involutive equality algebras, *Soft Computing* 22 (2018) 7505-7517.

[3] S. Burris, H. P. Sankappanavar, A course in universal algebra, Springer Verlag, *New York* (1981).

[4] J. L. Castiglioni, M. Menni, W. J. ZuluagaBotero, A representation theorem for integral rings and its applications to residuated lattices, *Journal of Pure and Applied Algebra* 220 (2016) 3533-3566.

[5] L. C. Ciungu, Internal states on equality algebras, *Soft Computing* 19 (2015) 939-953.

[6] A. Dvureenskij, O. Zahiri, Pseudo equality algebras: Revision, *Soft Computing* 20 (2016) 2019-2101.

[7] F. Forouzesh, F. Sajadian, M. Bedrood, Inverse topology in MV-algebras, *Mathematica Bohemica* 144 (2019) 273-285.

[8] B. Ganji Saffar, Fuzzy n-fold obstinate and maximal (pre)filters of EQ-algebras, *Journal of Algebraic Hyperstructures and Logical Algebras* 2 (2021) 83-98.

[9] G. Georgescu, D. Cheptea, C. Muresan, Algebraic and topological results on lifting properties in residuated lattices, *Fuzzy Sets and Systems* 271 (2015) 102-132.

[10] S. Ghorbani, Monadic pseudo-equality algebras, *Soft Computing* 23 (2019) 12937-12950.

[11] S. Jenei, Equality algebras, *Studia Logica* 100 (2012) 1201-1209.

[12] S. Jenei, L. Kóródi, On the variety of equality algebras, *Fuzzy Logic and Technology* (2011) 153-155.

[13] CH. Liu, L. Xu, Prime LI-ideal spaces of MTL-algebras, *International Journal of Algebra* 14 (2020) 213-222.

[14] C. Luan, Y. Yang, Filter topologies on MV-algebras, *Soft Computing* 21 (2017) 2531-2535.

[15] J. R. Munkres, Topology, Dorling Kindersley, *India*, (2000).

[16] J. B. Nganou, S. F. T. Tebu, Topological FL_{ew}-algebras, *Journal of Applied Logic* 13 (2015) 259-269.

[17] S. Niazian, M. Aaly Kologani, R. A. Borzooei, Co-annihilators on equality algebras, *Miskolc Mathematical Notes*, under review.

[18] V. Novák, B. De Baets, EQ-algebras, *Fuzzy Sets and Systems* 160 (2009) 2956-2978.

[19] H. Ono, Substructural logics and residualted lattices: An introduction, *Trends in Logic* 20 (2003) 177-212.

[20] Paad A., Ideals in bounded equality algebras, *Filomat* 33 (2019) 2113-2123.

[21] B. Van Gasse, G. Deschrijver, C. Cornelis , E. E. Kerre, Filters of residuated lattices and triangle algebras, *Information Sciences* 180 (2010) 3006-3020.

[22] J. T. Wanga, P. F. Hec, A. Borumand Saeid, Stabilizers in MTL-algebras, *Journal of Intelligent and Fuzzy Systems* 35 (2018) 717-727.

[23] M. Ward, R. P. Dilworth, Residuated lattices, *Transactions of the AMS* 45 (1939) 335-354.

[24] J. Yang, X. L. Xin, P. F. He, Uniform topology on EQ-algebras, *Open Mathematics* 15 (2017) 354-364.

[25] J. Yang, X. L. Xin, P. F. He, On topological EQ-algebras, *Iranian Journal of Fuzzy Systems* 15 (2018) 145-158.

[26] F. Zebardast, R. A. Borzooei, M. Aaly Kologani, Results on equality algebras, *Information Sciences* 381 (2017) 270-282.

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