Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 15, No. 1, 2023 Article ID IJIM-1429, 13 pages
DOI: http://dx.doi.org/10.30495/ijim.2023.21178
Research Article

A Hybrid Approach for Systems of Integral Equations

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Received Date: 2021-04-21 Revised Date: 2021-06-16 Accepted Date: 2021-09-05

Abstract

In this paper, we present a computational method for solving systems of Volterra and Fredholm integral equations which is a hybrid approach, based on the block-pulse functions and third kind of the Chebyshev polynomials which we will refer to as (HBV), for short. By using the HBV method and operational matrices of integration, such systems can be reduced into a linear system of algebraic equations. The existence and uniqueness of the solutions are addressed. Some examples are provided to clarify the efficiency and accuracy of the method.

Keywords : Systems of Fredholm and Volterra integral equations; Hybrid Method; Existence and uniqueness; Operational Matrices.

1 Introduction

We consider the following non-linear systems of Volterra and Fredholm integral equations of the second kind:

$$Y(\xi) = F(\xi) + \int_0^{\xi} K(\xi, \eta, Y(\eta))d\eta, \quad 0 \le \xi \le T \tag{1.1}$$

and

$$Y(\xi) = F(\xi) + \int_0^1 K(\xi, \eta, Y(\eta))d\eta, \tag{1.2}$$

where

$$Y(\xi) = [y_1(\xi), y_2(\xi), ..., y_n(\xi)]^T,$$

$$F(\xi) = [f_1(\xi), f_2(\xi), ..., f_n(\xi)]^T,$$

$$K(\xi, \eta, Y(\eta)) = \begin{bmatrix} k_1(\xi,\eta,y_1(\eta),y_2(\eta),...,y_n(\eta)) \\ k_2(\xi,\eta,y_1(\eta),y_2(\eta),...,y_n(\eta)) \\ \vdots \\ k_n(\xi,\eta,y_1(\eta),y_2(\eta),...,y_n(\eta)) \end{bmatrix},$$

where $Y(\xi)$ is unknown, but $F(\xi)$ is assumed to be known and $K(\xi, \eta, Y(\eta))$ is a linear or non-linear operator of its arguments. Systems of linear integral equations and their solutions have great importance in science and engineering since they are powerful tools to model the different natural processes, also they appear as reformulations of other mathematical problems such as partial differential equations and ordinary differential equations. This is a motivation for solving this class of functional equations, [6], [7], [15], [16], [18]. Most physical problems, such as biological applications in population dynamics and genetics where impulses arise naturally or are caused by control, can be modeled by an inte-

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gral equation or a system of these equations [3], [4], [5], [17], [20]. Since it is difficult to obtain an analytic solutions of these systems, some numerical methods have been used to approximate the solution of integral equations, such as a combination of spectral methods and orthonormal or piecewise bases and wavelets. Among piecewise bases, the block-pulse functions are widely used to numerically solve integral equations where the domain of the solution of the problem must be divided into subintervals in order to reach a resiable accuracy, which increasing the computational costs leads to decreasing the accuracy of the method. For increasing the accuracy of the block-pulse method, researchers have presented different hybrid methods which obtain of combinating the block-pulse functions and orthogonal polynomials such as the Legendre [14], Bernstein [10], Bernoulli [13], Chebyshev polynomials [19]. In this paper, we use a hybrid method consisting of the block-pulse functions and the third-kind Chebyshev polynomials for the computation of the approximate solution of a class of the system of integral equations which this approach has not been used to solve this category of equations. The main idea of using an orthogonal basis is that the problem under study reduces into a system of linear or nonlinear algebraic equations by considering truncated series of orthogonal basis functions with unknown coefficients for the solution of problem and using the operational matrices. The advantages of this method are: decreasing the computational costs and errors.

The organization of this paper is: In Section 2, we describe the construction of hybrid functions. In Section 3, we present the operational matrices of the integration and product. In Section 4, we solve systems of integral equations by using the utilized hybrid method. In Section 5, the existence and uniqueness of the solution of systems of Volterra integral equations are addressed. In Section 6, we report numerical results by providing some examples to illustrate the effectiveness and reability of the method, as well. Finally, the discussion and conclusion will present in Section 7.

2 Construction of hybrid methods

In this section, we review the hybrid of the block pulse functions and third kind of the Chebyshev polynomials. Moreover, we explain the expansions of functions.

2.1 Hybrid of block pulse functions and third kind of the Chebyshev polynomials

The HBV functions on the closed interval $[0, T]$ are defined as follows,

$$H_{i,j}(\xi) = \begin{cases} \sqrt{\frac{2T}{N}}v_i(\frac{2N\xi}{T} - 2i + 1), & \frac{(i-1)T}{N} \leq \xi < \frac{iT}{N}, \\ 0, & otherwise, \end{cases} \tag{2.3}$$

with the weight function, $w_i(\xi) = w(2N\xi - 2i + 1)$, $i = 1, 2, ..., N$, $j = 0, 1, , ..., M - 1$, where N and M are the kinds of the block-pulse functions and third kind of the Chebyshev polynomials, respectively. $H_{i,j}(\xi)$ is a combination of the orthogonal third kind of the Chebyshev polynomials and the block-pulse functions, and generates a complete orthogonal system on $L^2[0, 1)$.

2.2 Expansions of functions

A function $y(\xi) \in L^2[0, 1)$ can be expanded in term of HBV functions as the following,

$$y(\xi) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} c_{i,j}H_{i,j}(\xi), \tag{2.4}$$

such that

$$\begin{aligned} c_{i,j} &= \frac{(y(\xi), H_{i,j}(\xi))}{(H_{i,j}(\xi), H_{i,j}(\xi))} \\ &= \frac{N^2}{\pi}\int_0^1 w_i(\xi)H_{i,j}(\xi)y(\xi)d\xi, \end{aligned} \tag{2.5}$$

where $(.,.)$ denotes an inner product on $L^2 \in [0, 1]$ having $w_i(\xi)$ as the weight function. In practice, infinite series (2.4) will be replaced into the following form,

$$y(\xi) \cong \sum_{i=1}^{N} \sum_{j=0}^{M-1} c_{i,j} H_{i,j}(\xi) = C^T H(\xi), \tag{2.6}$$

the vectors C and $H(\xi)$ are as the following,

$$\begin{aligned} C &= [c_{1,0}, ..., c_{1,M-1}, ...c_{N,M-1}]^T, \\ H(\xi) &= [H_{1,0}, ..., H_{1,M-1}, ..., H_{N,M-1}]^T. \end{aligned} \tag{2.7}$$

The kernel $k(\xi, \eta) \in L^2([0,1] \times [0,1])$ can be approximated as follows,

$$k(\xi, \eta) \approx H^T(\eta) K H(\xi), \tag{2.8}$$

where K is a $NM \times NM$ known matrix with the following entries,

$$K_{i,j} = \frac{(H_i(\xi), (k(\xi,\eta), H_j(\xi))}{(H_i(\xi), H_j(\xi))(H_i(\eta) H_j(\eta))}, \quad i, j = 1, 2, ..., NM. \tag{2.9}$$

3 Operational matrices of integration and product

The HBV method describes how a system of integral equations is converted into a system of linear or non-linear algebraic equations based on two matrices of integration and product and the basis vector.

3.1 Operational matrix of integration

In this subsection, we will compute the integral of the H vector, which is important for solving Volterra integral equations. To better describe, computations of $H_6(\xi)$ are performed as the following,

$$\begin{cases} H_{10}(\xi) = 1, \\ H_{11}(\xi) = 8\xi - 3, & 0 \le \xi < \frac{1}{2}, \\ H_{12}(\xi) = 64\xi^2 - 40\xi + 5, \end{cases}$$

$$\begin{cases} H_{20}(\xi) = 1, \\ H_{21}(\xi) = 8\xi - 7, & \frac{1}{2} \le \xi < 1, \\ H_{22}(\xi) = 64\xi^2 - 104\xi + 41, \end{cases} \tag{3.10}$$

where,
$H_6(\xi) = [H_{10}, H_{11}, H_{12}, H_{20}, H_{21}, H_{22}]$.

Also, by integrating (3.10) and presenting in matrix form, we obtain the following approximations, that is applied for the third kind of the Chebyshev wavelets [21]. For the present method,

$$\begin{aligned} \int_0^\xi H_{10}(t)dt &= \begin{cases} \xi, & 0 \le \xi < \frac{1}{2}, \\ \frac{1}{2}, & \frac{1}{2} \le \xi < 1 \end{cases} \\ &= \frac{3}{8} H_{10} + \frac{1}{8} H_{11} + \frac{1}{2} H_{20} \\ &= \begin{bmatrix} \frac{3}{8} & \frac{1}{8} & 0 & \frac{1}{2} & 0 & 0 \end{bmatrix} H_6(\xi), \end{aligned}$$

$$\begin{aligned} \int_0^\xi H_{11}(t)dt &= \begin{cases} 4\xi^2 - 3\xi, & 0 \le \xi < \frac{1}{2}, \\ \frac{-1}{2}, & \frac{1}{2} \le \xi < 1, \end{cases} \\ &= \frac{-1}{2} H_{10} + \frac{-1}{16} H_{11} + \frac{1}{16} H_{12} \\ &+ \frac{-1}{2} H_{20} \\ &= [\, \tfrac{-1}{2} \; \tfrac{-1}{16} \; \tfrac{1}{16} \; \tfrac{-1}{2} \; 0 \; 0 \,] H_6(\xi), \end{aligned}$$

also, we have

$$\int_0^\xi H_{12}(t)dt = [\, \tfrac{5}{24} \; \tfrac{-1}{16} \; -\tfrac{1}{48} \, \tfrac{1}{6} \; 0 \; 0 \,] H_6(\xi) + \tfrac{1}{24} H_{13}(\xi),$$

$$\int_0^\xi H_{20}(t)dt = [\, 0 \; 0 \;\; 0 \; \tfrac{3}{8} \; \tfrac{1}{8} \; 0 \,] H_6(\xi),$$

$$\int_0^\xi H_{21}(t)dt = [\, 0 \; 0 \;\; 0 \; \tfrac{-1}{2} \; -\tfrac{1}{16} \; \tfrac{1}{16} \,] H_6(\xi),$$

$$\int_0^\xi H_{22}(t)dt = [\, 0 \; 0 \; 0 \; \tfrac{5}{24} \; -\tfrac{1}{16} \; -\tfrac{1}{48} \,] H_6(\xi) + \tfrac{1}{24} H_{23}(\xi).$$

The above approximations can be written in the matrix form as follows,

$$\int_0^\xi H_6(t)dt = P_{6\times 6}(\xi) + H_6^*(\xi), \tag{3.11}$$

where

$$P_{6\times 6} = \frac{1}{4} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & 0 & 2 & 0 & 0 \\ -2 & -\frac{1}{4} & \frac{1}{4} & -2 & 0 & 0 \\ \frac{5}{6} & -\frac{1}{4} & -\frac{1}{12} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -2 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{5}{6} & -\frac{1}{4} & -\frac{1}{12} \end{bmatrix},$$

and $H^*(\xi) = \frac{1}{24}(0 \;\; 0 \;\; H_{13}(\xi) \;\; 0 \;\; 0 \;\; H_{23}(\xi))^T$. In fact, the matrix $P_{6\times 6}$ can be written as,

$$P_{6\times 6} = \frac{1}{4} \begin{bmatrix} L_{3\times 3} & J_{3\times 3} \\ O_{3\times 3} & L_{3\times 3} \end{bmatrix},$$

where,

$$L_{3\times 3}=\frac{1}{4}\begin{bmatrix}\frac{3}{2} & \frac{1}{2} & 0\\ -2 & -\frac{1}{4} & \frac{1}{4}\\ \frac{5}{6} & -\frac{1}{4} & -\frac{1}{12}\end{bmatrix},$$

$$J_{3\times 3}=\frac{1}{4}\begin{bmatrix}2 & 0 & 0\\ -2 & 0 & 0\\ \frac{2}{3} & 0 & 0\end{bmatrix}. \tag{3.12}$$

For $M \geq 4$ one has

$$P=\frac{1}{N^2}\begin{bmatrix}L & J & \cdots & J\\ 0 & L & \cdots & J\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & L\end{bmatrix}, \tag{3.13}$$

where J and L are two $M \times M$ matrices as the following.
If M is even:

$$J=\begin{bmatrix}\tau_1 & 0 & \cdots & 0\\ -\tau_1 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \tau_{\frac{M}{2}} & 0 & \cdots & 0\\ -\tau_{\frac{M}{2}} & 0 & \cdots & 0\end{bmatrix}, \tag{3.14}$$

where $\tau_i=\frac{2}{2i-1},\ i=1,2,...,\frac{M}{2}$.
If M is odd:

$$J=\begin{bmatrix}\tau_1 & 0 & \cdots & 0\\ -\tau_1 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -\tau_{\frac{M+1}{2}-1} & 0 & \cdots & 0\\ -\tau_{\frac{M+1}{2}-1} & 0 & \cdots & 0\\ \tau_{\frac{M+1}{2}} & 0 & \cdots & 0\end{bmatrix}, \tag{3.15}$$

where $\tau_i=\frac{2}{2i-1},\ i=1,2,...,\frac{M+1}{2}$, and

$$H^*(\xi)=\frac{1}{N^2}(\lambda_1\ \ \lambda_2\ \ \lambda_3\ \ ...\ \ \lambda_N)^T \tag{3.16}$$

where

$$\lambda_i=\frac{1}{2M}(0\ \ 0\ \ 0\ \ ...\ \ 0\ \ H_{iM}),\quad i=1,2,...,N. \tag{3.17}$$

$$L=\begin{bmatrix}\frac{3}{2} & \frac{1}{2} & 0 & \cdots & 0 & 0\\ -2 & -\frac{1}{4} & \frac{1}{4} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \frac{(-1)^{m-2}2M-3}{(M-1)(M-2)} & 0 & 0 & \cdots & -\frac{1}{2(M-1)(M-2)} & \frac{1}{2(M-1)}\\ \frac{(-1)^{m-1}2M-1}{M(M-1)} & 0 & 0 & \cdots & -\frac{1}{2(M-1)} & -\frac{1}{2M(M-1)}\end{bmatrix}. \tag{3.18}$$

In general, the integration of the vector $H(\xi)$, defined in (2.7), can be presented as follows,

$$\int_0^{\xi} H(t)dt = PH(\xi)+H^*(\xi). \tag{3.19}$$

3.2 The product operational matrix of HBV functions

In this section, we will derive the integration of the inner product of two H vectorss in (2.7), which is important for solving Fredholm integral equations. Let

$$E=\int_0^1 H(\xi)H^T(\xi)d\xi, \tag{3.20}$$

where, E is a $NM \times NM$ nonsingular symmetric matrix with $N=2, M=3$.

$$E=\frac{1}{4}\begin{bmatrix}2 & -2 & \frac{2}{3} & 0 & 0 & 0\\ -2 & \frac{14}{3} & \frac{-10}{3} & 0 & 0 & 0\\ \frac{2}{3} & -\frac{10}{3} & \frac{86}{16} & 0 & 0 & 0\\ 0 & 0 & 0 & -\frac{494}{105} & \frac{2182}{315} & -\frac{1622}{315}\\ 0 & 0 & 0 & \frac{922}{315} & -\frac{1622}{315} & \frac{25402}{3465}\\ 0 & 0 & 0 & -\frac{782}{315} & \frac{11542}{3465} & -\frac{19102}{3465}\end{bmatrix}. \tag{3.21}$$

Also, we obtain the product $H^T(\xi)$ and $H(\xi)$, which is important for solving Volterra integral equations. Let

$$\int_0^{\xi} H^T(t)H(t)Cdt = C^*H(\xi)+H^*(\xi), \tag{3.22}$$

where C^* is the product operational matrix, and $H^*(\xi)$ is introduced in (3.16). For $N=2$ and $M=3$, the matrix C^* can be written as,

$$C^*_{6\times 10}=\begin{bmatrix}\beta_1 & \vartheta_1 & 0\\ 0 & \beta_2 & \vartheta_2\end{bmatrix},$$

where

$$\beta_i=\begin{bmatrix}\gamma_{i0} & \gamma_{i1} & \gamma_{i2}\\ \gamma_{i1} & \gamma_{i0}-\gamma_{i1}+\gamma_{i2} & \gamma_{i1}-\gamma_{i2}\\ \gamma_{i2} & \gamma_{i1}-\gamma_{i2} & \gamma_{i0}-\gamma_{i1}+\gamma_{i2}\end{bmatrix},$$

$$\vartheta_i=\begin{bmatrix}0 & 0\\ \gamma_{i2} & 0\\ \gamma_{i1}-\gamma_{i2} & \gamma_{i2}\end{bmatrix},\qquad i=1,2.$$

4 The approximate solution of a system of integral equations

Consider systems of integral equations (1.1) and (1.2), let us approximate the functions by the proposed approach as follows,

$$\begin{aligned} &f_i(\xi) \simeq F_i^T H(\xi), \qquad y_i(\xi) \simeq C_i^T H(\xi), \\ &k_{i,j}(\xi,\eta) \simeq H^T(\xi) K_{i,j} H(\eta), \\ &i = 1,2,...,N, \;\; j = 1,2,...,M-1. \end{aligned} \tag{4.23}$$

Substituting these approximations into systems (1.1) and (1.2) leads to:

$$\begin{aligned} C_i^T H(\xi) &\simeq F_i^T H(\xi) \\ &+ \int_0^{\xi} H^T(\xi) K_{i,j} H(\eta) H^T(\eta) C_i d\eta \\ &\simeq F_i^T H(\xi) + H^T(\xi) K_{i,j}[C_i^* H(\xi) \\ &+ H^*(\xi)], \;\; i = 1,2,...,N, \\ &j = 1,2,...,M-1, \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} C_i^T H(\xi) &\simeq F_i^T H(\xi) \\ &+ \int_0^{1} H^T(\xi) K_{i,j} H(\eta) H^T(\eta) C_i d\eta \\ &\simeq F_i^T H(\xi) + H^T(\xi) K_{i,j} E C_i, \\ &i = 1,2,..,N, \;\; j = 1,2,...,M-1, \end{aligned} \tag{4.25}$$

By solving (4.24) and (4.25) the coefficients of C_i, $i = 1,2,...,N$ will be obtained. Also the error function $e(y_i(\xi))$ can be constructed as follows,

$$e(y_i(\xi)) = \left| y_i(\xi) - \sum_{i=1}^{N} \sum_{j=0}^{M-1} c_{i,j}^T H_{i,j}(\xi) \right|. \tag{4.26}$$

If we set $\xi = \xi_j, \xi_j \in [0,1]$, the error values, at ξ_j , can be obtained.
Given that the hybrid method presents an approximation for a solution to systems (1.1) and (1.2), these equations must satisfy approximation (2.6).

Theorem 4.1. *Suppose that $y(\xi)$ is a function with square-integrable second derivative defined on $[0,1]$ that its second derivative is bounded, i.e. $|y''(\xi)| \leq A$, for some constant A, then series (2.6) converges uniformly to $y(\xi)$,*

$$\|y(\xi) - C^T H(\xi)\|^2 \leq \frac{\pi A^2}{8} \sum_{i=N+1}^{\infty} \sum_{j=M}^{\infty} \frac{1}{i^5 (j-1)^4}$$

where $C^T H(\xi)$ is the truncated H expansion of $y(\xi)$.

Proof. See [12].
An upper bound for the approximate function $y^*(\xi)$ is achieved in the following theorem:

Theorem 4.2. *Let us take $y \in H^r_{(1-t)^{r-\frac{1}{2}}(1+t)^{r+\frac{1}{2}}}(-1,1)$, and $I_M y$ be the approximation solution obtained from the suggested method, then*

$$\|I_M y - y\| \leq C M^{-r} \times \left(\int_{-1}^{1} (1-t)^{r-\frac{1}{2}} (1+t)^{r+\frac{1}{2}} (\frac{d^r y(t)}{dt^2})^2 dt \right)^{\frac{1}{2}},$$

where C is a positive constant and $H^r_{\chi}(\Omega)$ is a Chebyshev weighted Sobolev space.

Proof. See [9].
Now, we provide a suitable stability analysis which theoretically justifies stability of the proposed method for the numerical solution when $K(\xi,\eta,Y(\eta)) = k(\xi,\eta)Y(\eta)$ where $k(\xi,\eta)$ is a continuous and bounded function, i.e. $\| k(\xi,\eta) \| \leq K$. Suppose P_N be the space of all algebraic polynomials of degree up to N. $\mathcal{P}_N : L^2[0,1) \longrightarrow P_N$ which is an orthonormal projection such that for any $Y \in L^2[0,1)$,

$$(Y - \mathcal{P}_N Y, \phi) = 0, \quad \forall \phi \in P_N.$$

The estimation of the truncation error holds at the following inequalities

$$\begin{aligned} &\| Y - \mathcal{P}_N Y \|_{L^2[0,T)} \leq \\ &C N^{-k} \max_{0 \leq i < N} \| Y(\xi) \|_{H^k_{\omega}[\frac{(i-1)T}{N}, \frac{iT}{N})}, \\ &\| Y - \mathcal{P}_N Y \|_{\infty} \leq \\ &C \left(1 + O(\frac{M^{-k}}{\sqrt{N}}) \right) \| Y(\xi) \|_{L^2[0,T)}, \quad k \geq 0. \end{aligned} \tag{4.27}$$

Theorem 4.3. *(Stability) Let $Y^*(\xi)$ be the hybrid approximation to the exact solution of the system of integral equations (1.1). Assume that the function $F(\xi)$ is continuous. Also suppose that $\tilde{Y} \in P_N$ and $\tilde{F} \in C[0,1]$ are the error of Y and F respectively, and $1 - KT > 0$. Then, we have $\| \tilde{Y} \| \leq \mathcal{C} \| \tilde{F} \|$ where $\mathcal{C}$ is the stability constant.*

Proof. $Y^*(\xi)$ and $Y^*(\xi) + \tilde{Y}(\xi)$ satisfy the following equations.

$$Y^*(\xi) = \mathcal{P}_N F(\xi) + P_N \int_0^{\xi} k(\xi,\eta) Y^*(\eta) d\eta, \tag{4.28}$$

$$Y^*(\xi) + \tilde{Y}(\xi) = \mathcal{P}_N F(\xi) + \tilde{F}(\xi) + \mathcal{P}_N \int_0^{\xi} k(\xi,\eta)(Y^*(\eta) + \tilde{Y}(\eta)) d\eta, \tag{4.29}$$

where $\xi \in [0,T]$. Subtracting (4.28) form (4.29) we get,

$$\tilde{Y}(\xi) = \mathcal{P}_N \tilde{F}(\xi) + \mathcal{P}_N \int_0^{\xi} k(\xi,\eta) \tilde{Y}(\eta) d\eta, \tag{4.30}$$

and then

$$\| \tilde{Y} \| \leq \| \mathcal{P}_N \tilde{F} \| + \| \mathcal{P}_N \int_0^{\xi} k(\xi,\eta) \tilde{Y}(\eta) d\eta \| \tag{4.31}$$

Since $||\mathcal{P}_N|| = 1$ one has

$$\begin{aligned} \| \tilde{Y} \| &\leq \| \tilde{F} \| + \| \int_0^{\xi} k(\xi,\eta) \tilde{Y}(\eta) d\eta \| \\ &\leq \| \tilde{F} \| + KT \| \tilde{Y} \| . \end{aligned}$$

So,

$$(1 - KT) \| \tilde{Y} \| \leq \| \tilde{F} \|$$

or

$$\| \tilde{Y} \| \leq \frac{1}{1 - KT} \| \tilde{F} \| .$$

By setting $\mathcal{C} = 1/(1 - KT)$, the resired result is achieved. ■

5 The existence and uniqueness of the solution

To prove the existence and uniqueness of the solution of system (1.1), we extend the simple iterations:

$$Y_n(\xi) = F(\xi) + \int_0^{\xi} K(\xi,\eta) Y_{n-1}(\eta) d\eta, \tag{5.32}$$

with $Y_0(\xi) = F(\xi)$.
For simplicity's sake it is convenient to introduce

$$\Psi_n(\xi) = Y_n(\xi) - Y_{n-1}(\xi), \quad n = 1, 2, \dots , \tag{5.33}$$

with $\Psi_0(\xi) = F(\xi)$.
Consider, the equation resulted from equation (5.32), if n is replaced by $n-1$, subtraction of this equation from (5.32) results in;

$$\Psi_n(\xi) = \int_0^{\xi} K(\xi,\eta) \Psi_{n-1}(\eta) d\eta, \quad n = 1, 2, \dots . \tag{5.34}$$

Also from (5.33)

$$Y_n(\xi) = \sum_{i=0}^{n} \Psi_i(\xi). \tag{5.35}$$

In the following theorem we use this recurrrent formula to prove the existence and uniqueness of the solution under the only restriction that $K(\xi,\eta)$ and $F(\xi)$ are continuous.

Theorem 5.1. *If $F(\xi)$ and $K(\xi,\eta)$ are continuous in $0 \leq \eta < \xi \leq T$ then the system of Volterra integral equations of the second kind (1.1) has a unique continuous solution for $0 \leq \xi \leq T$.*

Proof. There exist constants f and k such that:

$$\|F(\xi)\| \leq f, \quad 0 \leq \xi \leq T,$$

$$\|K(\xi,\eta)\| \leq k, \quad 0 \leq \eta < \xi \leq T,$$

we first prove by induction that:

$$\|\Psi_n(\xi)| \leq \frac{(k\xi)^n f}{n!}, \quad 0 \leq \xi \leq T, \quad n = 1, 2, \dots . \tag{5.36}$$

If we assume that (5.36) is true for $n-1$, then we have from (5.34)

$$\begin{aligned} \|\Psi_n(\xi)\| &\leq \frac{f k^n}{(n-1)!} \int_0^{\xi} \eta^{n-1} d\eta \\ &= \frac{f k^n \xi^n}{n!} = \frac{(k\xi)^n f}{n!}. \end{aligned}$$

Since (5.36) is obviously true for $n = 0$, it holds for all n. These bounds make it obvious that the sequence $Y_n(\xi)$ in (5.35) converges uniformly and we can write

$$Y(\xi) = \sum_{i=0}^{\infty} \Psi_i(\xi). \tag{5.37}$$

We now show that $Y(\xi)$ satisfies equation (5.32). By uniform convergence of (5.37) we can interchange the kind of integration and summation in the following expression, to obtain:

$$\int_0^\xi K(\xi,\eta)\sum_{i=0}^\infty \Psi_i(\eta)d\eta$$
$$=\sum_{i=0}^\infty \int_0^\xi K(\xi,\eta)\Psi_i(\eta)d\eta$$
$$=\sum_{i=0}^\infty \Psi_{i+1}(\xi)$$
$$=\sum_{i=0}^\infty \Psi_i(\xi)-F(\xi).$$

This proves that $Y(\xi)$ defined by (5.37) satisfies equation (5.32). $\Psi_i(\xi)$ is clearly continuous. Therefore $Y(\xi)$ is continuous, since it is the limit of a uniformly convergent sequence of continuous functions.
To show that $Y(\xi)$ is the only continuous solution, suppose that there exists another continuous solution $Y^*(\xi)$ of (5.32), then

$$Y(\xi)-Y^*(\xi)=\int_0^\xi K(\xi,\eta)(Y(\eta)-Y^*(\eta))d\eta. \tag{5.38}$$

Since $Y(\xi)$ and $Y^*(\xi)$ are both continuous there exists a constant β such that,

$$\|Y(\xi)-Y^*(\xi)\|\leq \beta,\quad 0\leq\xi\leq T.$$

Substituting this into (5.38) gives,

$$\|Y(\xi)-Y^*(\xi)\|\leq \frac{(k\xi)^n\beta}{n!},\qquad 0\leq\xi\leq T.$$

Obviously $\frac{(k\xi)^n\beta}{n!}\longrightarrow 0$ as $n\to\infty$ for any ξ, which implies that

$$Y(\xi)=Y^*(\xi),\quad 0\leq\xi\leq T.$$

■

6 Numerical Examples

For showing the efficiency of the suggested numerical method, we consider the following examples. In the tables, the absolute error of y is denoted by AE_y. Since everys arbitrary interval $[a,b]$ can be converted to $[0,1]$ by change of variable $t=(\xi-a)/(b-a), \xi\in[a,b]$, the illustrate examples are considered over the interval $[0,1]$. The results will be compared with those of some existing methods.

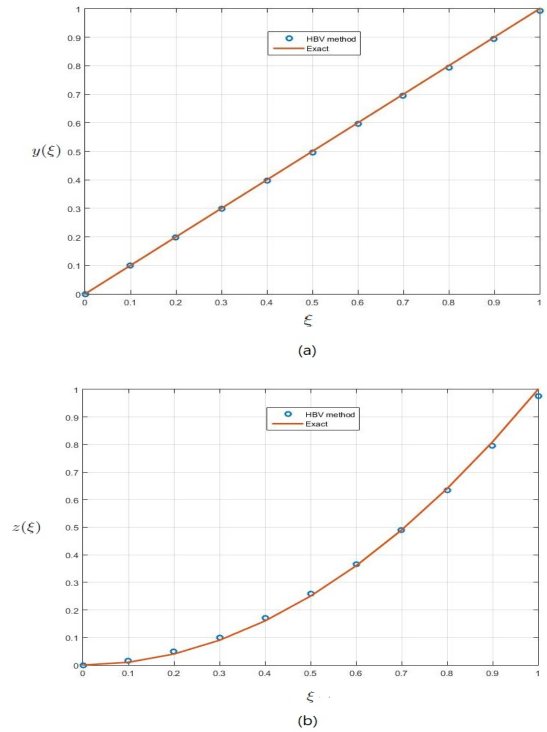

Figure 1: Exact and approximate solutions for (a) $y(\xi)$, (b) $z(\xi)$ for $N=1$ and $M=3$ of Example 6.1

Example 6.1. *In this example, we solve the following non-linear system of Fredholm integral equations [14]:*

$$\begin{cases} y(\xi)=\frac{23}{35}\xi+\displaystyle\int_0^1 \xi\eta^2(y^2(\eta)+z^2(\eta))d\eta, \\ z(\xi)=\frac{11}{12}\xi^2+\displaystyle\int_0^1 \xi^2\eta(y^2(\eta)-z^2(\eta))d\eta, \end{cases}$$

where $\xi\in[0,1]$. The exact solutions are $y(\xi)=\xi$ and $z(\xi)=\xi^2$. Let us take the following approximations,

$$\begin{cases} \frac{23}{35}\xi\simeq, F_1^TH(\xi), \\ \frac{11}{12}\xi^2\simeq F_2^TH(\xi), \\ \xi\eta^2\simeq H^T(\xi)K_1(\xi,\eta)H(\eta), \\ \xi^2\eta\simeq H^T(\xi)K_2(\xi,\eta)H(\eta), \\ y(\xi)\simeq C_1^TH(\xi), \\ z(\xi)=C_2^TH(\xi). \end{cases}$$

Applying the presented method, the following

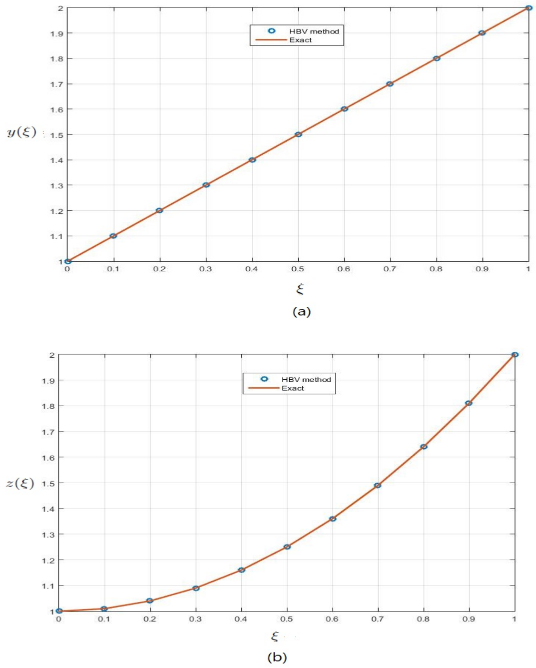

Figure 2: Exact and approximate solutions for (a) $y(\xi)$, (b) $z(\xi)$ for $N = 1$ and $M = 3$ of Example 6.2

system of algebraic equations will be obtained,

$$
\begin{cases}
C_1^T H(\xi) \simeq F_1^T H(\xi) \\
+\int_0^1 H^T(\xi) K_1(\xi,\eta) H(\eta) \\
\times (H^T(\eta) C_1 C_1^T H(\eta) \\
+H^T(\eta) C_2 C_2^T H(\eta)) d\eta, \\
\quad \simeq F_1^T H(\xi) \\
+H^T(\xi) K_1(\xi,\eta) E(C_1 C_1^T + C_2 C_2^T) \\
\times \int_0^1 H(\eta)) d\eta,
\end{cases}
$$

$$
\begin{cases}
C_2^T H(\xi) \simeq F_2^T H(\xi) \\
+\int_0^1 H^T(\xi) K_2(\xi,\eta) H(\eta) (H^T(\eta) C_1 C_1^T \\
\times H(\eta) - H^T(\eta) C_2 C_2^T H(\eta)) d\eta, \\
\quad \simeq F_2^T H(\xi) \\
+H^T(\xi) K_2(\xi,\eta) E(C_1 C_1^T - C_2 C_2^T) \\
\times \int_0^1 H(\eta)) d\eta.
\end{cases}
$$

Some numerical results are presented in Figure 1. Table 2 compares the absolute error of the present method with HBPF-GQR and HLBPF in [8] and [14], respectively. These figure and Table show a good agreement of numerical results of the suggested method in comparison with two other methods. To study the stability of the system, we add the perturbation value as $\varepsilon = 10^{-4}$ to the source function $F(\xi)$, then we solve the new system and compute the maximum absolute errors. As seen from Table 1, the error values remain constant.

Example 6.2. *In this example, we study the following non-linear system of Fredholm integral equations [14]:*

$$
\begin{cases}
y(\xi) = 1 - \frac{17}{20}\xi - \frac{7}{6}\xi^2 + \int_0^1 \xi\eta^2 y^3(\eta) d\eta \\
\quad +\int_0^1 \xi^2 \eta z^2(\eta) d\eta, \\
z(\xi) = 1 - \frac{17}{12}\xi + \xi^2 - \frac{31}{10}\xi^3 \\
\quad +\int_0^1 \xi\eta y^2(\eta) d\eta \\
\quad +\int_0^1 \xi^3 \eta z^4(\eta) d\eta,
\end{cases}
$$

where $\xi \in [0,1]$. The exact solutions are $y(\xi) = \xi + 1$ and $z(\xi) = \xi^2 + 1$. Let us take,

$$
\begin{aligned}
&1 - \tfrac{17}{20}\xi - \tfrac{7}{6}\xi^2 \simeq F_1^T H(\xi), \\
&1 - \tfrac{17}{12}\xi + \xi^2 - \tfrac{31}{10}\xi^3 \simeq F_2^T H(\xi), \\
&\xi\eta^2 \simeq H^T(\xi) K_1 H(\eta), \\
&\xi^2\eta \simeq H^T(\xi) K_2 H(\eta), \\
&\xi\eta \simeq H^T(\xi) K_3 H(\eta), \\
&\xi^3\eta \simeq H^T(\xi) K_4 H(\eta), \\
&y(\xi) \simeq C_1^T H(\xi), \\
&z(\xi) \simeq C_2^T H(\xi),
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 \xi\eta^2 y^3(\eta) d\eta \simeq \int_0^1 H^T(\xi) K_1 H(\eta) H^T(\eta) \\
&\times C_1 C_1^T H(\eta) H B^T(\eta) C_1 d\eta \\
&\simeq H^T(\xi) K_1 E C_1 C_1^T E C_1,
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 \xi^2 \eta z^2(\eta) d\eta \simeq \int_0^1 H^T(\xi) K_2 H(\eta) H^T(\eta) \\
&\times C_2 C_2^T H(\eta) d\eta \simeq H^T(\xi) K_2 E C_2 C_2^T \\
&\times \int_0^1 H^T(\eta) d\eta,
\end{aligned}
$$

Table 1: Maximum absolute errors before and after considering perturbation $\varepsilon = 10^{-4}$ for $N = 1$and $M = 3$ of Example 6.1

AE_y		AE_z	
Before	After	Before	After
0.00704	0.00704	0.0247	0.0247

Table 2: Values of absolute errors for presented method, HBPF-GQR, and HLBPF of Example 6.1

	HBV	HBPF-GQR [8]	HLBPF [14]
ξ	$N = 3, M = 4$	$n = 3, M = 4$	$M = 3, N = 4$
	(AE_y, AE_z)	(AE_y, AE_z)	(AE_y, AE_z)
0.2	$(1.408e-7, 8.381e-9)$	$(1.230e-7, 7.801e-9)$	$(1.250e-5, 1.185e-8)$
0.4	$(2.816e-9, 1.009e-8)$	$(4.026e-8, 1.279e-8)$	$(2.500e-5, 4.741e-8)$
0.6	$(4.224e-8, 5.142e-8)$	$(7.105e-8, 7.330e-9)$	$(3.750e-5, 1.066e-7)$
0.8	$(5.633e-9, 6.476e-9)$	$(5.724e-8, 1.196e-8)$	$(5.000e-5, 1.896e-7)$

$$\int_0^1 \xi\eta y^2(\eta)d\eta \simeq \int_0^1 H^T(\xi)K_3H(\eta)H^T(\eta) \times C_1C_1^TH(\eta))d\eta \simeq H^T(\xi)K_3EC_1C_1^T \times \int_0^1 H^T(\eta)d\eta,$$

$$\int_0^1 \xi^3\eta z^4(\eta)d\eta \simeq \int_0^1 H^T(\xi)K_4H(\eta)H^T(\eta) \times C_1C_1^TH(\eta)d\eta \simeq H^T(\xi)K_4EC_2C_2^TEC_2C_2^T \times \int_0^1 H^T(\eta)d\eta.$$

Some numerical results are presented in Figure 2. In Table 4 the absolute errors of present method compare with HBPF-GQR and HLBPF [8] and [14], respectively. To study the stability of the system, we add the perturbation value as $\varepsilon = 10^{-6}$ to the source function $F(\xi)$, then we solve the new system and compute the maximum absolute errors. As seen from Table 3, after adding the perturbation, the error values remain small.

Example 6.3. *Consider the following non-linear system of Volterra integral equations of the second kind [2]:*

$$\begin{cases} y(\xi) = \cos(\xi) - \frac{1}{2}\sin^2(\xi) + \displaystyle\int_0^\xi y(\eta)z(\eta)d\eta, \\ z(\xi) = \sin(\xi) - \xi + \displaystyle\int_0^\xi (y^2(\eta) + z^2(\eta))d\eta, \end{cases}$$

where $\xi \in [0,1]$and the exact solutions are $y(\xi) = \cos(\xi)$, $z(\xi) = \sin(\xi)$. We apply our method for solving this problem. Some numerical results are presented in Figure 3. Also we compare the absolute errors computed by the present method, and BPFs [2] in Table 6, then we add the perturbation values as $\varepsilon = 10^{-4}$ to the source function $F(\xi)$, then we solve the new system and compute the maximum absolute errors which results are seen in Table 5.

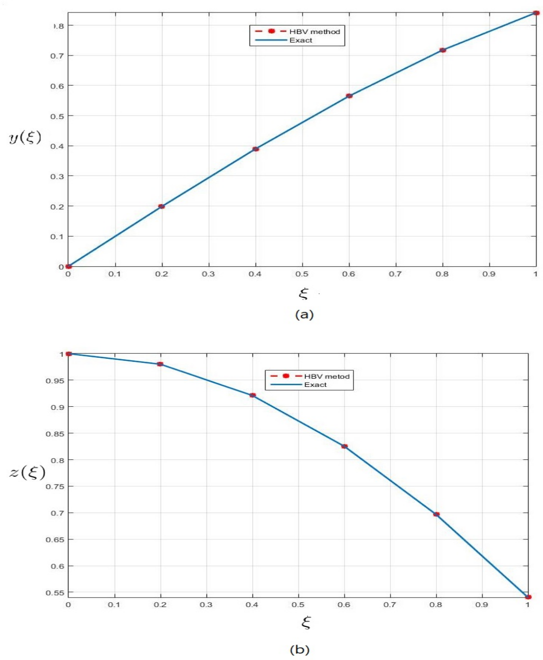

Figure 3: Exact and approximate solutions for (a)$y(\xi)$, (b) $z(\xi)$ for $N = 2$ and $M = 3$ of Example 3.

Table 3: Maximum absolute errors before and after considering perturbation $\varepsilon = 10^{-6}$, $N = 1$, and $M = 3$ of Example 6.2

AE_y		AE_z	
Before	After	Before	After
$8.30e-7$	$2.41e-3$	$1.32e-5$	$1.00e-1$

Table 4: Absolute errors of HBV method, HBPF-GQR, and HLBPF for $(y(\xi), z(\xi))$ of Example 6.2

	HBV	HBPF-GQR [8]	HLBPF [14]
ξ	$N=3, M=4$	$N=3, M=4$	$M=3, N=4$
0.2	$(0,0)$	$(3.603e-7, 2.630e-7)$	$(4.105e-5, 2.000e-5)$
0.4	$(0,0)$	$(2.935e-7, 7.165e-7)$	$(6.782e-5, 1.647e-5)$
0.6	$(0,0)$	$(4.573e-7, 1.020e-7)$	$(8.032e-5, 3.428e-5)$
0.8	$(0,0)$	$(1.584e-6, 4.208e-7)$	$(7.854e-5, 1.563e-4)$

Table 5: Maximum absolute errors before and after considering perturbation $\varepsilon = 10^{-6}$, $N = 2$, and $M = 3$ of Example 6.3

AE_y		AE_z	
Before	After	Before	After
$1.21e-10$	$2.41e-7$	0.00	$3.28e-3$

Table 6: Absolute errors for HBV method and BPFs for $(y(\xi), z(\xi))$ of Example 6.3

	HBV	HBPFs [2]
ξ	$N=2, M=4$	$h=0.1$
0.2	$(4.075e-7, 3.651e-7)$	$(8.022e-7, 4.561e-6)$
0.4	$(1.563e-7, 6.946e-7)$	$(3.243e-6, 9.614e-6)$
0.6	$(8.237e-7, 6.146e-7)$	$(7.206e-6, 1.588e-5)$
0.8	$(3.268e-7, 7.461e-8)$	$(1.256e-5, 2.450e-5)$

Example 6.4. *Consider the following Fredholm integral equations system [1]:*

$$\begin{cases} y(\xi) = 2e^{\xi} + \frac{e^{\xi+1}-1}{\xi+1} + \int_0^1 e^{\xi-\eta} y(\eta) d\eta \\ -\int_0^1 e^{(\xi+2)\eta} z(\eta) d\eta, \\ z(\xi) = e^{\xi} + e^{-\xi} + \frac{e^{\xi+1}-1}{\xi+1} \\ -\int_0^1 e^{\xi\eta} y(\eta) d\eta - \int_0^1 e^{\xi+\eta} z(\eta) d\eta, \end{cases}$$

where $\xi \in [0,1]$. The exact solutions are $y(\xi) = e^{\xi}$ and $z(\xi) = e^{-\xi}$. Table 8 illustrates the comparison between the exact solutions and numerical solutions given by the proposed method (HBV) for different values of N and M with BPFs method [11]. To study the stability of the system, we add the perturbation values $\varepsilon = 10^{-6}$ to the source function $F(\xi)$ for $N = 2$ and $M = 4$, then we solve the new system and compute the maximum absolute errors. As seen from Table 7, after inserting perturbations the error values remain small.

7 Conclusion

It has already been proved that hybrid approaches are very effective devices for solving systems of integral equations of the second kind. We used such a method, that is a combination of block-pulse functions and third kind of the chebyshev polynomials for solving linear and nonlinear systems of Volterra and Fredholm integral equations of the second kind. The efficiency

Table 7: Maximum absolute errors before and after considering perturbation $\varepsilon = 10^{-6}$, $N = 1$, and $M = 4$ of Example 6.4

AE_y		AE_z	
Before	After	Before	After
$1.62e-7$	$1.31e-3$	$7.8e-8$	$3.12e-4$

Table 8: Absolute errors for HBV method for different values of N and M and BPFs for $(y(\xi), z(\xi))$ of Example 6.4

ξ	Exact	HBV method		BPFs [11] method
		$N = 1, M = 4$	$N = 2, M = 3$	$m = 32$
0.2	$(1.22140, 0.81873)$	$(1.221403, 0.818726)$	$(1.221403, 0.818711)$	$(1.22496, 0.81636)$
0.4	$(1.49182, 0.67032)$	$(1.491831, 0.670310)$	$(1.491826, 0.670320)$	$(1.47776, 0.67682)$
0.6	$(1.82212, 0.54881)$	$(1.82206, 0.548837)$	$(1.82211, 0.548815)$	$(1.83910, 0.54386)$
0.8	$(2.22554, 0.44933)$	$(2.225477, 0.449351)$	$(2.225546, 0.449329)$	$(2.2190, 0.45010)$

and accuracy of the proposed method, for solving such equations, are approved by some illustrative examples. The results show that the present method is very accurate even for small values of N and M, and the errors are very small. Moreover, the existence and uniqueness of the solutions of the system of Volterra integral equations are addressed. After adding some perturbations to the source functions in the examples, the values of maximum absolute errors remained constant or small, this confirms the stability of the proposed approach. Using this method for solving systems of Fredholm and Volterra integral equations of the first kind is still a subject of research.

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