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Research Article

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A Novel Shifted Jacobi Operational Matrix for Solution of Nonlinear Fractional Variable-Order Differential Equation with Proportional Delays

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Abstract

This work presents the generalized nonlinear multi-terms fractional variable-order differential equation with proportional delays. In this paper, a novel shifted Jacobi operational matrix technique is introduced to solve a class of these equations mentioned, so that the main problem becomes a system of algebraic equations that we can solve numerically. The suggested technique is successfully developed for the aforementioned problem. Comprehensive numerical tests are provided to demonstrate the generality, efficiency, accuracy of presented scheme and the flexibility of this technique. The numerical experiments compared it with other existing methods such as Reproducing Kernel Hilbert Space method ($RKHSM$). Comparing the results of these methods as well as comparing the current method ($NSJOM$) with the true solution, indicating the validity and efficiency of this scheme. Note that the procedure is easy to implement and this technique will be considered as a generalization of many numerical schemes. Furthermore, the error and its bound are estimated.

Keywords : Nonlinear multi-terms differential equations; Fractional variable-order with proportional delays; Shifted Jacobi operational matrix; Caputo differential operator.

1 Introduction

THe applications and analysis of fractional order calculus is an active and the fastest growing region for research in the last three decades. It has currently become an important tool owing to their wide usages in diverse scientific disciplines, such as chemistry, physics, regular changes in thermodynamics, blood circulation phenomenons, biophysics, electro-dynamics of complex medium, capacitor theory, polymer rheology, dynamical systems, fitting of experimental data, etc. ([5, 22, 31, 43] and references therein). The increasing development of appropriate and efficient method to solve FDEs has aroused more interest of researchers in this field. Nonlinear phenomenons have diverse usages in different regions of engineering and sci-

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ence such as fluid flow, mechanics, thermic systems and other fields of applications[26]. Clearly, solving the nonlinear FDEs is relatively difficult compared to its linear type. In this regard, analytical methods such as, concept of Lambert function, homotopy perturbation (HPM), new iterative methods and Adomian decomposition, have been used in the recent literature widely [2, 9, 14, 15, 27, 53, 50] . On the other hand, some researchers like Diethelm et. al. [23, 22] have developed standard numerical schemes, for example Adams-Bashforth techniques have been used numerically for integration of FDEs.

Putting up of delay in the differential equations of fractional order creates new perspectives especially in the field of bioengineering [17], because in bioengineering, the realization of the dynamics that happen in biological tissues, is improved by fractional derivatives [17, 28].

In mathematical sciences, the differential equations of delay are a kind of differential equations in that the derivative of an unknown function at a definite time is presented in terms of the values of the function at prior times. The delay differential equations are also called systems of time-delay, deed-time systems or aftereffect systems, equations of differential-difference type, systems of hereditary, equations with deviating arguments [26].

Fractional delay differential equations are different from the ordinary type in which the derivative at any time related to the solution (and when the equations are neutral then depends on the derivative) at previous times. Many real-world events can be modeled as the delay fractional differential equations [43]. The fraction order of differential equations have many application in various scientific disciplines by modeling of various problem such as economy, electro dynamics, biology, control, finance, chemistry, physics and so on, for further reading, interested parties can refer toreader to the [26, 38, 11, 46, 47, 44, 3] resouces.

In recent years, Margado et al in [36] analyzed numerical solution and approximated it for $FDDE$s. The stability areas of systems of $FDDE$s is examined via Cermak et al in [13]. Stability for systems of $FDDE$s by means of Gr$\ddot{u}$nwalds approach is analyzed by Lazarovic and Spansic [33]. Daftardar-Gejji et al in [16] have proposed a New Predictor-Corrector method ($NPCM$) and new iteration technique presented by Daftardar-Gejji and Jafari [18] , for solving FDEs numerically. Bhalekar and Daftardar-Gejji proposed A Predictor-Corrector method to solve non-linear fractional-order delay differential equations in [10]. In [52], the author generalized the algorithm of Adams-Bashforth-moulton presented in [22, 23, 21] to solve the delay FDEs. Varsha et al [17], have presented a new technique for solving non-linear $FDDE$s. Ghasemi et al [26], have employed Reproducing kernel Hilbert Space method for solving nonlinear $FDDE$s. Jhing and Daftardar-Gejji have provided a new numerical scheme to solve FDDEs [28] and Khodabandehlo et al in [29, 30] have proposed a novel shifted Jacobi operational matrix method for nonlinear variable-order FDDEs with periodic and anti periodic condition.

Moreover, the spectral methods that fundamentally related to an orthogonal polynomials set, are used for solving the FDEs. One of the most famous of them, is the classic Jacobi polynomials which shown as follow:

$$P_m^{(\alpha,\beta)}(t)\ (\alpha > -1,\ \beta > -1,\ m \geq 0).$$

These polynomials have been employed broadly in mathematical analysis and practical applications due to it has the benfits of achieving the numeric solutions in parameters α and β. Therefore, the systematic study of Jacobi polynomials with general indexes α and β will be useful and obviously, this case, in addition to extending the time interval $t \in [0, I]$, can be considered as one of the goals and novelties of this version [24]. Furthermore, recently interest of researchers has increased in this field (field of variable fractional differential equations) [49, 34]. So,many techniques are used to find the numeric solution of these equations [37, 25, 35].

Now, the goal of current work is to generalize the orthogonal polynomials in the base of solution. In fact, this technique is introduced in

[24] and we present a Novel Shifted Jacobi Operational Matrix for the fractional derivatives to solve a class of nonlinear multi-terms differential equations of fractional variable-order with proportional delays which as follow:

$$\begin{aligned} &D^{\eta(t)}w(t) = \\ &F(t, w(t), w(p_1t), w(p_2t), ..., w(p_kt)), \\ &0 \le t \le T, \\ &w^{(i)}(0) = \lambda_i, i = 0, 1, \cdots, n-1, \\ &0 < p_m < 1, \ \ m = 1, 2, \cdots, k\,. \end{aligned} \tag{1.1}$$

where $D^{\eta(t)}$ is the Caputo's derivative of variable-order fractional.

note

If $\eta(t)$ is constant, then equation (1.1) will be as follow:

$$\begin{aligned} &D^{\eta}w(t) \ = \ F\,(t,\ w(t),\ w(p_1t), \\ &w(p_2t),\ ...,\ w(p_kt))\,,\ 0 \le t \le T, \\ &w^{(i)}(0) = \lambda_i,\ \ i = 0, 1, \cdots, n-1. \end{aligned} \tag{1.2}$$

Also note that: we can use many polynomials such as Legendre polynomials, Gegenbauer polynomials, Fibonacci polynomials, all Chebyshev polynomials, Lucas, Vieta-Lucas, and etc in our new suggestion technique. The numerical results obtained for the mentioned equation in this study reveal that the present method is of highly accuracy. By focusing on numerical experiments gained by this method with other available methods, and comparing them, we can find that the proposed scheme capable of solving the variable-order $FDDE$, playing role of a powerful effective and practical numerical technique.

2 Fundamentals and preliminaries

At the beginning of this part of the article, we review some of the fundamental and most important features of theory of fractional calculus. Then recall some important traits of the Jacobi polynomials which help us for developing the suggested technique. So we refuse to include duplicate concepts in other related articles and unnecessary content but readers who are interested can refer to [8, 48, 20].

2.1 The derivative of fractional order

Different definitions are provided and used for the fractional derivative, but the three most usual are the Caputo definitions, Grünwald-Letincov and Riemann-Liouville. This article is based on Caputo definition because , in the initial conditions, only the Caputo's definition has the same form as integer-order DEs.

Definition 2.1. *The right and left-sided Caputo fractional derivatives of order $\eta\,(m-1 < \eta \le m\,)$ are detemined as*

$$D^{\eta}_{-}w(t) = \frac{(-1)^m}{\Gamma(m-\eta)} \int_t^T \frac{U'(s)}{(s-t)^{\eta-m+1}} ds,$$

$$D^{\eta}_{+}w(t) = \frac{1}{\Gamma(m-\eta)} \int_0^t \frac{U'(s)}{(t-s)^{\eta-m+1}} ds. \tag{2.3}$$

that

$$D^{\eta}_{+}t^j = \begin{cases} 0\ ,\ for\ j \in M_0\ and\ j < \lceil \eta \rceil, \\ \dfrac{\Gamma(j+1)t^{j-\eta}}{\Gamma(j-\eta+1)}, j \in M_0, j > \lceil \eta \rceil. \end{cases} \tag{2.4}$$

and

$$D^{\eta}_{-}(T-t)^j = \begin{cases} 0\ ,\ for\ j \in M_0\ and\ j < \lceil \eta \rceil, \\ \dfrac{(-1)^j\Gamma(j+1)(T-t)^{j-\eta}}{\Gamma(j-\eta+1)}, \\ for\ j \in M_0\ and\ j > \lceil \eta \rceil. \end{cases} \tag{2.5}$$

where $\lceil . \rceil$ is the ceiling function and $M_0 = \{0, 1, 2, \cdots\}$.

Also

$$D^{\eta}_{\pm}(\gamma\varphi(t) + \delta\phi(t)) = \gamma D^{\eta}_{\pm}(\varphi(t)) + \delta D^{\eta}_{\pm}(\phi(t)).$$

where γ and δ are constants.

Definition 2.2. *The Caputo derivative with fractional variable-order $\eta(t)$ for $w(t) \in C^m[0,T]$ are given respectively as [34, 12]:*

$$D^{\eta(t)}w(t) = \frac{1}{\Gamma(1-\eta(t))} \int_{0+}^t \frac{w'(s)}{(t-s)^{\eta(t)}} ds + \frac{w(0^+) - w(0^-)}{\Gamma(1-\eta(t))} t^{-\eta(t)}. \tag{2.6}$$

At the beginning time and for $0 < \eta(t) < 1$, *we have:*

$$D^{\eta(t)}w(t) = \frac{1}{\Gamma(1-\eta(t))}\int_{0^+}^{t}\frac{w'(s)}{(t-s)^{\eta(t)}}ds. \quad (2.7)$$

also, if a *and* b *are constant then*

$$D^{\eta(t)}(a\,w_1(t) + b\,w_2(t)) = a\,D^{\eta(t)}w_1(t) + b\,D^{\eta(t)}w_2(t)\,. \quad (2.8)$$

According to Eqution (2.6), *we have:*

$$D^{\eta(t)}C \;=\; 0,\; C \quad \textit{is a constant}. \quad (2.9)$$

On the other hand

$$D^{\eta(t)}t^k = \begin{cases} 0, & for\; k = 0, \\ \dfrac{\Gamma(k+1)}{\Gamma(k+1-\eta(t))}t^{k-\eta(t)} & for\; k = 1, 2, \cdots. \end{cases} \quad (2.10)$$

2.2 Shifted Jacobi polynomials and their properties

Suppose $P_m^{(\alpha,\beta)}(x)$; $\alpha > -1\,, \beta > -1$ as the $m-$th degree Jacobi orthogonal polynomial in x defined on $[-1,\ 1]$.

As any classical orthogonal polynomials, $P_m^{(\alpha,\beta)}(x)$ form an orthogonal system with respect to weight function $\omega^{(\alpha,\beta)}(x) = (1+x)^\alpha\,(1-x)^\beta$, in other words [8]:

$$\int_{-1}^{1} P_i^{(\alpha,\beta)}(x)\; P_j^{\alpha,\beta)}(x)\omega^{(\alpha,\beta)}\,dx \;=\; h_j^{(\alpha,\beta)}\delta_{i,j}, \quad (2.11)$$

where

$$h_j^{(\alpha,\beta)} \;=\; \frac{2^{\alpha+\beta+1}\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{(2j+\alpha+\beta+1)j!\;\Gamma(j+\beta+\alpha+1)},$$

and $\delta_{i,j}$ is the function of Kronecker. also the $i-$th order Jacobi polynomial has the following analytical form [24]

$$P_i^{(\alpha,\beta)}(t) = \sum_{k=0}^{i} \frac{\Gamma(\alpha+i+1)\Gamma(\alpha+i+1+\beta+k)(\frac{t-1}{2})^k}{\Gamma(\alpha+\beta+i+1)\Gamma(\alpha+1+k)k!\,(i-k)!}, \quad (2.12)$$

the polynomials presented in Equation (2.12) can be generated as follow:

$$d_{1,i}^{\alpha,\beta}P_i^{(\alpha,\beta)}(t) \;= d_{2,i}^{\alpha,\beta}P_{i-1}^{(\alpha,\beta)}(t) - d_{3,i}^{\alpha,\beta}P_{i-2}^{(\alpha,\beta)}(t), \quad i = 2, 3, \cdots \quad (2.13)$$

where

$$d_{1,i}^{\alpha,\beta} = 2i(\alpha+i+\beta)(\alpha+2i-2+\beta),$$

$$d_{2,i}^{\alpha,\beta} = (\alpha+2i-1+\beta)(\alpha^2-\beta^2 +(\alpha+2i+\beta)(\alpha+2i+\beta-2)t),$$

$$d_{3,i}^{\alpha,\beta} = 2(\alpha+i-1)(\beta+i-1)(\alpha+2i+\beta).$$

That start values as follow

$$P_0^{(\alpha,\beta)}(t) = 1$$

$$and\; P_1^{(\alpha,\beta)}(t) = \frac{1}{2}[(\alpha+\beta+2)\,t + (\alpha-\beta)].$$

If we implement the shift of variable $x = (\frac{2t}{T} - 1)$, then we can use the polynomial of Eq. (2.12) on the interval $0 \le t \le T$. Therefore, we have constructed the shifted Jacobi orthogonal polynomials $P_n^{(\alpha,\beta)}(\frac{2t}{T} - 1)$ which denoted via $P_{T,i}^{(\alpha,\beta)}(t)$. Then $P_{T,i}^{(\alpha,\beta)}(t)$ form an orthogonal system with $\omega_T^{(\alpha,\beta)}(t) = t^\alpha\,(T-t)^\beta$ as the weight function for $0 \le t \le T$ with follow orthogonality trait:

$$\int_0^T P_{T,i}^{(\alpha,\beta)}(t)\; P_{T,j}^{\alpha,\beta)}(t)\omega_T^{(\alpha,\beta)}\,dt \;=\; h_{T,j}^{(\alpha,\beta)}\delta_{i,j}, \quad (2.14)$$

where

$$h_{T,j}^{(\alpha,\beta)} \;=\; (\frac{T}{2})^{\alpha+\beta+1}\,h_j^{(\alpha,\beta)}.$$

Also the $i-$th order Shifted Jacobi polynomial has the following analytical form [24]

$$P_{T,i}^{(\alpha,\beta)}(t) = \sum_{k=0}^{i}(-1)^{i-k} \frac{(\alpha+i)!\,\Gamma(\alpha+k+i+1+\beta)\,t^k}{\Gamma(\alpha+i+1+\beta)(\alpha+k)!\,k!\,(i-k)!\,T^k}, = \sum_{k=0}^{i}\frac{(i+\beta)!\,\Gamma(\alpha+k+i+\beta+1)(T-t)^k}{\Gamma(\alpha+i+1+\beta)(\beta+k)!\,k!\,(i-k)!\,T^k}. \quad (2.15)$$

And in the endpoint values are given as

$$P_{T,i}^{(\alpha,\beta)}(0) = (-1)^i \frac{\Gamma(\alpha+i+1)}{\Gamma(\alpha+1)\Gamma(k+1)},$$
$$P_{T,i}^{(\alpha,\beta)}(T) = \frac{\Gamma(\beta+i+1)}{\Gamma(\beta+1)\Gamma(k+1)}.$$

Note that we can estimated the $P_{T,i}^{(\alpha,\beta)}(t)$ by the recurrence formula, we refer the interested reader to [24].

note

The Jacobi's shifted orthogonal polynomials constitute infinite number of orthogonal polynomials such as the shifted Chebyshev polynomials of the first, second, third and fourth kinds $T_{T,i}^{(\alpha,\beta)}(t)$, $U_{T,i}^{(\alpha,\beta)}(t)$, $V_{T,i}^{(\alpha,\beta)}(t)$ and $W_{T,i}^{(\alpha,\beta)}(t)$, respectively; the shifted Gegenbauer polynomials $G_{T,i}^{(\alpha,\beta)}(t)$, and the shifted Legendre polynomials $L_{T,i}^{(\alpha,\beta)}(t)$. These polynomials, which are all orthogonal, are related to $P_{T,i}^{(\alpha,\beta)}(t)$as follow:

$$L_{T,i}^{(\alpha,\beta)}(t) = P_{T,i}^{(0,0)}(t),$$

$$G_{T,i}^{(\alpha,\beta)}(t) = \frac{\Gamma(i+1)\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2}+i)} P_{T,i}^{(\alpha-\frac{1}{2},\beta-\frac{1}{2})}(t),$$

$$T_{T,i}^{(\alpha,\beta)}(t) = \frac{\Gamma(i+1)\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+i)} P_{T,i}^{(-\frac{1}{2},-\frac{1}{2})}(t),$$

$$U_{T,i}^{(\alpha,\beta)}(t) = \frac{\Gamma(i+2)\Gamma(\frac{1}{2})}{2\Gamma(\frac{3}{2}+i)} P_{T,i}^{(\frac{1}{2},\frac{1}{2})}(t),$$

$$V_{T,i}^{(\alpha,\beta)}(t) = \frac{2^{2i}(\Gamma(i+1))^2}{\Gamma(2i+1)} P_{T,i}^{(-\frac{1}{2},\frac{1}{2})}(t),$$

$$W_{T,i}^{(\alpha,\beta)}(t) = \frac{2^{2i}(\Gamma(i+1))^2}{\Gamma(2i+1)} P_{T,i}^{(\frac{1}{2},-\frac{1}{2})}(t).$$

3 Function Approximation by Shifted Jacobi Polynomials

Suppose the function $w(t)$ is square integrable with respect to $\omega_T^{(\alpha,\beta)}(t)$ in $[0,\ T]$, thus, we can expand it as follows:[8, 24]

$$w(t) = \sum_{i=0}^{\infty} a_i P_{T,i}^{(\alpha,\beta)}(t), \tag{3.16}$$

where a_i (the coefficients of the series) are obtained by

$$a_i = \frac{1}{h_{T,j}^{(\alpha,\beta)}} \int_0^T \omega_T^{(\alpha,\beta)}\, P_{T,i}^{(\alpha,\beta)}(t)\, w(t)dt, \quad i = 0, 1, \cdots. \tag{3.17}$$

So, we can estimate the approximate solution via considering $(N+1)$-terms of the presented series in Eq. (3.16) and we will have

$$w(t) \simeq w_N(t) = \sum_{i=0}^{N} a_i P_{T,i}^{(\alpha,\beta)}(t) = A^T \Phi_{T,N}(t), \tag{3.18}$$

where $A = [a_0, a_1, \cdots, a_N]^T$, and $\Phi_{T,N}(t) = [P_{T,0}^{(\alpha,\beta)}(t), P_{T,1}^{(\alpha,\beta)}(t), \cdots, P_{T,N}^{(\alpha,\beta)}(t)]^T$.

Here, we suppose that

$$S(t) = [1,\ t,\ t^2,\ t^3, \cdots, t^N]^T. \tag{3.19}$$

By equation (3.18), the vector $\Phi_{T,N}(t)$ can be presented as

$$\Phi_{T,N}(t) = B_{(\alpha,\beta)} S(t). \tag{3.20}$$

where $B_{(\alpha,\beta)}$ is a square matrix (of order $(N+1)\times(N+1)$) that given as follow

$$b_{i+1,j+1} = \begin{cases} \dfrac{(-1)^{i-j}(\alpha+i)!\,(\alpha+\beta+j+i)!}{(\alpha+\beta+i)!\,(\alpha+j)!\,(j!)(i-j)!\ T^j}, i \geq j, \\ 0\,,\ otherwise. \end{cases} \tag{3.21}$$

for $0 \leq i, j \leq N$.

Let $N=4$, $\alpha = \beta = 0$, then B as follows

$$B_{(0,0)} = \frac{1}{T^i} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 1 & -6 & 6 & 0 & 0 \\ -1 & 12 & -30 & 20 & 0 \\ 1 & -20 & 90 & -140 & 70 \end{bmatrix} \tag{3.22}$$

Hence, using Eq. (3.20), we get

$$S(t) = B_{(\alpha,\beta)}^{-1}\, \Phi_{T,N}(t). \tag{3.23}$$

note

Note that, we can obtain this matrix B for all orthogonal polynomials as well. For instance, let $N=4, \beta=\frac{-1}{2}, \alpha=\frac{1}{2}$ then the orthogonal polynomials will be of the fourth kind shifted Chebyshev type, hence B(square matrix of order 4×4) for these polynomials as follows

$$B_{(\frac{1}{2},\frac{-1}{2})} = \frac{1}{T^i}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & 0 \\ 1 & -12 & 16 & 0 & 0 \\ -1 & 24 & -80 & 64 & 0 \\ 1 & -40 & 240 & -448 & 256 \end{bmatrix} \tag{3.24}$$

4 Novel Shifted Jacobi Polynomials Operational Matrix($NSJOM$)

Operational matrix, which are applied in different areas of numerical analysis and to solve problems of different types and topics are of especial importance such as integral equations, DEs, integro-DEs, ordinary and partial FDEs and etc [37, 8, 12, 1, 54, 32, 40, 19, 6, 41, 42, 4, 7]. In this part, we investigate the ($SJOM$) of fractional variable-order to support the numerical solution of Eq. (1.1). Therefore, we convert the problem into the system of algebraic of equations which is solved numerically in collocation points.

At first, we deduce $D^{\eta(t)}\Phi_{T,N}(t)$, as follows: according to the previous content, we have: $\Phi_{T,N}(t) = B_{(\alpha,\beta)}S(t)$, thus

$$D^{\eta(t)}\Phi_{T,N}(t) = D^{\eta(t)}(B_{(\alpha,\beta)}S(t)) = B_{(\alpha,\beta)}D^{\eta(t)}[1,t,\cdots,t^N]^T, \tag{4.25}$$

Combining Eqs.(2.10) and (4.25), it gives

$$\begin{aligned} D^{\eta(t)}\Phi_{T,N}(t) &= B_{(\alpha,\beta)}D^{\eta(t)}(S(t)) \\ &= B_{(\alpha,\beta)}[0, \frac{\Gamma(2)t^{(1-\eta(t))}}{\Gamma(2-\eta(t))}, \frac{\Gamma(3)t^{(1-\eta(t))}}{\Gamma(3-\eta(t))}, \\ &\quad \cdots, \frac{\Gamma(N+1)t^{(N-\eta(t))}}{\Gamma(N+1-\eta(t))}]^T \\ &= B_{(\alpha,\beta)}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & q_2(t) & 0 & \cdots & 0 \\ 0 & 0 & q_3(t) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q_{N+1}(t) \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^N \end{bmatrix} \\ &= B_{(\alpha,\beta)}Q(t)S(t), \end{aligned} \tag{4.26}$$

where

$$Q(t) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & q_2(t) & 0 & \cdots & 0 \\ 0 & 0 & q_3(t) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q_{N+1}(t) \end{bmatrix}. \tag{4.27}$$

and $q_i(t) = \frac{\Gamma(i)t^{-\eta(t)}}{\Gamma(i-\eta(t))}$Using Eq. (3.23), then

$$D^{\eta(t)}\Phi_{T,N}(t) = B_{(\alpha,\beta)}Q(t)B^{-1}_{(\alpha,\beta)}\Phi_{T,N}(t). \tag{4.28}$$

The operational matrix of $D^{\eta(t)}\Phi_{T,N}(t)$, is $B_{(\alpha,\beta)}Q(t)\ B^{-1}_{(\alpha,\beta)}$.

Here, we estimate the variable-order fractional of the calculated function that obtained in Eq. (3.18) as follows

$$\begin{aligned} D^{\eta(t)}w(t) \simeq D^{\eta(t)}(A^T\Phi_{T,N}(t)) = \\ A^T D^{\eta(t)}\Phi_{T,N}(t) = \\ A^T B_{(\alpha,\beta)}Q(t)\ B^{-1}_{(\alpha,\beta)}\Phi_{T,N}(t). \end{aligned} \tag{4.29}$$

By using Eq. (4.29), hence the equation (1.1) turned into

$$A^T B_{(\alpha,\beta)} Q(t)\, B^{-1}_{(\alpha,\beta)} \Phi_{T,N}(t) = F\left(t,\ A^T\Phi_{T,N}(t),\ A^T\Phi_{T,N}(p_1 t), A^T\Phi_{T,N}(p_2 t), \ldots, A^T\Phi_{T,N}(p_k t)\right),\ 0 \le t \le T\,, \tag{4.30}$$

with condition

$$A^T \Phi^{(i)}_{T,N}(0) = \lambda_i,\ \ i = 0, 1, \cdots, n-1.$$

that $\Phi^{(i)}_{T,N}(0)$ is the i-th derivative of each element of $\Phi_{T,N}(t)$ at point $t = 0$.

Finally, we use t_j ($j = 0, 1, 2, \cdots, m.$) where they are the roots of $P^{(\alpha,\beta)}_{T,m+1}(t)$. Therefore Eq. (4.30) converted into the following form

$$A^T B_{(\alpha,\beta)} Q(t_j)\, B^{-1}_{(\alpha,\beta)} \Phi_{T,N}(t_j) = F\left(t_j,\ A^T\Phi_{T,N}(t_j), A^T\Phi_{T,N}(p_1 t_j), A^T\Phi_{T,N}(p_2 t_j), \ldots, A^T\Phi_{T,N}(p_k t_j)\right),$$

$$0 \le t \le T\,,\ \ j = 0, 1, 2, \cdots, m. \tag{4.31}$$

So, we can solve the system in Eq. (4.31) with the conditions mentioned numerically for determining the vector A. Therefore, the numerical solution that presented in Eq. (3.18) can be obtained.

5 Error Analysis

In this part of paper, we use the Lagrange interpolation polynomials to stimate an upper bound for the absolute error. On other hand, via applyin the current technique ($NSJOM$) with error approximation and the residual correction scheme [39, 45], an effiective error approximation will be obtained for the variable-order FDEs.

5.1 Error bound

Now, consider the smooth function $w(t)$ on $[0,\ T]$ and assume that $w_N(t) \in \prod_N^{\alpha,\beta}$ is the best approximation for it. Our goal is to gain an analytical form of error norm for $w_N(t)$ via expanding it into Jacobi polynomials. Let

$$\prod_N^{\alpha,\beta} = span\left\{P^{(\alpha,\beta)}_{T,i}(t),\ i = 0, 1, 2, \cdots, N\right\}.$$

Here, by using the definition and concept of the best approximation, one can write

$$\forall v_N(t) \in \prod_N^{\alpha,\beta}$$
$$\|w(t) - w_N(t)\|_\infty \le \|w(t) - v_N(t)\|_\infty. \tag{5.32}$$

Supose the interpolating polynomials at node points t_i ($i = 0, 1, \cdots, m$) be $v_N(t)$, (where t_i are the roots of $P^{(\alpha,\beta)}_{T,m+1}(t)$). It is obvious that $v_N(t)$ satisfies in the above inequality. Then by the Lagrange interpolation polynomials formula and its error formula, we will have

$$w(t) - v_N(t) = \frac{w^{(N+1)}(\xi)}{(N+1)!} \prod_{j=0}^{N} (t - t_j)\,, \tag{5.33}$$

where $0 < \xi < T]$, and hence we will have

$$\| w(t) - v_N(t) \|_\infty \le \max_{0\le t\le T} \mid w^{(N+1)}(\xi) \mid \frac{\| \prod_{j=0}^{N}(t - t_j) \|_\infty}{(N+1)!}\,. \tag{5.34}$$

We note that $w(t)$ on $[0,\ T]$ is smooth, therefore, there is a constant C_1, as

$$\max_{0\le t\le T} \mid w^{(N+1)}(\xi) \mid \le C_1\,. \tag{5.35}$$

We minimize the factor $\| \prod_{j=0}^{N}(t - t_j) \|_\infty$ as shown below:

One-to-one mapping $t = \frac{T}{2}(x+1)$ between the interval $[-1,\ 1]$ and $[0,\ T]$ is used to conclude that [8, 51]

$$\begin{aligned} &\min_{0\le t_j\le T} \max_{0\le t\le T} \mid \prod_{j=0}^{N}(t - t_j) \mid \\ &= \min_{-1\le x_j\le 1} \max_{-1\le x\le 1} \mid \prod_{j=0}^{N} \frac{T}{2}(x - x_j) \mid \\ &= \left(\frac{T}{2}\right)^{N+1} \min_{-1\le x_j\le 1} \max_{-1\le x\le 1} \mid \prod_{j=0}^{N}(x - x_j) \mid \\ &= \left(\frac{T}{2}\right)^{N+1} \min_{-1\le x_j\le 1} \max_{-1\le x\le 1} \mid \frac{P^{(\alpha,\beta)}_{N+1}(x)}{\mu^{(\alpha,\beta)}_N} \mid, \end{aligned} \tag{5.36}$$

where $\mu_N^{(\alpha,\beta)} = \dfrac{\Gamma(2N+\alpha+\beta+1)}{2^N N!\,\Gamma(N+\alpha+\beta+1)}$ is the last factor of $P_{N+1}^{(\alpha,\beta)}(x)$ and x_j are the roots of $P_{N+1}^{(\alpha,\beta)}(x)$. It is clear that

$$\max_{-1\le x\le 1} \mid P_{N+1}^{(\alpha,\beta)}(x) \mid =$$

$$P_{N+1}^{(\alpha,\beta)}(1) = \frac{\Gamma(\beta+N+2)}{\Gamma(\beta+1)(N+1)!}\,,$$

Using Eqs.(5.35) and (5.36), gives the following result

$$\| w(t) - w_N(t) \|_\infty \le C_1 \frac{(\frac{T}{2})^{N+1}\Gamma(\beta+N+2)}{\mu_N^{(\alpha,\beta)}((N+1)!)^2\,\Gamma(\beta+1)}\,. \tag{5.37}$$

Therefore, we estimate an upper bound for absolute error between the true and numerical solutions.

5.2 Error function estimation

In this subsection, we have introduced the error approximation based on the error function of residual of the presented technique and the estimate solution (3.18) is refined via the residual correction scheme. The error approxmation of residual was employed to estimate the error of some schemes for various equations [1, 45, 55, 56].

At first, we denote $e_N(t) = w_N(t) - w(t)$ be the function of error for the $NSJOM$ estimation $w_N(t)$ to $w(t)$, that $w(t)$ is the true solution of Eq. (1.1).

Therefore, $w_N(t)$ satisfies the below equation

$$\begin{aligned} D^{\eta(t)}w_N(t) &= F\,(t,\, w_N(t),\, w_N(p_1t), \\ & w_N(p_2t),\, ...,\, w_N(p_kt)) + R_N(t), \\ & 0\le t\le T\,, w_N^{(i)}(0)=\lambda_i, \\ & i=0,1,\cdots,n-1 \end{aligned} \tag{5.38}$$

where $R_N(t)$ is the function of residual of Eq. (1.1), which is estimated by replacing the $w_N(t)$ with $w(t)$ in Eq. (1.1).

By subtracting Eq. (1.1) from Eq. (5.38), the error problem is constructed in the form of

$$\begin{aligned} D^{\eta(t)}e_N(t) &= R_N(t) + R_{N,F}(t),\ 0\le t\le T\,, \\ e_N^{(i)}(0) &= 0,\ \ i=0,1,\cdots,n-1 \end{aligned} \tag{5.39}$$

where

$$\begin{aligned} & R_{N,F}(t) = \\ & F\,(t,\, w_N(t),\, w_N(p_1t), w_N(p_2t), ..., w_N(p_kt)) \\ & -F\,(t,\, w(t),\, w(p_1t),\, w(p_2t),\, ...,\, w(p_kt)). \end{aligned} \tag{5.40}$$

Thus, the (5.39) can be solved like the way it was presented in the previous section and we obtain the following estimation to $e_N(t)$.

$$\mathbf{e}_N(t) = \sum_{i=0}^{N} d_i P_{T,i}^{(\alpha,\beta)}(t) = D^T\Phi_{T,N}(t), \tag{5.41}$$

then the maximum absolute error can be obtained approximately by

$$\mathbf{E}_N(t) = \max\{\mathbf{e}_N(t)\,, 0\le t\le T\}. \tag{5.42}$$

The above estimation of error, is influenced by the rate of expansions convergence in Jacobi polynomials. Thus, the rates of sesible convergence in temporal discretizations, are providad by it [8, 56].

It is worth to mention here that if the exact solution of the problem (1.1) is unknown, in real practical experiment, we have trouble with computing $R_{N,F}$. But, the replacement strategy is to be approximated by its bound, say $D|e_N(t)|$ with D as positive constant. In fact, it is possible by supposing that the non-linear term F satisfies Lipschitz condition with respect to its all arguments.

6 Numerical experiences

In this section, based on the previous discussion, several numerical examples are provided to demonstrate the accuracy, efficiency, applicability, generality and validity of the presented technique. In all examples, the results of the present method are computed by Mathematica 10 software. In order to test our scheme, we compared it with Reproducing kernel Hilbert space method in terms of absolute errors which defined as: $\mid w_{exact}(t) - w_n(t) \mid$.

Collation of the results gained by this technique with the accurate solution of each example shows that this new technique has a better agreement than other methods. The stability, consistency and easy implementation of this technique cause this method to be more applicable and reliable.

Example 6.1. *[26] Consider the below delay fractional order equation with* $0 < \eta \leq 1$

$$\begin{aligned} D^{\eta} w(t) &= \frac{\Gamma(4) w(t)^{3-\eta}}{\Gamma(4-\eta)} \\ &- w(t) + w(\frac{t}{2}) + \frac{7}{8} t^3, \end{aligned} \tag{6.43}$$

$$w(0) = 0,$$

Note that $w(t) = t^3$ *is the exact solution and* $0 \leq t \leq T$, $T = 1$, $\eta = 0.5$.

By the concepts presented in Section 4, we consider the approximate solution with $(N+1)$ finite terms that presented in Eq. (3.18) for this problem and substitute in main problem. Then by using Eq. (4.29), this problem is converted to form of the Eq. (4.30), finally using t_i, therefore, a equations system emerges that we can solve it via known numerical methods to detect the unknown vector A. The solution of the Eq. (6.43) approximated using new method compared to other method($RKHS$), is in the best agreement with the accurate solution. The absolute errors (at some nodal points) of this technique and method in [26] are recorded and compared in Table 1. From this Table, it is observed that the numerical results which obtained via our technique, are much closer to the true solution and we gained an excellent approximation for the accurate solution by employing current scheme and it was found the our method in comparison with mentioned method is better with view to utilization, accuracy and more time efficiency. Fig. 1 compares the exact and approximated solution which confirms the reliability of $NSJOM$ method. Moreover, in Fig. 2 we draw the absolute error for this instance. Not that the Figs. 1 and 2 show a proper agreement between estimate and true solution. In this instance for $N = 3$ and $N = 5$, we have $A = [0.25, 0.45, 0.25, 0.05]^T$, $A = [0.25, 0.45, 0.25, 0.05, 0, 0]^T$, respectively.

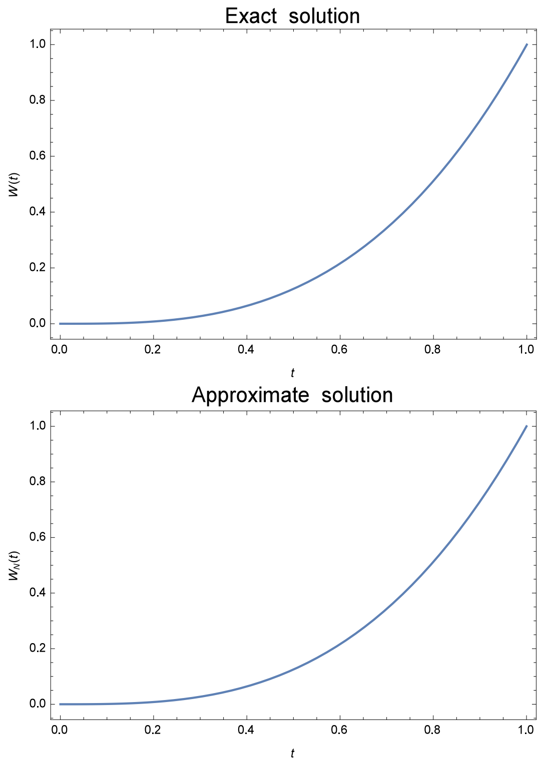

Figure 1: Comparison between estimate solution(w_3) of $NSJOM$ method and accurate solution for Example 6.1

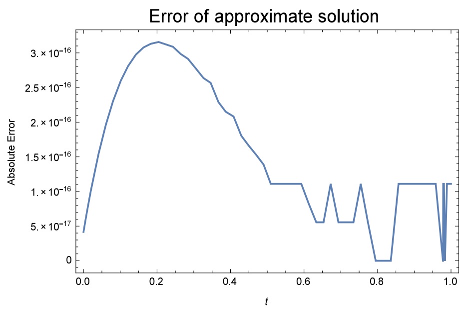

Figure 2: The absolute errors comparison between estimate solution(w_3) and accurate solution for Example 6.1

Example 6.2. *[26] Consider the below delay*

Table 1: Comparison the absolute errors between outcoms in [26] and our outcoms with $\alpha = 0, \beta = 0$ and $T = 1$ for Example 6.1.

t ∈ [0,T]	New technique $N = 3$	New technique $N = 5$	$RKHS$ $(w(t) \in W_2^2)$,$N = 12$	$RKHS$ $(w(t) \in W_2^4)$,$N = 12$
0	9.020×10^{-17}	0	0	0
0.1	2.775×10^{-16}	2.168×10^{-19}	4.210×10^{-3}	2.206×10^{-7}
0.2	3.087×10^{-16}	1.734×10^{-18}	1.047×10^{-2}	9.889×10^{-7}
0.3	2.636×10^{-16}	0	1.185×10^{-2}	3.412×10^{-6}
0.4	2.082×10^{-16}	1.387×10^{-17}	5.009×10^{-3}	1.952×10^{-5}
0.5	1.665×10^{-16}	2.775×10^{-17}	1.112×10^{-3}	5.622×10^{-5}
0.6	1.665×10^{-16}	0	1.739×10^{-3}	1.204×10^{-4}
0.7	2.220×10^{-16}	0	2.125×10^{-3}	2.253×10^{-4}
0.8	2.220×10^{-16}	1.110×10^{-16}	3.269×10^{-2}	3.754×10^{-4}
0.9	2.220×10^{-16}	1.110×10^{-16}	6.290×10^{-2}	5.800×10^{-4}
1.0	0	2.220×10^{-16}	5.563×10^{-2}	8.478×10^{-4}

fractional order equation for $1 < \eta \leq 2$

$$\begin{aligned} D^{\eta} w(t) &= \frac{\Gamma(3) w(t)^{2-\eta}}{\Gamma(3-\eta)} \\ &+ \frac{3}{4} w(t) + w(\frac{t}{2}) - t^2, \end{aligned} \quad (6.44)$$

$$w(0) = 0,\ w'(0) = 0,$$

In above problem $w(t) = t^2$ *is the exact solution and* $0 \leq t \leq T$, $T = 1$, $\eta = 2$.

Using the process mentioned in Ex.1, we get the solution of this problem. The solution of the Eq. (6.44) approximated using current method compared to $RKHS$ scheme, is in the best agreement with the true solution. The absolute errors (at some nodal points) of this technique and method in [26] are presented and compared in Table 2 . Also in Table 3 , we reported the absolute errors of current thechnique for Ex2 with $\eta = 1.7$. From these Tables, it is observed that the numerical outcomes which obtained via our technique, are much closer to the true solution and we gained an excellent estimation for the accurate solution by employing new technique and it was found the current method in comparison with mentioned method is better with view to utilization, accuracy and more time efficiency. Fig. 3 compares the exact and approximated solution which confirms the reliability of $NSJOM$ method. Moreover, in Fig. 4 we draw the absolute error for this instance for $\eta = 1.7$. Not that the Figs. 3 and 4 show a proper agreement between estimate and accurate solution. In this instance for $N = 2$ and $N = 5$, we have $A = [0.33333, 0.5, 0.166667]^T$, $A = [0.33333, 0.5, 0.166667, 0, 0, 0]^T$, respectively.

Example 6.3. *As the general example, consider the variable order* $FDDE$ *for* $1 < \eta(t) \leq 2$

$$\begin{aligned} D^{\eta(t)} w(t) &= \frac{\Gamma(3) w(t)^{2-\eta(t)}}{\Gamma(3-\eta(t))} \\ &+ \frac{3}{4} w(t) + w(\frac{t}{2}) - t^2, \end{aligned} \quad (6.45)$$

$$w(0) = 0,\ w'(0) = 0,$$

Note that $w(t) = t^2$ *is the exact solution and* $0 \leq t \leq T$, $T = 1$, $\eta(t) = \frac{t}{5}$.

By the concepts presented in Section 4, *we consider the approximate solution with* $(N + 1)$ *finite terms that presented in Eq.* (3.18) *for this problem and substitute in main problem. Then by using Eq.* (4.29), *this problem is converted to form of the Eq.* (4.30), *finally using* t_i, *therefore, a equations system emerges that we can solve it via known numerical methods to detect the unknown vector* A. *The solutions of the Eq.* (6.45) *approximated for different values of* α *and* β, *via new method compared to other methods, is in the best agreement with the true solution. The absolute and relative errors (at some nodal points) of this method, also the CPU time needed*

Table 2: Comparison the absolute errors between outcomes in [26] and our outcomes with $\alpha = 0, \beta = 0$ and $T = 1$ for Example 6.2

t ∈ [0,T]	New technique $N = 2$	New technique $N = 5$	$(w(t) \in W_2^5), N = 20$ $(w(t) \in W_2^5), N = 20$	$(w(t) \in W_2^6), N = 20$ $(w(t) \in W_2^6), N = 20$
0	0	0	0	0
0.1	0	0	1.306×10^{-6}	7.908×10^{-6}
0.2	0	0	1.810×10^{-5}	9.124×10^{-5}
0.3	0	0	3.045×10^{-5}	1.523×10^{-5}
0.4	0	0	2.767×10^{-5}	1.387×10^{-5}
0.5	0	0	2.306×10^{-5}	1.656×10^{-5}
0.6	0	0	3.364×10^{-5}	1.686×10^{-5}
0.7	0	0	4.021×10^{-5}	2.012×10^{-5}
0.8	0	0	4.737×10^{-5}	1.871×10^{-5}
0.9	0	0	5.082×10^{-5}	2.543×10^{-5}
1.0	0	0	5.952×10^{-5}	2.985×10^{-5}

Table 3: Absolute errors of $w(t)$ with $\alpha = 0, \beta = 0, \eta = 1.7$ and $T = 1$ for Example 6.2

$t \in [0, T]$	New technique,$N = 2$	New technique, $N = 5$
0	0	7.982×10^{-17}
0.1	0	9.367×10^{-17}
0.2	0	1.526×10^{-16}
0.3	0	2.359×10^{-16}
0.4	0	3.053×10^{-16}
0.5	0	3.885×10^{-16}
0.6	0	4.996×10^{-16}
0.7	0	5.551×10^{-16}
0.8	0	6.661×10^{-16}
0.9	0	8.882×10^{-16}
1.0	0	8.882×10^{-16}
CPU time	$0\ s$	$0.078001\ s$

Table 4: Absolute errors of $w(t)$ with $\alpha = 0, \beta = 0$ and $T = 1$ for Example 6.3

$t \in [0, T]$	*New technique* , $N = 2$	*New technique* , $N = 4$
0	0	0
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1.0	0	0
CPU time	$0\ s$	$9.516061\ s$

for our method are shown in Tables. 4 − 9. From these Tables, it is observed that the numerical outcomes which obtained via our technique, are much closer to the true solution and

Table 5: Relative errors of $w(t)$ with $\alpha = 0, \beta = 0$ and $T = 1$ for Example 6.3

$t \in [0, T]$	$New\ technique\ , N = 2$	$New\ technique\ , N = 4$
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1.0	0	0
CPU time	0 s	9.516061 s

Table 6: Absolute errors of $w(t)$ with $\alpha = 1, \beta = 1$ and $T = 1$ for Example 6.3

$t \in [0, T]$	$New\ technique\ , N = 2$	$New\ technique\ , N = 4$
0	0	0
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1.0	0	0
CPU time	0.031200 s	9.890463 s

Table 7: Relative errors of $w(t)$ with $\alpha = 1, \beta = 1$ and $T = 1$ for Example 6.3

$t \in [0, T]$	$New\ technique\ , N = 2$	$New\ technique\ , N = 4$
0.1	0	0
0.2	0	0
0.3	0	0
0.4	0	0
0.5	0	0
0.6	0	0
0.7	0	0
0.8	0	0
0.9	0	0
1.0	0	0
CPU time	0.031200 s	9.890463 s

we gained an excellent approximation for the accurate solution by employing current scheme and it was found the our technique in comparison with mentioned method($RKHS$) is better with view to utilization, accuracy and more time efficiency. Fig. 5 compares the accurate and estimated solution which confirms the reliability of $NSJOM$ method. Moreover, in Fig. 6 we draw the abso-

Table 8: Relative errors of $w(t)$ with $\alpha = \frac{1}{2}, \beta = \frac{1}{2}$ and $T = 1$ for Example 6.3

$t \in [0, T]$	*New technique*, $N = 2$	*New technique*, $N = 4$
0	5.551×10^{-17}	0
0.1	2.255×10^{-17}	0
0.2	5.551×10^{-17}	0
0.3	3.498×10^{-16}	0
0.4	4.718×10^{-16}	0
0.5	7.771×10^{-16}	0
0.6	1.165×10^{-15}	0
0.7	1.554×10^{-15}	0
0.8	2.109×10^{-15}	0
0.9	2.664×10^{-15}	0
1.0	3.330×10^{-15}	0
CPU time	$0\ s$	$8.985651\ s$

Table 9: Relative errors of $w(t)$ with $\alpha = \frac{1}{2}, \beta = \frac{1}{2}$ and $T = 1$ for Example 6.3

$t \in [0, T]$	*New technique*, $N = 2$	*New technique*, $N = 4$
0.1	8.330×10^{-16}	0
0.2	9.362×10^{-16}	0
0.3	9.221×10^{-16}	0
0.4	1.004×10^{-15}	0
0.5	1.389×10^{-15}	0
0.6	2.320×10^{-15}	0
0.7	3.531×10^{-15}	0
0.8	4.089×10^{-15}	0
0.9	4.224×10^{-15}	0
1.0	5.551×10^{-15}	0
CPU time	$0\ s$	$8.985651\ s$

lute error with $\alpha = 1$, $\beta = 1$ for this instance. Not that the Figs. 5 and 6 show a proper agreement between approximate and accurate solution. In this instance, we have

$$If\ \alpha = 0,\ \beta = 0\ and\ N = 2,\ then$$

$$A = [0.33333, 0.5, 0.16667]^T,$$

$$if\ \alpha = 0,\ \beta = 0\ and\ N = 4,\ then$$

$$A = [0.33333, 0.5, 0.16667,$$

$$4.03011 \times 10^{-14}, 1.43774 \times 10^{-14}]^T,$$

$$if\ \alpha = \frac{1}{2},\ \beta = \frac{1}{2}\ and\ N = 2,\ then$$

$$A = [0.3125, 0.33333, 0.1]^T,$$

$$if\ \alpha = \frac{1}{2},\ \beta = \frac{1}{2}\ and\ N = 4,\ then$$

$$A = [0.3125, 0.33333, 0.1,$$

$$7.63278 \times 10^{-16}, 2.01228 \times 10^{-16}]^T,$$

$$if\ \alpha = 1,\ \beta = 1\ and\ N = 2,\ then$$

$$A = [0.3, 0.25, 0.06667]^T,$$

$$if\ \alpha = 1,\ \beta = 1\ and\ N = 4,\ then$$

$$A = [0.3, 0.25, 0.06667,$$

$$-9.22873 \times 10^{-15}, -2.61943 \times 10^{-15}]^T.$$

7 Conclusions

In the current paper, we have proposed the novel shifted Jacobi operational matrix (NSJOM)

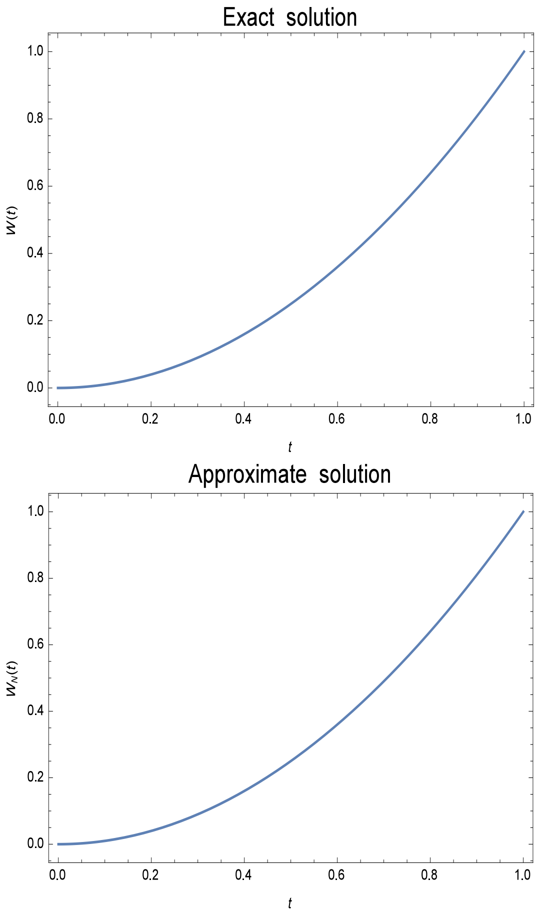

Figure 3: Comparison between estimate solution(w_2) of $NSJOM$ method and accurate solution for Example 6.2

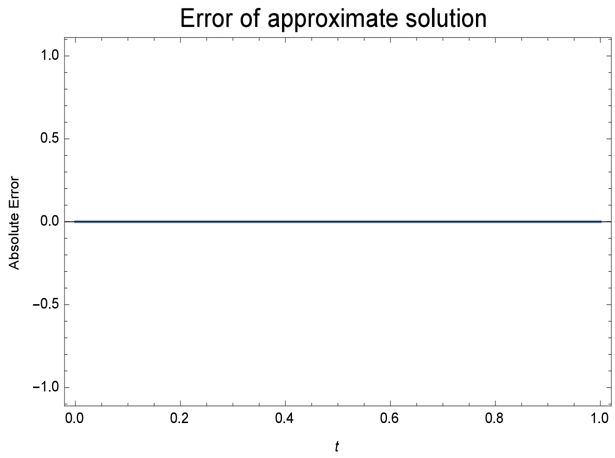

Figure 4: The absolute errors comparison between estimate solution (w_2) and accurate solution for Example 6.2

scheme for the generalized non-linear variable-order FDE with proportional delays via converting the main problem to an algebraic equations system that this system is solved using the appropriate numerical methods. We have shown that the proposed method has good convergence, is easy to implement and its concepts are simple. The obtained results are excellent compared to other method. Finally, the numerical results have been reported to clarify the efficiency and validity of this technique.

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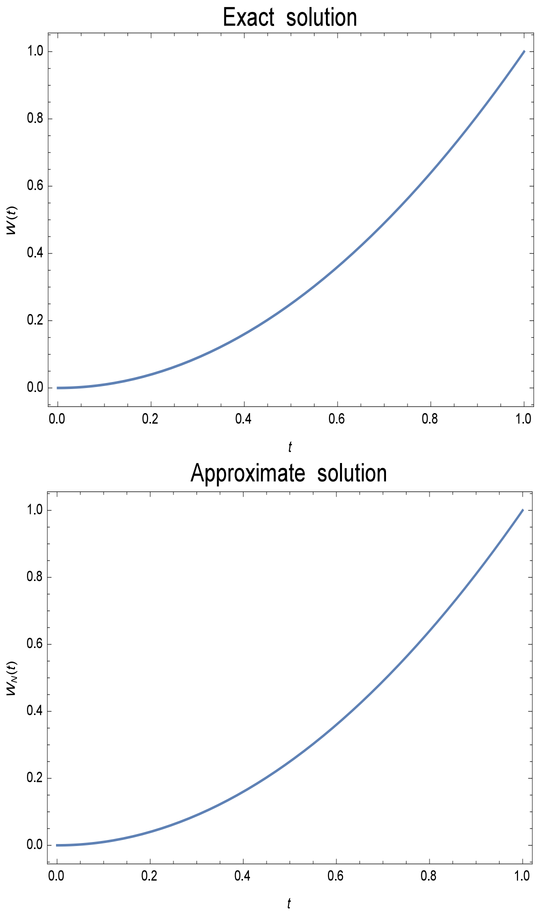

Figure 5: Comparison between estimate solution(w_2) of *NSJOM* method and accurate solution for Example 6.3. ($\eta(t) = 0.5\,t$).

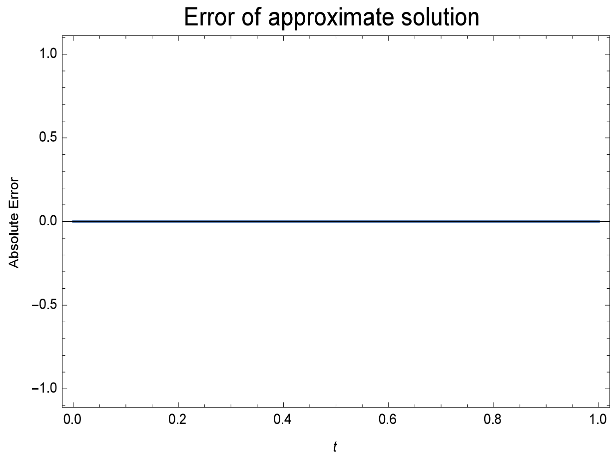

Figure 6: The absolute errors comparison between estimate solution(w_2) and accurate solution for Example 6.3 ($\eta(t) = 0.5\,t$).

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