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Research Article

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A New Efficient Method for Solving System of Fuzzy Volterra Integral Equations Based on Fibonacci Polynomials

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Abstract

Here, based on the Fibonacci polynomials, a new collocation method is presented in order to solve the system of linear fuzzy Volterra integral equations of the second kind. By using this method, these systems are reduced to a linear system of algebraic equations which are simply solvable. Also, the error analysis and existence of the solution of the suggested method are discussed. Finally, to show the importance and application of this method, we have used some rational examples. The method is computationally very attractive and gives very accurate results. Easy implementation and simple operations are the essential features of the Fibonacci polynomials.

Keywords : Fuzzy; Volterra integral equations; Collocation method; Convergence; Fibonacci polynomials.

1 Introduction

THe solutions of integral and integro-differntial equations play a important role in the fields of Science, such as elasticity, plasticity, heat and mass transfer, oscillation theory, fluid dynamics, filtration theory, electrostatics, electrodynamics, biomechanics, glass forming process, electrical engineering, economics, and medicine [14, 19].

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [41, 42], Dubois and Prade [12, 13]. It was Brunner and Kauthen who introduced the collocation method to solve Fredholm-Volterra integral equations [10, 21]. Later, the essential arithmetic structure for fuzzy numbers was studied by Tanaka, Mizumoto and Nahmias [30, 31, 32]. The Fibonacci collocation method has been used to find the approximate solutions of differential, integral, and integro-differential equations [17, 23, 24, 28, 29]. The fuzzy method is suitable for systems with non-specific models, and therefore, suits well to a process where the model is undetermined or indefinite and especially to systems with unknown or complex dynamics [40]. Since, many important problems are expressed by fuzzy number rather than crisp case, then it is impor-

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tant to expand mathematical numerical methods and model that would formulated general fuzzy integral equations and solve them. The study of these equations started with the investigations by Kaleva [20], Nanda [33], Goetschel, Seikkala [36] and Voxman [18] for the fuzzy Volterra integral equations. Also, the existence and uniqueness of the solution of these equations are proved using the fixed point theorems like the Banach's principle and the Darbo's theorem [6, 7, 8].
A problem which has long been of fundamental interest in classical analysis is the expansion of a given function $v(y)$ in a series of the form:

$$v(y) = \sum_{m=0}^{\infty} a_m p_m(y),$$

where $p_m(y)$ is a prescribed sequence of polynomials and the coefficients a_m are the numbers correspond to $v(x)$. The innumerable investigations on expansions of "arbitrary" functions have led to many important convergence and summability theorems and various interesting results in the theory of approximation [29].
In this paper, based on the Fibonacci polynomials, a new collocation method is presented to solve the system of linear fuzzy Volterra integral equations of the second kind. By using this method, these systems are reduced to a linear system of algebraic equations which are simply solvable. The method is computationally very attractive and gives very accurate results.
In Section 2, we introduce the Fibonacci polynomials and some of its features. In the third section, some basic definitions and notations of fuzzy number are mentioned. In Section 4 we study the fuzzy systems of Volterra integral equations of the second kind. Then we explain the collocation method formed on Fibonacci polynomials to solve fuzzy Volterra integral equations in Section 5. Then, the existence theorem of the solution for the linear fuzzy Volterra integral equations are proved in Section 6. Convergence analysis is established in Section 7. This new method is used to numerical examples in next section. Finally, conclusion is presented in Section 9.

2 Fibonacci polynomials

2.1 The Fibonacci polynomials

The Fibonacci polynomials $F_n(x)$ for $n = 1, 2, \cdots$, studied by Catalan [17] are defined by the recursion

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \quad n \geq 1,$$

with preliminary conditions $F_1(x) = 1$ and $F_2(x) = x$. They are given by the obvious formula

$$F_{n+1}(x) = \sum_{i=0}^{\left[\frac{n}{2}\right]} \binom{n-i}{i} x^{n-2i}, \quad n \geq 0,$$

where

$$\left[\frac{n}{2}\right] = \begin{cases} \frac{(n-1)}{2}, & n \quad even, \\ \frac{n}{2}, & n \quad odd. \end{cases}$$

Note that $x = 0$ is the only real root of $F_{2n}(0) = 0$, while $F_{2n+1}(0) = 1$ is without real roots. Furthermore for $x = k \in \mathbb{N}$, the elements of the k-Fibonacci sequences is obtained [16].
The Fibonacci polynomials have generating function [21]

$$\begin{aligned} W(x,y) &= \frac{y}{1-y^2-yx} \\ &= \sum_{n=1}^{\infty} F_n(x) y^n \\ &= y + xy^2 + (x^2+1)y^3 + (x^3+2x)y^4 + \cdots. \end{aligned}$$

The Fibonacci polynomials are normalized so that $F_n(1) = F_n$, where F_n is nth Fibonacci number. The first few Fibonacci polynomials are

$$\begin{aligned} F_1(x) &= 1, \\ F_2(x) &= x, \\ F_3(x) &= x^2+1, \\ F_4(x) &= x^3+2x, \\ F_5(x) &= x^4+3x^2+1. \end{aligned}$$

This equations can be written in the matrix form as follows

$$F(x) = GH(x), \tag{2.1}$$

where

$$F(x) = [F_1(x), F_2(x), \cdots, F_{m+1}(x)]^T,$$
$$H(x) = [1, x, x^2, \cdots, x^m]^T,$$

and G is the lower triangular matrix with input the coefficients appearing in the expansion Fibonacci polynomials at increasing powers of x. When $m = 7$ we have

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 & 0 & 1 & 0 & 0 \\ 1 & 0 & 6 & 0 & 5 & 0 & 1 & 0 \\ 0 & 4 & 0 & 10 & 0 & 6 & 0 & 1 \end{bmatrix}.$$

Notice that in matrix G, the non-zero inputs are made exactly the diagonals of the Pascal triangle precisely and the sum of the elements in the same row gives the classical Fibonacci sequence. Further, matrix G is irreverible so, x^n may be written as linear composition of Fibonacci polynomials [16].

2.2 Approximation of function

Suppose that a function $f(x)$ can be expressed in terms of the Fibonacci polynomials. In particular, only the first-$(m+1)$-term of Fibonacci polynomials are considered. Hence, the function $f(x)$ can be written in the matrix form $f(x) = CF(x)$, where $C = [c_1, c_2, \cdots, c_{m+1}]$. So, from Eq.(2.1), we have

$$f(x) \simeq CGH(x).$$

First, by truncated Taylor series we can approximate the function $u(x,t)$. In the next step we apply the truncated Fibonacci series as follows [23]

$$u(x, y) \simeq \sum_{p=0}^{m}\sum_{q=0}^{m} \mathbf{u}_{pq} x^p y^q, \tag{2.2}$$

$$u(x, y) \simeq \sum_{p=0}^{m}\sum_{q=0}^{m} u_{pq} F_p(x) F_q(y), \tag{2.3}$$

where

$$\mathbf{u}_{pq} = \frac{1}{p!\,q!} \frac{\partial^{p+q} u(0,0)}{\partial x^p \partial y^q}, \quad p, q = 0, 1, \cdots, m.$$

We can write Eqs.(2.2) and (2.3) in the matrix form as follows:

$$u(x, y) = H^T(x) \mathbf{u}_{pq} H(y), \tag{2.4}$$

and

$$u(x, y) = F^T(x) u_{pq} F(y). \tag{2.5}$$

From relations (2.4) and (2.5), we have

$$H^T(x)\mathbf{u}_{pq}H(y) = F^T(x)u_{pq}F(y)$$
$$\rightarrow H^T(x)\mathbf{u}_{pq}H(y) = H^T(x)G^T u_{pq} G H(y).$$

Also

$$\begin{aligned} B_x &= \int_a^x F(t)F^T(t)dt \\ &= \int_a^x G H(t) H^T(t) G^T dt \\ &= G K_x G^T, \quad x \in [a, b], \end{aligned} \tag{2.6}$$

where

$$K_x = \int_a^x H(t)H^T(t)dt = [k_{i,j}],$$
$$k_{i,j} = \frac{x^{i+j+1} - a^{i+j+1}}{i+j+1}, \quad i, j = 0, 1, \ldots, m. \tag{2.7}$$

3 Preliminaries in fuzzy calculus

From the different definitions of the concept of fuzzy number we choose for this article the definitions below:

Definition 3.1. *([3]). The parametric form of fuzzy number $\tilde{y}$ is given by an ordered pair of functions $(\underline{y}, \overline{y})$ Which satisfying to the following conditions*

i) $\overline{y} : r \rightarrow y_r^- \in \mathbb{R}$ *is a left-continuous, bounded and non-decreasing function on* $[0, 1]$.

ii) $\underline{y} : r \rightarrow y_r^+ \in \mathbb{R}$ *is a left-continuous, bounded and non-increasing function on*$[0, 1]$.

iii) $\underline{y} \leq \overline{y}$ *for all* $0 \leq r \leq 1$.

Furthermore, for any fuzzy numbers $\tilde{y} = (\underline{y}(r), \overline{y}(r))$ *and* $\tilde{x} = (\underline{x}(r), \overline{x}(r))$ *we have* *(i)* $\underline{y(r) \oplus x(r)} = \underline{y}(r) \oplus \underline{x}(r)$ *and* $\overline{y(r) \oplus x(r)} = \overline{y}(r) \oplus \overline{x}(r)$.

(ii) $\underline{y(r) \ominus x(r)} = \underline{y}(r) \ominus \underline{x}(r)$ *and* $\overline{y(r) \ominus x(r)} = \overline{y}(r) \ominus \overline{x}(r)$.

(iii)

$$t.\tilde{y} = \begin{cases} (t\underline{y}(r), t\overline{y}(r)), & t \geq 0, \\ (t\underline{y}(r), t\overline{y}(r)), & t < 0. \end{cases}$$

(iv)

$$\tilde{y} \odot \tilde{x} = \begin{cases} (\underline{y.x})(r) = \max\{\underline{y}(r).\underline{x}(r), \underline{y}(r).\overline{x}(r), \\ \qquad \overline{y}(r).\underline{x}(r), \overline{y}(r).\overline{x}(r)\}, \\ \\ (\overline{y.x})(r) = \min\{\underline{y}(r).\underline{x}(r), \underline{y}(r).\overline{x}(r), \\ \qquad \overline{y}(r).\underline{x}(r), \overline{y}(r).\overline{x}(r)\}. \end{cases}$$

Definition 3.2. *For arbitrary fuzzy numbers* $\tilde{y} = (\underline{y}(r), \overline{y}(r))$ *and* $\tilde{x} = (\underline{x}(r), \overline{x}(r))$*, the quantity*

$$D(\tilde{y}, \tilde{x}) = \max\Big\{ \sup_{r\in[0,1]} |\underline{y}(r) - \underline{x}(r)|, \sup_{r\in[0,1]} |\overline{y}(r) - \overline{x}(r)| \Big\},$$

is the distance between $\tilde{y}$ *and* $\tilde{x}$ *[2].*

The above metric is equivalent to the metrics defined by Puri, Ralescu and Kaleva [20, 34]. It is proven that (E^1, D) is a complete metric space [35].
In the following, we define the integral of a fuzzy function using the Riemann integral concept, is defined.

Definition 3.3. *The fuzzy function* $\tilde{y} : [a, b] \to E^1$ *is said to be continuous if for any fixed* $x_0 \in [a, b]$ *and* $\varepsilon > 0$*, there exists some* $\delta > 0$ *such that if* $|x - x_0| < \delta$*, then* $D(\tilde{y}(x), \tilde{y}(x_0)) < \varepsilon$.

Definition 3.4. *Let* $\tilde{y} : [a, b] \to E^1$.
(Definite integral of $y(x)$ *on* $[a, b]$ *is* $\int_a^b y(x)dx = \lim_{\mathcal{N}\to 0} R_q = \sum_{i=0}^{n} \tilde{y}(\eta)(x_i - x_{i-1})$ *for each partition* $q = \{x_0, x_1, \cdots, x_n\}$ *of* $[a, b]$ *and arbitrary* $\eta_i : x_{i-1} \leq \eta_i \leq x_i$, $i = 1, 2, \cdots, m$ *and* $\mathcal{N} = \max_{1\leq i\leq n}|x_i - x_{i-1}|$.*) If the fuzzy function* $\tilde{y}(x)$ *is continuous whit respect to metric* D*, then its definite integral exists. Also,*

$$\underline{\int_a^b y(x; r)dx} = \int_a^b \underline{y}(x; r)dx,$$

$$\overline{\int_a^b y(x; r)dx} = \int_a^b \overline{y}(x; r)dx,$$

where $(\underline{y}(x, r), \overline{y}(x, r))$ *is the parametric form of* $y(x)$ *[11]. It should be noted that the fuzzy integral can also be defined by using the Lebesgue-type approach [20].*

Definition 3.5. *([39]).* $\tilde{y} : [a, b] \to E^1$ *is fuzzy-Riemann integrable to* $I(\tilde{y}) \in E^1$ *if for any* $\epsilon > 0$*, there exists* $\delta > 0$ *such that for any division* $Z = \{[f, g]; \xi\}$ *of* $[a, b]$ *with the norms* $\Delta(Z) < \delta$*, we have*

$$D\Big(\sum_Z^{*} (g - f) \odot \tilde{y}(\xi), I(\tilde{y}) \Big) < \epsilon,$$

where $\sum_Z^*$ *denotes the fuzzy summation.*

Lemma 3.1. *([4]). If* $\tilde{f}$ *and* $\tilde{g} : [a, b] \subseteq R \to E^1$ *are fuzzy continuous function, then the function* $\mathbb{F} : [a, b] \to R_+$ *by* $\mathbb{F}(x) = D(\tilde{f}(x), \tilde{g}(x))$ *is continuous on* $[a, b]$*, and*

$$D\Big(\int_a^b \tilde{f}(x)dx, \int_a^b \tilde{g}(x)dx \Big) \leq \int_a^b D(\tilde{f}(x), \tilde{g}(x))dx.$$

Theorem 3.1. *([5, 9]).*

(i) The pair $(E^1, \oplus)$ *is a commutative semigroup with* $\tilde{0} = \chi\{0\}$ *zero element.*

(ii) For fuzzy numbers which are not crisp, there is no opposite element (that is, $(E^1, \oplus)$ *cannot be a group).*

(iii) The function of $\|.\|_F$: $E^1 \to R$ *by* $\|f\|_F = D(\tilde{0}, \tilde{f})$ *has the usual properties of norm, that is,* $\|\tilde{f}\|_F = 0$ *if and only if* $\tilde{f} = \tilde{0}$, $\|\alpha \odot \tilde{f}\| = |\alpha|\|\tilde{f}\|_F$ *and* $\|\tilde{f} \oplus \tilde{g}\|_F \leq \|\tilde{f}\|_F + \|\tilde{g}\|_F$.

(iv) $|\|\tilde{f}\|_F - \|\tilde{g}\|_F| \leq D(\tilde{f}, \tilde{g})$ *and* $D(\tilde{f}, \tilde{g}) \leq \|\tilde{f}\|_F + \|\tilde{f}\|_F$ *for any* $\tilde{f}, \tilde{g} \in E^1$.

Definition 3.6. *([26]). The fuzzy linear system*

$$A \odot \tilde{x} = B \odot \tilde{x} \oplus \tilde{y}, \tag{3.8}$$

where $\tilde{y}$ *is a fuzzy number vector and* $A = (a_{ij})$*, and* $B = (b_{ij})$ *for* $1 \leq i, j \leq m$*, are crisp coefficient matrices, is called dual fuzzy linear system.*

4 Fuzzy Volterra integral equations

Here, we consider the system of Volterra integral equations as follows:

$$V(y) = F(y) + \lambda \int_a^y K(y,t)V(t)dt, \qquad (4.9)$$

where

$$V(y) = [v_1(y), v_2(y), \cdots, v_m(y)]^T,$$
$$F(y) = [f_1(y), f_2(y), \cdots, f_m(y)]^T,$$
$$K(y,t) = [k_{i,j}(y,t)], \quad i,j = 1,2,\cdots,m,$$

where λ and a are suitable constants, $v_i(y)$, $i = 1,2,\cdots,m$ is indeterminate function, while $f_i(y)$ and kernels $k_{i,j}(y,t)$, $i,j = 1,2,\cdots,m$ are known functions. Let $\tilde{F}(y)$ and $\tilde{V}(y)$ be the parametric form of $F(y)$ and $V(y)$, respectively (see Definition 3.1). The fuzzy Volterra integral equations of the second kind for $y \in [a,b]$ are as follows:

$$\tilde{V}(y) = \tilde{F}(y) \oplus \lambda \int_a^y K(y,t) \odot \tilde{V}(t)dt, \qquad (4.10)$$

where

$$\tilde{V}(y) = [\tilde{v}_1(y), \tilde{v}_2(y), \cdots, \tilde{v}_m(y)]^T,$$
$$\tilde{v}_i(y) = (\underline{v}_i(y,s), \overline{v}_i(y,s)),$$
$$\tilde{F}(y) = [\tilde{f}_1(y), \tilde{f}_2(y), \cdots, \tilde{f}_m(y)]^T,$$
$$\tilde{f}_i(y) = (\underline{f}_i(y,s), \overline{f}_i(y,s)),$$

$$k_{i,j}(y,t) \odot \underline{v}_j(t,s) = \begin{cases} k_{i,j}(y,t)\underline{v}_j(t,s), k_{i,j}(y,t) \geq 0 \\ k_{i,j}(y,t)\overline{v}_j(t,s), k_{i,j}(y,t) < 0 \end{cases},$$

and

$$k_{i,j}(y,t) \odot \overline{v}_j(t,s) = \begin{cases} k_{i,j}(y,t)\overline{v}_j(t,s), k_{i,j}(y,t) \geq 0 \\ k_{i,j}(y,t)\underline{v}_j(t,s), k_{i,j}(y,t) < 0 \end{cases}.$$

5 Collocation method based on Fibonacci operational matrix

Now, we have solved the fuzzy system of linear Volterra integral equations by collocation method based on Fibonacci polynomials. Let $P(y) = [p_1(y), p_2(y), \cdots, p_m(y)]$ be a Fibonacci polynomial. Then, we have

$$\sum_{i=1}^m a_i p_i(y) = f_j(y) \oplus \sum_{i=1}^m \Big(\lambda \int_a^y k_{i,j}(y_i,t) \odot a_i p_i(y)\Big) dt, \ j = 1,\cdots,m. \qquad (5.11)$$

Let us consider

$$v_m(y) = \sum_{i=1}^m a_i F_i(y).$$

We assume that the system (4.10) has a unique solution. Therefore we have

$$\begin{bmatrix} (\underline{f}_1(y,s), \overline{f}_1(y,s)) \\ (\underline{f}_2(y,s), \overline{f}_2(y,s)) \\ \vdots \\ (\underline{f}_m(y,s), \overline{f}_m(y,s)) \end{bmatrix} = \begin{bmatrix} (\underline{v}_1(y,s), \overline{v}_1(y,s)) \\ (\underline{v}_2(y,s), \overline{v}_2(y,s)) \\ \vdots \\ (\underline{v}_m(y,s), \overline{v}_m(y,s)) \end{bmatrix} - \begin{bmatrix} (\underline{Z}_1, \overline{Z}_1) \\ (\underline{Z}_2, \overline{Z}_2) \\ \vdots \\ (\underline{Z}_m, \overline{Z}_m) \end{bmatrix}, \qquad (5.12)$$

where

$$(\underline{Z}_i, \overline{Z}_i) = \Big(\sum_{j=1}^m \int_a^y k_{i,j}(y,t)\underline{v}_j(t,s)dt, \sum_{j=1}^m \int_a^y k_{i,j}(y,t)\overline{v}_j(t,s)dt \Big), \ i = 1,2,\cdots,m. \qquad (5.13)$$

By using Eq.(2.5), we can approximate functions $(\underline{v}_i(y,s), \overline{v}(y,s))$, and $k_{i,j}(y,t), i,j = 1,2,\cdots,m$ as follows

$$\underline{v}_i(y,s) = F^T(y)\check{V}_i F(s), \qquad (5.14)$$
$$\overline{v}_i(y,s) = F^T(y)\check{V}_i F(s), \qquad (5.15)$$
$$k_{i,j}(y,t) = F^T(y)k_{i,j}F(t). \qquad (5.16)$$

Applying Eqs.(5.14)-(5.16), in Eq.(5.13) we have

$$(\underline{Z}_i, \overline{Z}_i) = \Big(\sum_{j=1}^m \int_a^y F^T(y)k_{i,j}F(t)F^T(t)V_jF(s)dt, \sum_{j=1}^m \int_a^y F^T(y)k^{i,j}F(t)F^T(t)\check{V}_jF(s)dt \Big),$$
$$= \Big(F^T(y) \sum_{j=1}^m k_{i,j} \Big(\int_a^y F(t)F^T(t)dt \Big) V_jF(s), F^T(y) \sum_{j=1}^m k_{i,j} \Big(\int_a^y F(t)F^T(t)dt \Big) \check{V}_jF(s) \Big),$$

therefore, according to Eqs.(2.6) and (2.7), we have

$$(\underline{Z}_i, \overline{Z}_i) = \Big(F^T(y)\sum_{j=1}^{m} k_{i,j}B_yV_jF(s), \\ F^T(y)\sum_{j=1}^{m} k_{i,j}B_y\check{V}_jF(s)\Big).$$

So we can rewrite the system (4.10) in the following form

$$\mathcal{V} = \mathcal{F} + \mathcal{K}\mathcal{B}_y\mathcal{V}, \tag{5.17}$$

where

$$\mathcal{K}\mathcal{B}_y = \begin{bmatrix} k^{1,1}B_y & k^{1,2}B_y & k^{1,3}B_y & \cdots & k^{1,m}B_y \\ k^{2,1}B_y & k^{2,2}B_y & k^{2,3}B_y & \cdots & k^{2,m}B_y \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k^{m,1}B_y & k^{m,2}B_y & k^{m,3}B_y & \cdots & k^{m,m}B_y \end{bmatrix},$$

$$\mathcal{V} = \begin{bmatrix} (V_1, \check{V}_1) \\ (V_2, \check{V}_2) \\ (V_3, \check{V}_3) \\ \vdots \\ (V_m, \check{V}_m) \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} (F_1, \check{F}_1) \\ (F_2, \check{F}_2) \\ (F_3, \check{F}_3) \\ \vdots \\ (F_m, \check{F}_m) \end{bmatrix},$$

and

$$\underline{f}_i(y,s) = F^T(y)F_iF(s),$$
$$\overline{f}_i(y,s) = F^T(y)\check{F}_iF(s), \quad i = 1,2,\cdots,m.$$

Thus, we can write system (5.17) as follows

$$(Z - \mathcal{K}\mathcal{B}_y)\mathcal{V} = \mathcal{F}, \tag{5.18}$$

where Z is an identity matrix. After solving this linear system, we approximate the solution of system (5.12) using V_i and $\check{V}_i$, $i = 1,2,\cdots,m$ in Eqs.(5.14) and (5.15), respectively.

6 Existence of the solution

Here, before proving the existence theorem for the linear fuzzy Volterra integral equation (4.10), we give some useful notations. Given an $n \times n$ matrix, or a vector, P with (i,j) arriving $p_{i,j}$, we define $|P|$, the absolute value of P, to be the $n \times n$, matrix whose (i,j) arriving is $|p_{ij}|$. Also, given two $n \times m$ matrices P and Q, we will write $P \leq Q$ if and only if $p_{ij} \leq q_{ij}$ for all i and j. Therefore, we can show [38]

(i) If $|P| \leq |Q|$ then $\|P\|_\infty \leq \|Q\|_\infty$.

(ii) $\|P\|_\infty = \|\,|P|\,\|_\infty$.

Theorem 6.1. *Suppose that $\tilde{f}_i(y)$, $i = 1,2,\cdots,m$, $y \in [a,b]$ be the fuzzy continuous functions and $k_{i,j}(y,t)$, $i,j = 1,2,\cdots,m$ be continuous for $a \leq y,t \leq b$, and $\tau_{i,j} = \max_{a \leq y,t \leq b} |k_{i,j}(y,t)|$. Let $\tau_1 = \max\{k_{i,j},\ i,j = 1,2,\cdots,m\}$ and $A = \{\tilde{g} : [0,1] \to E^1 : \tilde{g}$ is continuous$\}$ be a space of fuzzy continuous functions with the metric $D^*(\tilde{g},\tilde{h}) = \sup_{a \leq y \leq b} D(\tilde{g}(y),\tilde{h}(y))$ (that is called the uniform distance between fuzzy number value functions). If $\mathcal{B} = m(b-a)\tau_1 < 1$, then the fuzzy system (4.10) has an unique solution $\tilde{V}^*$ in A^m which can be obtained by the following successive approximations method*

$$\tilde{V}_0(y) = \tilde{F}(y),$$

$$\tilde{V}_m(y) = \tilde{F}(y) \oplus \int_a^y k(y,t) \odot \tilde{V}_{m-1}(t)dt,$$
$$a \leq y,t \leq b, \quad m \geq 1.$$

Furthermore, the sequence of successive approximations, $\tilde{F}_m$ converges to the solution $\tilde{F}^*$. Moreover, the following error bound holds

$$\|D\Big(\tilde{V}^*(t), \tilde{V}_m(t)\Big)\|_\infty \leq \frac{\mathcal{B}^{n+1}}{1-\mathcal{B}}\tau_0,$$

where

$$\tau_0 = \sup_{y \in [a,b]} \|D(\tilde{0}, \tilde{f}_i(y))\|_\infty,\ i = 1,2,\cdots,m.$$

Proof. See [42]. □

7 Convergence analysis

Now, we show that our numerical method converges to the exact solution.

Theorem 7.1. *Let $\tilde{v}(y)_{i,M}$ and $\tilde{v}_i(y)$, $i = 1,2,\cdots,m$ be the A.S and E.S of system (4.10), respectively. If $k_{i,j}(y,t)$, $i,j = 1,2,\cdots,m$ and $a \leq y,t \leq b$ are bounded and continuous, then $\tilde{v}(y)_{i,M} \to \tilde{v}_i(y)$, as $M \to \infty$.*

Proof. Take

$$\omega = \max_{a \leq y,t \leq b} |k_{i,j}(y,t)| < \infty,$$
$$i,j = 1,2,\cdots,m.$$

Therefore, we have

$$D(\tilde{v}_i(y), \tilde{v}_{i,M}) = D\Big(\sum_{j=1}^{m}\int_a^y k_{i,j}(y,t) \odot \tilde{v}_j(t)dt, \\ \sum_{j=1}^{m}\int_a^y k_{i,j}(y,t) \odot \Big(\sum_{p=1}^{M*} a_{j,m} \odot F_m(t)\Big)dt\Big) \\ \leq \omega\sum_{j=1}^{m}\int_a^y D\Big(\tilde{v}_j(t), \sum_{m=1}^{M*} a_{j,m} \odot F_m(t))\Big)dt.$$

So, we have

$$\lim_{M\to\infty} D(\tilde{v}_i(y), \tilde{v}_{i,M}(y)) \leq$$

$$\omega \lim_{M\to\infty} \sum_{j=1}^{m} \int_a^y D\Big(\tilde{v}_j(t), \sum_{m=1}^{M*} a_{j,m} \odot F_m(t)\Big) dt.$$

By using Eq.(2.1), we deduce that

$$\tilde{v}_i(y) = \lim_{M\to\infty} \sum_{m=1}^{M*} a_{j,m} \odot F_m(x).$$

Thus

$$\lim_{M\to\infty} D\Big(\tilde{v}_i(y), \sum_{m=1}^{M*} a_{i,m} \odot F_m(y)\Big) \to 0,$$

$$i = 1, \cdots, m.$$

Finally, since ω is bounded value, we conclude the expected result as follows

$$\lim_{M\to\infty} D\Big(\tilde{v}_i(y), \tilde{v}_{i,M}(x)\Big) \to 0,$$

$$i = 1, \cdots, m.$$

□

8 Some numerical examples

To show the effectiveness of the proposed method, we consider three examples of equations of fuzzy Volterra integral with the proposed method. All numerical calculations are performed using Mathematica program. As can be seen from Tables 1, 2 and 3, the absolute error (A.E) of $(\underline{v}(y,s), \overline{v}(y,s))$ is represented by the presented method and Figures 1, 2, 3, 4, 5 and 6 display exact (E.S) and approximate solutions (A.S) and absolute error functions obtained from the presented method for $m = 8, 9$ and $m = 7$ and $y = 0.5, 0.2$ and $y = 0.8$, respectively.

Example 8.1. *Assume in the following fuzzy Volterra integral equation*

$$\underline{f}(y,s) = \frac{1}{2}y^2 e^y(3s^2 - 1) + e^y(3s^2 - 1) + (1 - 3s^2),$$
$$\overline{f}(y,s) = 2y^2 e^y + e^y(3 - s^3) + (s^3 - 3),$$

and kernel functions

$$k(y,t) = t, \quad \lambda = \frac{-1}{2}.$$

The exact solution in this case is given by

$$V(y,s) = \Big(\underline{v}(y,s), \overline{v}(y,s)\Big)$$
$$= \Big(ye^y(3s^2 - 1), ye^y(3 - s^2)\Big).$$

The results are shown in Table 1 and Figures 1 and 2 for $m = 8$ and $y = 0.5$.

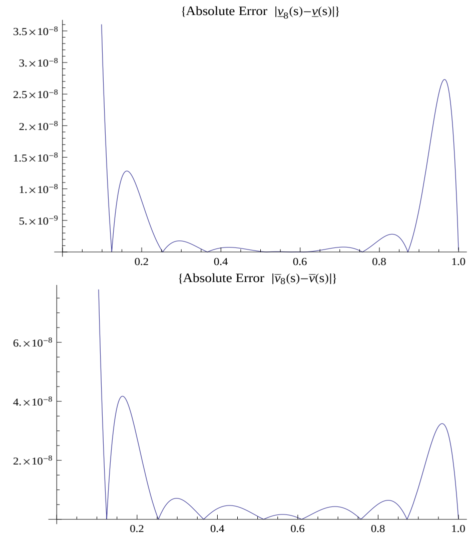

Figure 1: A.E functions from the presented method for $m = 8$ of Example (8.1)

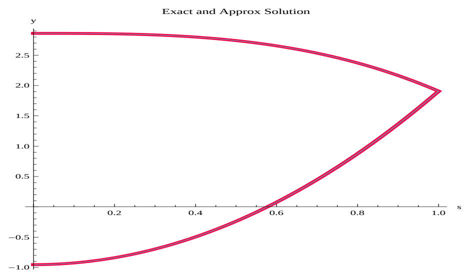

Figure 2: E.S and A.S functions from the presented method for $m = 8$ of Example (8.1)

Example 8.2. *Assume in the following fuzzy Volterra integral equation*

$$\underline{f}(y,s) = (s^2 + s - 1)(y + y^2 + \frac{1}{12}y^4 + 2\cos y + \sin y - 2),$$
$$\bar{f}(y,s) = (2 - s)(y^2 + \frac{1}{12}y^4 + 2\cos y - 2) + (2 - s)(y + \sin y),$$

and kernel functions

$$K(y,t) = (y - t)^2$$

and $\lambda = -1$. We get

$$V(y,s) = \Big(\underline{v}(y,s), \overline{v}(y,s)\Big)$$
$$= \Big((\sin y + y)(s^2 + s - 1), (y + \sin y)(2 - s)\Big),$$

that is the exact solution. The results for $m = 9$ and $y = 0.2$ are shown in Table 2 and Figures 3 and 4.

Table 1: The errors of $(\underline{v}(y,s), \overline{v}(y,s))$ of Example 8.1 for $m = 8$.

s	A.S $\underline{v}_8(y,s)$	A.S $\overline{v}_8(y,s)$	A.E $\lvert\underline{v} - \underline{v}_8\rvert$	A.E $\lvert\overline{v} - \overline{v}_8\rvert$
0.	-0.8243610	0.4730800	3.66058×10^{-10}	1.098170×10^{-9}
0.1	-0.7996300	0.3898200	3.55076×10^{-10}	1.061200×10^{-9}
0.2	-0.7254370	0.3016100	3.22131×10^{-10}	1.022030×10^{-9}
0.3	-0.6017830	0.2035200	2.67222×10^{-10}	9.78472×10^{-10}
0.4	-0.4286680	0.0905800	1.90350×10^{-10}	9.28322×10^{-10}
0.5	-0.2060900	0.9578600	9.15145×10^{-11}	8.69387×10^{-10}
0.6	0.0659489	0.8004000	2.92847×10^{-11}	7.99470×10^{-10}
0.7	0.3874490	0.6132700	1.72047×10^{-10}	7.16375×10^{-10}
0.8	0.7584120	0.3915200	3.36773×10^{-10}	6.17906×10^{-10}
0.9	0.1788400	0.1302000	5.23462×10^{-10}	5.01865×10^{-10}
1.	0.6487200	0.8243610	7.32115×10^{-10}	3.66057×10^{-10}

Table 2: The errors of $(\underline{v}(y,s), \overline{v}(y,s))$ of Example 8.2 for $m = 9$.

s	A.S $\underline{v}_9(y,s)$	A.S $\overline{v}_9(y,s)$	A.E $\lvert\underline{v} - \underline{v}_9\rvert$	A.E $\lvert\overline{v} - \overline{v}_9\rvert$
0.	-0.9794260	1.9588500	2.65836×10^{-10}	5.31671×10^{-10}
0.1	-0.8716890	1.8609100	2.36593×10^{-10}	5.05087×10^{-10}
0.2	-0.7443630	1.7629700	2.02035×10^{-10}	4.75804×10^{-10}
0.3	-0.5974500	1.6650200	1.62160×10^{-10}	4.51921×10^{-10}
0.4	-0.4309470	1.5670800	1.16968×10^{-10}	4.25337×10^{-10}
0.5	-0.2448560	1.4691400	6.64589×10^{-11}	3.98753×10^{-10}
0.6	-0.0391770	1.3712000	1.06334×10^{-11}	3.72170×10^{-10}
0.7	0.1860910	1.2732500	5.05087×10^{-11}	3.45586×10^{-10}
0.8	0.4309470	1.1753100	1.16968×10^{-10}	3.19003×10^{-10}
0.9	0.6953920	1.0773700	1.88743×10^{-10}	2.92419×10^{-10}
1.	0.9794260	0.9794260	2.65836×10^{-10}	2.65836×10^{-10}

Table 3: The errors of $(\underline{v}(y,s), \overline{v}(y,s))$ of Example 8.3 for $m = 7$.

s	A.S $\underline{v}_7(y,s)$	A.S $\overline{v}_7(y,s)$	A.E $\lvert\underline{v} - \underline{v}_7\rvert$	A.E $\lvert\overline{v} - \overline{v}_7\rvert$
0.	0.4794260	1.4382800	5.12281×10^{-8}	1.53684×10^{-7}
0.1	0.4319620	1.3898500	4.61565×10^{-8}	1.48510×10^{-7}
0.2	0.3873760	1.3385600	4.13923×10^{-8}	1.43029×10^{-7}
0.3	0.3485420	1.2815000	3.72429×10^{-8}	1.36933×10^{-7}
0.4	0.3183990	1.2158200	3.40155×10^{-8}	1.29915×10^{-7}
0.5	0.2996410	1.1386400	3.20176×10^{-8}	1.21667×10^{-7}
0.6	0.2953260	1.0470700	3.15565×10^{-8}	1.11882×10^{-7}
0.7	0.3082710	0.9382360	3.29397×10^{-8}	1.00253×10^{-7}
0.8	0.3413510	0.8092700	3.64744×10^{-8}	8.64731×10^{-8}
0.9	0.3974440	0.6572920	4.24681×10^{-8}	7.02338×10^{-8}
1.	0.4794260	0.4794260	5.12281×10^{-8}	5.12281×10^{-8}

Example 8.3. *Assume the following fuzzy Volterra integral equation with*

$$\underline{f}(y,s) = (s^3 - s + 1)(e^y - \cos y),$$
$$\overline{f}(y,s) = (3 - s^3 - s)(\cos y - e^y),$$

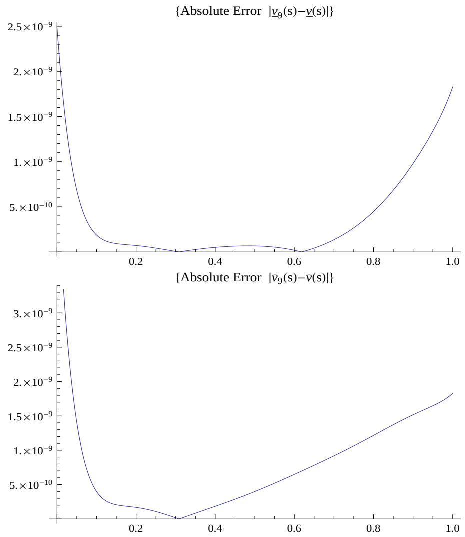

Figure 3: A.E functions from the presented method for $m = 9$ of Example (8.2)

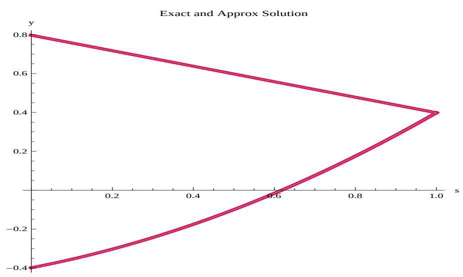

Figure 4: E.S and A.S functions from the presented method for $m = 9$ of Example (8.2)

and kernel functions

$$K(y,t) = e^{y-t}, \quad \lambda = -2.$$

The exact solution in this case is given by

$$\begin{aligned} V(y,s) &= \Big(\underline{v}(y,s), \overline{v}(y,s)\Big) \\ &= \Big(\sin y(s^3 - s + 1), (3 - s - s^3)\sin y\Big). \end{aligned}$$

The results are shown in Table 3 and Figures 5 and 6 for $m = 7$ and $y = 0.8$.

9 Conclusions

In this work, we have studied a numerical scheme to solve system of fuzzy Volterra integral equations of the second kind based on Fibonacci polynomials. Moreover, using a collocation method based on Fibonacci polynomials, we obtained the existence of the solution in Theorem (6.1). In Theorem (7.1), we prove the convergence of the suggested method. Collocation

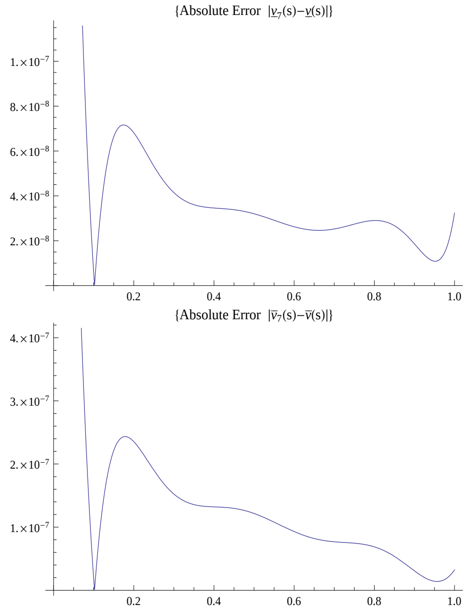

Figure 5: A.E functions from the presented method for $m = 7$ of Example (8.3)

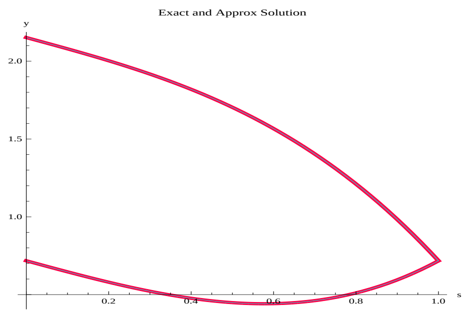

Figure 6: E.S and A.S functions from the presented method for $m = 7$ of Example (8.3)

method is simply applicable and is more efficient to approximate the solution of systems of integral equation (4.10). Also, the introduced method is quick and simple to compute. Finally, in the above presented numerical examples, we saw that the suggested method well preforms for these systems and convergence result is confirmed.

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