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Research Article

A New Method for Solving Multi-Dimensional Fredholm Integral Equations and Its Convergence Analysis

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Abstract

The present study is focused on the development of an approximate solution for multi-dimensional Fredholm integral equations of the second type. To this end, the expansion method was utilized which reduced the multi-dimensional integral equation to a partial differential one. By constructing boundary conditions, this partial differential equation was further reduced to an algebraic equation that could be easily solved with diverse approaches. Furthermore, some theorems were proved for convergence analysis. At last, the efficiency of the method was illustrated through several numerical examples.

Keywords : Multi-dimensional expansion; Multi-dimensional linear and nonlinear Fredholm integral equations; Boundary conditions

1 Introduction

The past fifty years have witnessed significant advancement in the analytical and numerical solutions for various types of linear and nonlinear integral equations. Numerous problems could be modeled in different fields of science as a Fredholm integral equation. Various numerical methods have been developed to solve one and multi-dimensional integral equations (2.6)-(4.14). Further details on analytical solution methods can be found elsewhere [3, 8].

Here, the second type of multi-dimensional linear and nonlinear Fredholm integral equations was considered:

$$f(t) = g(t) + \lambda \int_I k(t,s) f(s) ds \,, t \in I, \qquad (1.1)$$

$$f(t) = g(t) + \lambda \int_I k(t,s) V(f(s)) ds \,, t \in I, \qquad (1.2)$$

where λ is a constant, $I = \underbrace{[a,b] \times \ldots \times [a,b]}_{n\ times} \subseteq \mathbb{R}^n$, and $f(s)$ denotes the unidentified function, $f, g, k \in C^n(I) \cdot V(f(s))$ is a nonlinear continuous function of $f(t)$.

The existence and uniqueness of the solution for a two-dimensional (NIE) were explored by the degenerate kernel approach [1], [2].

Existence and uniqueness of Eqs. (1.1) - (1.2) can be found in Refs. [6, 9, 10, 12, 17].

The development of a high-order numerical

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scheme is of crucial importance as the essential characteristics of the model can be extensively applied to the real world.
In this research, a numerical scheme is proposed based on a simple and fast approach to cope with the difficulties of solving multi-dimensional integral equations.
The rest of the paper is organized as follows: Section 2 addresses the solution of the linear multi-dimensional integral equation; while the solution of the nonlinear multi-dimensional integral equation is presented in Section 3.
The solution of $f(s)$ in Eqs. (1.1) - (1.2) can be expanded as follows:

$$f(s) = f(t) + \sum_{k=1}^{m} \frac{1}{k!} D^k f(t) \cdot \Big(\underbrace{h \cdot h, \cdots, h}_{k \ times} \Big) + R_m(t,c), \tag{1.3}$$

or

$$f(s) = f(t) + \sum_{k=1}^{m} \frac{1}{k!} \left(\frac{\partial}{\partial t_1} + \cdots + \frac{\partial}{\partial t_n} \right)^k f(t) \cdot \underbrace{(h \cdot h, \cdots, h)}_{ktimes} + R_m(t,c),$$

The above equation is the Taylor expansion of $f(s)$ function around the t point. In this equation, the variable of s-t is changed with h. Noteworthy, n shows the dimensional of the integral equation, while k represents the terms of Taylor expansion varying from 0 to m. D^k also denotes the k-times derivative of $f(s)$ at t point. More over, $k \leq m$.
R_m is the truncation error of Taylor expansion and c stands for a point on line segment of s-t.

2 Solution for linear multi-dimensional integral equation

By substituting the first m terms of Eq. (1.3) by $f(s)$ in Eq. (1.1) and ignoring the term $\int_I k(t,s) R_m(t,c) ds$, one can find:

$$f(t) - \lambda \int_I k(t,s) \Bigg(f(t) + \sum_{k=1}^{m} \frac{1}{k!} \left(\frac{\partial}{\partial t_1} + \cdots + \frac{\partial}{\partial t_n} \right)^k f(t) \cdot \underbrace{(h \cdot h, \cdots, h)}_{k \ times} \Bigg) ds \simeq g(t). \tag{2.4}$$

Therefore, Eq. (2.4) can be developed into a partial differential equation. Nonetheless, this partial differential equation requires suitable boundary conditions.
To construct the boundary conditions, both sides of Eq. (1.1) were first differentiated to reach the following differential equations:

$$\begin{cases} \frac{\partial f}{\partial t_1} = \frac{\partial g}{\partial t_1} + \lambda \int_I \frac{\partial k(t,s)}{\partial t_1} f(s) ds, \\ \vdots \\ \frac{\partial^i f(t)}{\partial t_j^i} = \frac{\partial^i g}{\partial t_j^i} + \lambda \int_I \frac{\partial^i k(t,s)}{\partial t_j^i} f(s) ds. \end{cases} \tag{2.5}$$

Where $i = 1, \ldots, m$, $j = 1, \ldots, n$. Next, $f(t)$ is the first term of Eq. (1.3) which replaced $f(s)$.

$$\frac{\partial^i}{\partial t_j^i} f(t) \simeq \frac{\partial^i}{\partial t_j^i} g(t) + \lambda \int_I \frac{\partial^i}{\partial t_j^i} k(t,s) f(s) ds. \tag{2.6}$$

Now, a combination of Eqs. (2.4) and (2.6) will leads to a linear equation.

3 Solution for nonlinear multi-dimensional integral equation

By substituting the first m terms of Eq. (1.3) with $f(s)$ in Eq. (1.1) and ignoring the term $\int_I k(t,s) R_m(t,c) ds$, we will have:

$$f(t) - \lambda \int_I k(t,s) V(f(t) + \sum_{k=1}^{m} \frac{1}{k!} \left(\frac{\partial}{\partial t_1} + \cdots + \frac{\partial}{\partial t_n} \right)^k f(t) \cdot \underbrace{(h \cdot h, \cdots, h)}_{k \ times}) ds \simeq g(s). \tag{3.7}$$

Thus, Eq. (3.7) is a partial differential equation that requires proper boundary conditions.
To this end, both sides of Eq. (1.2) were differentiated to reach the following differential equations:

$$\begin{cases} \frac{\partial f}{\partial t_1} = \frac{\partial g}{\partial t_1} + \lambda \int_I \frac{\partial k(t,s)}{\partial t_1} V(f(s))ds, \\ \vdots \\ \frac{\partial^i f(t)}{\partial t_j^i} = \frac{\partial^i g}{\partial t_j^i} + \lambda \int_I \frac{\partial^i k(t,s)}{\partial t_j^i} V(f(s))ds. \end{cases} \quad (3.8)$$

Where $i = 1, \ldots, m$, $j = 1, \ldots, n$. Next, $f(s)$ is substituted by $f(t)$ to obtain,

$$\frac{\partial^i}{\partial t_j^i} f(t) \simeq \frac{\partial^i}{\partial t_j^i} g(t) + \lambda \int_I \frac{\partial^i}{k}(t,s) V(f(t))ds. \quad (3.9)$$

Now a combination of Eqs. (3.7) and (3.9) will be a nonlinear equation.
To further illustrate the method, it was utilized for $n = 2$:

$$f(x,y) - \lambda \int_a^b \int_a^b k(x,y,s,t) V\left(\sum_{i=0}^{m} \sum_{j=0}^{m} \frac{1}{i!\,j!} \frac{\partial^i \partial^j}{\partial x^i \partial y^j} f(x,y)(s-x)^i (t-y)^j \right) ds\ dt \simeq g(x,y). \quad (3.10)$$

The above partial differential equations are of m-order and require appropriate boundary conditions. which can be established by differentiating both sides of Eq. (1.1) for $n = 2$ to obtain the following differential equations:

$$\begin{cases} \frac{\partial f(x,y)}{\partial x} = \frac{\partial g(x,y)}{\partial x} \\ \qquad + \lambda \int_a^b \int_a^b \frac{\partial k(x,y,s,t)}{\partial x} V(f(s,t))ds\ dt, \\ \frac{\partial f(x,y)}{\partial y} = \frac{\partial g(x,y)}{\partial y} \\ \qquad + \lambda \int_a^b \int_a^b \frac{\partial k(x,y,s,t)}{\partial y} V(f(s,t))ds\ dt, \\ \qquad \vdots \\ \frac{\partial^i \partial^j}{\partial x^i \partial y^j} f(x,y) = \frac{\partial^i \partial^j}{\partial x^i \partial y^j} g(x,y) \\ + \lambda \int_a^b \int_a^b \frac{\partial^i \partial^j}{\partial x^i \partial y^j} k(x,y,s,t) V(f(s,t))ds\ dt. \end{cases} \quad (3.11)$$

The first term of Taylor's expansion of $f(s,t)$ is introduced in Eq. (1.3) for $n = 2$. $f(s,t)$ is then replaced by $f(x,y)$ in the above system for each $j = 1, \ldots, m$. In other words,

$$\frac{\partial^i \partial^j}{\partial x^i \partial y^j} f(x,y) \simeq \frac{\partial^i \partial^j}{\partial x^i \partial y^j} g(x,y) + \lambda \int_a^b \int_a^b \frac{\partial^i \partial^j}{\partial x^i \partial y^j} k(x,y,s,t) V(f(x,y))ds\ dt. \quad (3.12)$$

Now using a combination of Eqs. (3.10) and (3.12), one can deduce that

$$f(x,y) - \lambda \int_a^b \int_a^b k(x,y,s,t) V(A_{ij} + B_{ij})ds\ dt \simeq g(x,y). \quad (3.13)$$

Where

$$A_{ij} = \sum_{i=0}^{m} \sum_{j=0}^{m} \frac{1}{i!\,j!} \frac{\partial^i \partial^j}{\partial x^i \partial y^j} g(x,y),$$

And

$$B_{ij} = \lambda V(f(x,y)) \left[\sum_{i=0}^{m} \sum_{j=0}^{m} \frac{1}{i!\,j!} \int_a^b \int_a^b \frac{\partial^i \partial^j}{\partial x^i \partial y^j} k(x,y,s,t)ds\ dt \right] \times (s-x)^i (t-y)^j.$$

Note that Eq. (3.13) is a nonlinear equation that can offer the desired approximation $f_n(x, y)$. This equation can be solved by iterations of a nonlinear solver; here, Newtons method was utilized.

4 Convergence analysis

This section shows the convergence properties of the presented scheme. Let $(C[I], \|.\|)$ Be the space of all continuous functions on interval I with the following norm

$$\|g(t)\|= \max |g(t)|_{\forall t\in I}.$$

For the error associated with the proposed Taylor series expansion method, Eq. (3.7) is considered as:

$$f(t) = g(t) + \lambda \int_I k(t,s) V\left(\sum_{k=0}^{m} \frac{1}{k!} D^k f(t) \underbrace{(h,h,\dots,h)}_{k\ times} + R_m(t,C)ds\right). \quad (4.14)$$

By differentiating both sides of Eq. (1.2), we will arrive at:

$$f^{(i)}(t) = g^{(i)}(t) + \lambda \int_I k^{(i)}(t,s) V(f(s))ds. \quad (4.15)$$

For $i = 1,\dots,m$. $f(s)$ (Eq. (1.3)) can be applied in Eq. (4.15). In other words, Eq. (4.15) can be rewritten as:

$$f^{(i)}(t) = g^{(i)}(t) + \lambda \int_I k^{(i)}(t,s) V\left(\sum_{k=0}^{m} \frac{1}{k!} D^k f(t)\cdot(h,h,\dots,h) + R_m(t,C)ds\right)ds. \quad (4.16)$$

$\bar{f}^{(i)}(t)$ is the approximate solution of the abovementioned numerical method thus it can be used in Eq. (4.14)

$$\bar{f}^{(i)}(t) = g^{(i)}(t) + \lambda \int_I k^{(i)}(t,s) V\left(\sum_{k=0}^{m} \frac{1}{k!} D^k \overline{f(t)}(h,h,\dots,h)\right)ds. \quad (4.17)$$

The following equation is achieved based on Eqs. (4.16) and (4.17) and using the Lipshitz condition and the mean value theorem

$$\left(f^{(i)}(t) - \bar{f}^{(i)}(t)\right) - \lambda \int_I k^{(i)}(t,s)\frac{\partial V}{\partial f}(\theta_i) \sum_{k=0}^{m} \frac{1}{k!} D^k \left(f(t) - \overline{f(t)}\right)\cdot(h,h,\dots,h)ds$$
$$= \lambda \int_I k^{(i)}(t,s)\frac{\partial V}{\partial f}(\theta_i) R_m(t,c)ds. \quad (4.18)$$

For some θ_i, $s,t \in I$ and $i = 1,\dots,m$. let

$$\varepsilon_i \equiv f^{(i)}(t) - \bar{f}^{(i)}(t),$$

$$a_{ij} \equiv \delta_{ij} - \lambda \int_I k^{(i)}(t,s)\frac{\partial V}{\partial f}(\theta_i)(h,h,\dots,h)ds,$$

$$f_i \equiv \lambda \int_I k^{(i)}(t,s)\frac{\partial V}{\partial f}(\theta_i) R_m(t,c)ds.$$

For $i = 1,\dots,m$, $j = 1,\dots,n$. Then, for each $s,t \in I$ the error $\varepsilon_i(s)$ of Taylor series, the expansion method must satisfy the following matrix equation

$$A_n \tilde{\varepsilon}_n = F_n.$$

Where $A_n = [a_{ij}]$, $\tilde{\varepsilon}_n = [\varepsilon_i]$ and $F_n = [f_i]$ for $i = 1,\dots,m$, $j = 1,\dots,n$. Let $\|.\|$ denote a vector norm as well as its corresponding matrix norm, then

$$\|\tilde{\varepsilon}_n\| \le \|A_n^{-1}\|\|F_n\|.$$

5 Numerical examples

The accuracy and effectiveness of the method are explored in this section. The computations were carried out using Mathematica 7 software on a personal computer.

Example 5.1. *Consider a two-dimensional linear Fredholm integral equation [13]*

$$f(x,y) = 1 - \frac{1}{xy}\left(e^{-4(x+y)} - e^{-4y} - e^{-4x} + 1\right) + \int_0^4 \int_0^4 e^{-xs-yt} f(s,t)ds\ dt.$$

Where $(x,y) \in [0,4) \times [0,4)$. *The exact solution is* $f(x,y) = 1$.

This problem was considered in Ref. [13]. In 2010, Liang and Lin introduced a fast numerical scheme based on piecewise polynomial interpolation and quadrature rules. Their result involved 4 iterations and $N = 512$ (N^2 denotes the number of quadrature points) which showed the error of 2.00e - 10. The results of the current study are listed in Table 1.

Table 1: Numerical solution of Example 5.1 for $m = 1$

(x, y)	Absolute error
$(.5, .5)$	1.99×10^{-15}
$(.1, .1)$	4.55×10^{-15}
$(1.5, 1.5)$	1.33×10^{-15}
$(2, 2)$	0.
$(2.5, 2.5)$	0.
$(3, 3)$	1.11×10^{-16}
$(3.5, 3.5)$	3.33×10^{-16}
$(4, 4)$	1.11×10^{-16}
E_∞	4.55×10^{-15}

Example 5.2. *Consider a three-dimensional linear Fredholm integral equation*

$$f(x,y,z) = \frac{1}{10}(10\sqrt{x} - xyz) + \int_0^1\int_0^1\int_0^1 xyzrst f(r,s,t)drdsdt.$$

Where $(x,y,z) \in [0,1) \times [0,1) \times [0,1)$.
The exact solution is $f(x,y,z) = \sqrt{x}$. *The solution for* $f(x,y,z)$ *can be determined by expansion as described in Section 2 whose results are presented in Table 2 for* $m = 8, 10,$ *and* 20.

Table 2: Numerical solution of Example 5.2

(x, y, z)	Exact $f(x,y,z)$	Error $m = 8$
$(.2, 0, .2)$	0.447214	5.55112×10^{-17}
$(.4, 0, .4)$	0.632456	1.11022×10^{-16}
$(.6, 0, .6)$	0.774597	2.22045×10^{-16}
$(.7, .1, .3)$	0.836660	4.26189×10^{-6}
$(.8, 0, .8)$	0.894427	1.11022×10^{-16}
$(.8, .4, 0)$	0.894427	1.022×10^{-16}
$(1, 0, 1)$	1	9.1587×10^{-6}

(x, y, z)	Error $m = 10$	Error $m = 20$
$(.2, 0, .2)$	0.	0.[1]
$(.4, 0, .4)$	1.11022×10^{-16}	1.11022×10^{-16}
$(.6, 0, .6)$	0.	0.
$(.7, .1, .3)$	3.89667×10^{-6}	5.45793×10^{-6}
$(.8, 0, .8)$	0.	0.
$(.8, .4, 0)$	0.	0.
$(1, 0, 1)$	0.	0.

Example 5.3. *Consider a three-dimensional linear Fredholm integral equation*

$$5f(x,y,z) = \frac{5xyz - \sin x \sin y \sin z}{xyz} + \int_0^1\int_0^1\int_0^1 \cos xr \cos sy \cos zt \, f(r,s,t)drdsdt.$$

Where $(x,y,z) \in [0,1) \times [0,1) \times [0,1)$.
While $f(x,y,z) = 1$ *is an exact solution, the solution of* $f(x,y,z)$ *can be obtained by expansion as described in Section 2. The results are listed in Table 3 for* $m = 1$.

Table 3: Numerical solution of Example 5.3 for $m = 1$

(x, y, z)	Exact $f(x,y,z)$	Error
$(.1, .1, .1)$	1	1.64093×10^{-10}
$(.1, .3, .1)$	1	3.3884×10^{-13}
$(.3, .3, .3)$	1	5.08482×10^{-14}
$(.5, .5, .5)$	1	1.33227×10^{-15}
$(.7, .7, .7)$	1	2.24045×10^{-16}
$(.9, .9, .9)$	1	2.24045×10^{-16}
$(1, .4, .6)$	1	4.44089×10^{-16}

Example 5.4. *Consider the following three-dimensional nonlinear Fredholm integral equation*

$$f(x,y,z) = \frac{1}{72}(-7 - \cos 2 + 72xz\cos y - 5\sin 2) + \int_0^1\int_0^1\int_0^1 (s+t+r) f^2(r,s,t)drdsdt.$$

Where $(x, y, z) \in [0, 1) \times [0, 1) \times [0, 1)$.
$f(x, y, z) = xz \cos y$ *is the exact solution. The solution for* $f(x, y, z)$ *can be attained by Taylor-series presented in Section 2 whose results are listed in Table 4 for* $m = 8$.

Table 4: Numerical solution of Example 5.4 for $m = 8$

(x, y, z)	Exact $f(x, y, z)$	Error
$(0, 0, 0)$	0.	3.25834×10^{-8}
$(.2, .2, .2)$	0.0392027	1.03091×10^{-8}
$(.2, .4, 0)$	0	9.19683×10^{-10}
$(.4, .4, .4)$	0.14737	9.19683×10^{-10}
$(.6, .6, .6)$	0.297121	1.17736×10^{-9}
$(.8, .8, .8)$	0.445892	2.75135×10^{-8}
$(.9, .3, .1)$	0.0859803	1.37462×10^{-11}
$(1, 1, 1)$	0.540302	3.06287×10^{-7}

Example 5.5. *Consider a three-dimensional nonlinear Fredholm integral equation*

$$f(z, x, y) = \frac{1}{144}(-11 + 3\cos^3 1 + 144xz\cos y - 4\sin 2) + \int_0^1\int_0^1\int_0^1 (sr\sin t + 1) f^2(r, s, t)drdsdt.$$

Where $(x, y, z) \in [0, 1) \times [0, 1) \times [0, 1)$.
The exact solution is $f(x, y, z) = xz \cos y$. *The solution for* $f(x, y, z)$ *can be obtained by expansion. The results are presented in Table 5 for* $m = 10$.

Table 5: Numerical solution of Example 5.5 for $m = 10$

(x, y, z)	Exact $f(x, y, z)$	Error
$(0, 0, 0)$	0.	0.00156026
$(.2, .2, .2)$	0.0392027	0.00156026
$(.4, .4, .4)$	0.14737	0.00156026
$(.6, .6, .6)$	0.297121	0.00156026
$(.8, .8, .8)$	0.445892	0.00156026
$(1, 1, 1)$	0.540302	0.00156026

Example 5.6. *Consider the following two-dimensional nonlinear Fredholm integral equation [4]*

$$f(x, y) = x\cos y - \frac{1}{8} - \frac{7}{24}(\sin 1)(\cos 1) - \frac{1}{12}(\cos 1)^2 + \int_0^1\int_0^1 (s + t) f^2(s, t)dsdt.$$

Where $(x, y) \in [0, 1) \times [0, 1)$.

The exact solution is $f(x, y) = x \cos y$.
The present approach gives the absolute error of the order of 10^{-9} *for* $m = 10$. *The Mathematica software was utilized to implement the developed approach using the routine command of Findroot. In this routine command, the default iteration setting is at most* 100 *until the approach converges to the desired solution and the solution time is* 00 : 02.45.93. *Recently, this example has been solved by the Chebyshev collocation method in [4] with the best absolute error of* $E_\infty = 2.3e - 8$.

Table 6: Numerical solution of Example 5.6 for $m = 10$

(x, y)	Absolute error
$(0, 0)$	1.86084×10^{-9}
$(.2, .2)$	4.2823×10^{-10}
$(.4, .4)$	2.23158×10^{-11}
$(.6, .6)$	2.18681×10^{-11}
$(.8, .8)$	9.08×10^{-10}
$(1, 1)$	1.58983×10^{-9}
E_∞	3.51×10^{-9}

6 Conclusion

The current article proposed an expansion approach for solving multi-dimensional Fredholm integral equations. This technique is highly simple with low computational costs. Analytical solution of multi-dimensionalFredholm integral equations is a challenging task, requiring an approximate solution in many cases. Therefore, the developed technique can be extended for other classes of integral equations such as multi-dimensional Volterra integral equations and multi-dimensional mixed Volterra- Fredholm integral equations

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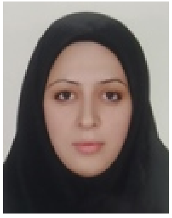

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