Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 15, No. 1, 2023 Article ID IJIM-1312, 10 pages
DOI: http://dx.doi.org/10.30495/ijim.2022.44028.1312
Research Article

Science and Research Branch (IAU)

Existence and Uniqueness Analysis for a Class of Singular Non-Linear Two-Point Boundary Value Problems by an Optimal Iterative Sequence

E. Shivanian *†

Received Date: 2019-04-11 Revised Date: 2019-10-24 Accepted Date: 2019-10-27

Abstract

In this paper, a constructive proof is presented to study the existence and uniqueness of the solutions to the following singular problem

$$y''(x) + ny'(x) + \frac{m}{x}y'(x) = f(x, y(x)),$$
$$0 < x \leq 1,$$

with the boundary conditions

$$y'(0) = 0, \quad Ay(1) + By'(1) = C.$$

It is assumed, in general, $f(x, y(x))$ be non-singular with respect to the independent variable x but it is allowed to be singular with respect to y. We apply the Picard iterative sequence by constructing integral equation whose Green's function is not negative. The convergence of this iterative sequence is then controlled by an embedded parameter. The fastest convergence occurs for an optimal embedded parameter which maximizes a special function. This optimization problem brings a sequence with high rate of the convergence to the unique solution in the finite region where $\frac{\partial f}{\partial y}$ has to be positive. Some illustrative examples are given to confirm the validity and reliability of this constructive theory.

Keywords : Singular boundary value problem; Constructive theorem; Existence and uniqueness; Convergence.

1 Introduction

Consider the two-point boundary value problems of the type

$$y''(x) + ny'(x) + \frac{m}{x}y'(x) = f(x, y(x)),$$
$$0 < x \leq 1, \quad m > 0, \; n \in \mathbb{R}, \tag{1.1}$$
$$y'(0) = 0, \quad Ay(1) + By'(1) = C, \tag{1.2}$$

*Corresponding author. shivanian@sci.ikiu.ac.ir, Tel:+98(912)6825371.

†Department of Applied Mathematics, Imam Khomeini International University, Qazvin, Iran.

which is more general than that considered by W.F. Ford, J. A. Pennline [9] due to the term $ny'(x)$ in where $n \in \mathbb{R}$. This BVP arises in biology and some diseases such as the study of various tumor growth problems. In boundary conditions (1.2), we have, in fact, $A \neq 0$, A and B have the same sign and $C \in \mathbb{R}$, but it is supposed, without loss of generality, $A > 0$ and $B, C \geq 0$ (if $C < 0$ then we can apply $y \to -y$). It is supposed that $f(x, y(x))$ be non-singular with respect to the independent variable $x \in [0,1]$ but it can be singular with respect to y.

We say $y(x)$ is the solution of the BVP (1.1)-(1.2) provided that three following conditions hold [9]

I. $y(x) \in C^1[0,1] \cap C^2(0,1]$.

II. $y(x)$ satisfies the boundary conditions (1.2).

III. For all $x \in [0,1]$, $(x, y(x))$ belongs to a domain of $f(x, y(x))$ i.e. a subset of $[0,1] \times (-\infty, \infty)$ where f is continuous in (x, y), and (1.1) is satisfied.

The nonlinear two-point boundary value problem (1.1)-(1.2) in some specific forms have received much attention in the past. In the well-known monograph by Keller [13], in the case $m = n = 0$, it was given some results by considering very strict assumptions. In fact, these assumptions, which were based on subtracting $k^2 y$ from both sides of (1.1), are the continuity, being non-negative and boundedness of $\frac{\partial f}{\partial y}$, and more k^2 was restricted to be greater than or equal to the bound on $\frac{\partial f}{\partial y}$.

Later, an application of the method of successive approximations for obtaining the solution of a nonlinear integral equation arising from a two-point boundary value problem (1.1)-(1.2), in the case $m = n = 0$ with specific boundary conditions, was illustrated in [15]. Then, J.A. Pennline [16] presented some constructive existence and uniqueness theorems for the problem (1.1)-(1.2) when $m = n = 0$. Furthermore, in the case of $m = 0$ and n replaced by a function of independent variable having a continuous derivative, J.A. Pennline [17] again established some constructive existence and uniqueness theorems. In these works, the assumptions are made to be restricted, i.e. previous assumption on $\frac{\partial f}{\partial y}$ was restricted within a finite region $0 \leq y \leq M$ or $|y| \leq M$, and further, the value for k^2 was smaller than before i.e. equal to half the bound on $\frac{\partial f}{\partial y}$.

On the other hand, the boundary value problems (1.1)-(1.2) involving the governing ordinary differential equation of (1.1) or slight generalizations of it, in the case of $n = 0$, have been investigated by Gatica et al. [10], Fink et al. [8], Baxley [3], Baxley and Gersdorff [4], Wang and Li[19], Tinio [18], Wang [20], [7], Agarwal and O'Regan [1] and also the authors of [11, 21]. In these works, the fixed point theory or approximation theory was prevalently used and their studies included assumptions that restrict $f(x, y)$ to be of one sign and usually continuous for $y \geq 0$. Later, in Refs [2, 12, 14], authors started to allow sign-changing nonlinearities but still they have required $f(x, y)$ to be continuous for $y \geq 0$. We notice that it has been assumed, in all these works, $n = 0$ and some particular types of boundary conditions (1.2) or some specific forms of governing differential equation (1.1).

Very recent investigation that is very close to the boundary value problem (1.1)-(1.2) has been studied by W.F. Ford and J.A. Pennline [9]. They have considered the nonlinear two-point boundary value problem (1.1)-(1.2), in the case of $n = 0$, and shown some constructive existence and uniqueness theorems with these assumptions that $f(x, y(x))$ be non-singular with respect to the independent variable x (but it can be singular with respect to y) and $\frac{\partial f}{\partial y}$ be continuous in some closed finite regions.

The aim of the present work is to provide some constructive existence and uniqueness theorems for the problem (1.1)-(1.2) as the same as those provided by W.F. Ford and J.A. Pennline in [9] by the same assumptions considered by them, i.e. $f(x, y(x))$ not only be allowed sign-changing but also it can be singular with respect to y. The same as their work, the only restriction for $f(x, y(x))$ is to be non-singular with respect to the independent variable x. Finally, our main requirement is that $\frac{\partial f}{\partial y}$ be continuous in some closed region $\mathbf{D} : [0,1] \times [y_L(x), y_U(x)]$, and our purpose is to search that region and prove that a unique solution to the boundary value problem (1.1)-(1.2) exists within it.

2 Constructing integral equation

We subtract $k^2 y$ from both sides of Eq. (1.1) so that the differential equation is converted to

$$\begin{aligned} & y''(x) + ny'(x) + \frac{m}{x}y'(x) - k^2 y(x) \\ & = f(x, y(x)) - k^2 y(x). \qquad (2.3) \end{aligned}$$

Let us, now, consider the homogeneous type of the above equation, i.e.

$$\begin{aligned} & \omega''(x) + n\omega'(x) + \frac{m}{x}\omega'(x) - k^2\omega(x) = 0, \\ & m > 0, \;\; n \in \mathbb{R}, \qquad (2.4) \end{aligned}$$

with two homogeneous conditions

$$\omega'(0) = 0, \quad A\omega(1) + B\omega'(1) = 0. \qquad (2.5)$$

Suppose that $u_m(x)$ and $v_m(x)$ are given as follow

$$\begin{aligned} & u_m(x) = \\ & e^{\frac{1}{2}x\left(-\sqrt{4k^2+n^2}-n\right)} L_\alpha^{m-1}\left(x\sqrt{4k^2+n^2}\right), \\ & \alpha = -\frac{m\left(\sqrt{4k^2+n^2}+n\right)}{2\sqrt{4k^2+n^2}}, \qquad (2.6) \\ & v_m(x) = \\ & e^{\frac{1}{2}x\left(-\sqrt{4k^2+n^2}-n\right)} U\left(\beta, m, \sqrt{4k^2+n^2}x\right), \\ & \beta = \frac{m\left(n+\sqrt{4k^2+n^2}\right)}{2\sqrt{4k^2+n^2}}, \qquad (2.7) \end{aligned}$$

where, L_α^{m-1} is generalized Laguerre polynomial which is related to hydrogen atom wave functions in quantum mechanics, further, $U(a, b, z)$ is the Hypergeometric function which is a second linearly independent solution to Kummer's equation and defined by

$$U(a, b, z) = \frac{1}{\Gamma(a)}\int_0^\infty t^{a-1}(t+1)^{b-a-1}e^{-tz}\mathrm{d}t. \qquad (2.8)$$

Now, we define

$$\begin{aligned} u(x) &= u_m(x), \qquad (2.9) \\ v(x) &= v_m(x) - S u_m(x), \\ S &= \frac{A v_m(1) + B v_m'(1)}{A u_m(1) + B u_m'(1)}. \qquad (2.10) \end{aligned}$$

It can be easily seen that $u(x)$ and $v(x)$ are the solutions of Eq. (2.4) so that they hold $u'(0) = u_m'(0) = 0$ and $Av(1) + Bv'(1) = 0$, respectively. We will give some properties of these functions in the next section.

Assume that $\varphi(x)$ satisfies

$$\varphi''(x) + n\varphi'(x) + \frac{m}{x}\varphi'(x) - k^2\varphi(x) = 0, \qquad (2.11)$$

$$\varphi'(0) = 0, \quad A\varphi(1) + B\varphi'(1) = C. \qquad (2.12)$$

In fact, $\varphi(x)$ can be expressed in terms of $u(x)$, as follows:

$$\varphi(x) = \frac{C u(x)}{A u(1) + B u'(1)}. \qquad (2.13)$$

Now, consider the differential equation (2.3) with homogeneous boundary conditions

$$y'(0) = 0, \quad Ay(1) + By'(1) = 0, \qquad (2.14)$$

then, for $k^2 \neq 0$, Eq. (2.3) can be converted to an equivalent integral equation by means of the Green's function appropriate to the operator on the left-hand side, as follows:

$$y(x) = \varphi(x) + \int_0^1 G(x, t)[k^2 y(t) - f(t, y(t))]\mathrm{d}t, \qquad (2.15)$$

where, $G(x, t)$ satisfies

$$G_{xx} + nG_x + \frac{m}{x}G_x - k^2 G = -\delta(x - t), \qquad (2.16)$$

$$G_x(0, t) = 0, \quad AG(1, t) + BG_x(1, t) = 0, \qquad (2.17)$$

in where $\delta(x)$ is the Dirac delta function. It can be easily discovered from the elementary theory of differential equations that $G(x, t)$ may be expressed as

$$G(x, t) = \frac{1}{W(t)}\left\{\begin{array}{ll} u(x)v(t), & x \leq t \\ v(x)u(t), & x \geq t, \end{array}\right. \qquad (2.18)$$

where $u(x)$ and $v(x)$ are given by Eqs. (2.9)-(2.10) and the Wronskian $W(t)$ is defined by

$$W(t) \equiv v(t)u'(t) - u(t)v'(t)$$

$$= v_m(t)u'_m(t) - u_m(t)v'_m(t)$$

$$= \frac{1}{2}e^{-t\left(\sqrt{4k^2+n^2}+n\right)} \times \tag{2.19}$$

$$\left(\begin{array}{c} m\left(\sqrt{4k^2+n^2}+n\right)\times \\ U\left(\frac{nm}{2\sqrt{4k^2+n^2}}+\frac{m}{2}+1, m+1,\right. \\ \left.\sqrt{4k^2+n^2}t\right) \\ \times L^{m-1}_{\frac{1}{2}m\left(-\frac{n}{\sqrt{4k^2+n^2}}-1\right)}\left(t\sqrt{4k^2+n^2}\right) \\ -2\sqrt{4k^2+n^2}\times \\ U\left(\frac{1}{2}m\left(\frac{n}{\sqrt{4k^2+n^2}}+1\right), m,\right. \\ \left.\sqrt{4k^2+n^2}t\right) \\ \times L^{m}_{-\frac{mn}{2\sqrt{4k^2+n^2}}-\frac{m}{2}-1}\left(t\sqrt{4k^2+n^2}\right) \end{array}\right)$$

$$= W(1)e^{n-nt}t^{-m}, \tag{2.20}$$

in where, obviously, $W(1)$ is in terms of n, m, k. In this section, we present some properties of the Green's function as theorems which are extremely important to our analyzing on (1.1)-(1.2).

Lemma 2.1. *Taking into account $A > 0$, $B \geq 0$, $m > 0$ and $n \in \mathbb{R}$, the functions $u(x)$ and $v(x)$, given by (2.9)-(2.10), satisfy the following properties*

- *The function $v(x)$ is a positive decreasing function of $x \in [0,1]$ and unbounded at the origin.*
- *Assuming m, n and k are so that $W(1)$ is positive, then $u(x)$ is positive increasing function of $x \in [0,1]$.*
- *Assuming m, n and k are so that $W(1)$ is negative, then $u(x)$ is negative decreasing function of $x \in [0,1]$.*

Proof. It is straightforward from the definitions (2.9)-(2.10) and Wronskian $W(t)$ at $t = 1$.

Theorem 2.1. *The Green's function $G(x,t)$ is always non-negative, i.e.*

$$\forall k^2, m > 0, \forall n \in \mathbb{R}, \forall x,t \in [0,1]: \quad G(x,t) \geq 0. \tag{2.21}$$

Proof. Please see the detailed proof in Refs. [6, 5].

Theorem 2.2. *The Green's function $G(x,t)$ is bounded in such a way that, i.e.*

$$\begin{aligned} &\forall k^2, m > 0, \forall n \in \mathbb{R}, \\ &\forall x \in [0,1]: \quad k^2 \int_0^1 G(x,t)\,dt \leq \mu(k) < 1. \end{aligned} \tag{2.22}$$

Proof. Suppose that $\omega(x)$ satisfies the differential equation (2.4), then it is easily seen that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[t^m e^{nt}\omega'(t)\right] = \\ &t^{m-1}e^{nt}\left(m\omega'(t) + nt\omega'(t) + t\omega''(t)\right) = \\ &t^{m-1}e^{nt}\left(k^2 t\omega(t)\right) = k^2 t^m e^{nt}\omega(t), \end{aligned} \tag{2.23}$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{t^m e^{nt}\omega'(t)}{W(1)e^n}\right] = \frac{k^2 t^m e^{nt}\omega(t)}{W(1)e^n}, \tag{2.24}$$

therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\omega'(t)}{W(t)}\right] = \frac{k^2\omega(t)}{W(t)}. \tag{2.25}$$

Now, with the help of this latest equation, we obviously have

$$k^2 \int_0^1 G(x,t)\mathrm{d}t = 1 + \frac{u(x)v'(1)}{W(1)}. \tag{2.26}$$

Lemma 2.1 confirms that $v'(1) < 0$ and $\frac{u(x)}{W(1)} \geq 0$ then $\frac{u(x)v'(1)}{W(1)} < 0$. On the other hands, Theorem 2.1 reveals $\int_0^1 G(x,t)\mathrm{d}t \geq 0$, therefore we conclude that

$$0 \leq \mu(k) = 1 + \frac{u(x)v'(1)}{W(1)} < 1, \tag{2.27}$$

and the proof is complete.

3 The region of existence and uniqueness

Consider a standard Picard sequence iteration as below which is based on Eq. (2.15)

$$y_{n+1}(x) = \varphi(x) + \int_0^1 G(x,t)[k^2 y_n(t) - f(t, y_n(t))]\mathrm{d}t, \quad n = 0, 1, 2, ... \tag{3.28}$$

or equivalently in the operator form

$$y_{n+1} = \varphi(x) + T[k^2 y_n - f_n], \quad n = 0, 1, 2, ... \tag{3.29}$$

where, $y_n = y_n(x)$, $f_n = f(x, y_n(x))$ and the operator $T : C[0,1] \to C[0,1]$ is defined as

$$T[z(x)] = \int_0^1 G(x,t)z(t)\mathrm{d}t, \quad x \in [0,1]. \tag{3.30}$$

Now, we show that (3.28) can converge uniformly in a finite region such as $\mathbf{D} : [0,1] \times [y_L(x), y_U(x)]$ so that it presents unique solution to the boundary value problem (1.1)-(1.2) within it.

Theorem 3.1. *Consider the boundary value problem (1.1)-(1.2) and suppose the following conditions hold:*

(a) $\frac{\partial f}{\partial y}$ *be continuous in* $\mathbf{D} : [0,1] \times [y_L(x), y_U(x)]$ *and satisfies* $0 \leq \frac{\partial f}{\partial y} \leq N_D$ *within it.*

(b) $y_0(x) = \frac{1}{2}[y_L(x) + y_U(x)]$ *and* $y_L(x) \leq y_n(x) \leq y_U(x), n = 1, 2, 3, \cdots$.

(c) The value of k^2 *satisfies* $k^2 \geq \frac{N_D}{2}$.

Then the Picard sequence iteration (3.29) converges uniformly to a $y(x)$ *as unique solution of the boundary value problem (1.1)-(1.2) in* $\mathbf{D}$.

Proof. Define

$$\Delta y_n(x) = y_n(x) - y_{n-1}(x), \tag{3.31}$$

$$\Delta f_n(x) = f(x, y_n(x)) - f(x, y_{n-1}(x)), \tag{3.32}$$

for $n = 1, 2, 3, \cdots$, then by subtracting two successive iterations, we obtain

$$\Delta y_{n+1} = T[k^2 \Delta y_n - \Delta f_n], \quad n = 1, 2, 3, \cdots. \tag{3.33}$$

Applying the mean value theorem to Δf_n, results in

$$\Delta y_{n+1} = T[(k^2 - \psi_n)\Delta y_n], \quad n = 1, 2, 3, \cdots. \tag{3.34}$$

where,

$$\psi_n(x) = \frac{\partial f}{\partial y}[x, y_n(x) - \theta(x)\Delta y_n(x)], \quad 0 \leq \theta(x) \leq 1. \tag{3.35}$$

Remembering $k^2 \geq \frac{N_D}{2}$ and $0 \leq \frac{\partial f}{\partial y} \leq N_D$ within $\mathbf{D}$, we conclude $-\frac{N_D}{2} \leq k^2 - \psi_n \leq k^2$ and then $0 \leq |k^2 - \psi_n| \leq k^2$. Therefore, Eq. (3.34) leads to

$$|\Delta y_{n+1}(x)| \leq k^2 \int_0^1 G(x,t)|\Delta y_n(t)|\mathrm{d}t \leq k^2 \|\Delta y_n\| \int_0^1 G(x,t)\mathrm{d}t, \tag{3.36}$$

where

$$\|y\| = \max_{0 \leq x \leq 1} |y(x)|. \tag{3.37}$$

Now, application of Theorem 2.2 yields

$$\|\Delta y_{n+1}\| \leq \mu(k)\|\Delta y_n\|, \tag{3.38}$$

in the other words,

$$\|\Delta y_{n+1}\| \leq [\mu(k)]^n \|\Delta y_1\|, \quad \mu(k) < 1. \tag{3.39}$$

Now, we have proved $\{y_n(x)\}$ is a Cauchy sequence with the norm defined by (3.37) then $y_n(x)$ converges uniformly to a function $y(x)$ that satisfies the integral equation (2.15) or equivalently the boundary value problem (1.1)-(1.2). We prove uniqueness of the solution by contradiction, suppose $y_1(x)$ and $y_2(x)$ are two solutions to (1.1). Then, they easily satisfy (2.15). Now, choosing $k^2 \geq \frac{N_D}{2}$ and using the exactly previous analysis that leads to (3.38), we obtain

$$\|y_1 - y_2\| \leq \mu(k)\|y_1 - y_2\|, \tag{3.40}$$

that is contradiction because $\mu(k) < 1$, this completes the proof.

Remark 3.1. If we see the Theorem 3.1, then we realize that rate of the convergence of Picard sequence iteration process depends on $\mu(k)$ which has been defined in Theorem 2.2, the smaller $\mu(k)$ the faster convergence. Therefore, we need to find the optimal value for $\mu(k)$ in order to make convergence fast. This is done by finding maximum value of $|u(x)|$ because we know by definition

$$\mu(k) = 1 + \frac{u(x)v'(1)}{W(1)},$$

$u(x)$ and $W(1)$ have the same sign and more $v'(1)$ is negative. Furthermore, to maximize $|u(x)|$ depends on k^2 and on the other hand it should be hold $k^2 \geq \frac{N_D}{2}$ from Theorem 3.1. We conclude that the optimal value $\mu(k)^*$ occurs for a special value k_*^2 which satisfies $k_*^2 \geq \frac{N_D}{2}$ too.

Our next step is to locate the regions where unique solutions of (1.1)-(1.2) exist. To this end, we suppose the boundary functions $y_L(x)$ and $y_U(x)$ are solutions to

$$y_L''(x) + ny_L'(x) + \frac{m}{x}y_L'(x) = F_U(x), \tag{3.41}$$

$$y_L'(0) = 0, \quad Ay_L(1) + By_L'(1) = C_L, \\ C_L \leq C, \tag{3.42}$$

and

$$y_U''(x) + ny_U'(x) + \frac{m}{x}y_U'(x) = F_L(x), \tag{3.43}$$

$$y_U'(0) = 0, \quad Ay_U(1) + By_U'(1) = C_U, \\ C_U \geq C, \tag{3.44}$$

in which $F_L(x)$ and $F_U(x)$ are given continuous functions. Obviously, these solutions satisfy also the integral equation (2.15), i.e.

$$y_L(x) = \varphi_L(x) + T[k^2 y_L(x) - F_U(x)], \tag{3.45}$$

$$y_U(x) = \varphi_U(x) + T[k^2 y_U(x) - F_L(x)], \tag{3.46}$$

where $\varphi_L(x)$ and $\varphi_U(x)$ are in terms of $\varphi(x)$, in the forms

$$\varphi_L(x) = \frac{C_L}{C}\varphi(x), \quad \varphi_U(x) = \frac{C_U}{C}\varphi(x). \tag{3.47}$$

A very special situation is when $F_L(x)$ and $F_U(x)$ are polynomials of degree one, where we can obtain easily $y_L(x)$ and $y_U(x)$ then find the aim finite region $\mathbf{D}$.

Example 3.1. *Assume $F_L(x)$ and $F_U(x)$ are zeros, then the solutions of (3.41)-(3.42) and (3.43)-(3.44) are found as follows:*

$$y_L(x) = \frac{C_L}{A}, \quad y_U(x) = \frac{C_U}{A}. \tag{3.48}$$

Example 3.2. *Assume $F_L(x)$ and $F_U(x)$ are polynomials of degree one, say $\alpha(1+m+nx)$, then the solutions of (3.41)-(3.42) and (3.43)-(3.44) are found as follows:*

$$y_L(x) = \frac{\alpha}{2}x^2 + \frac{2C_L - \alpha(A+2B)}{2A}, \\ y_U(x) = \frac{\alpha}{2}x^2 + \frac{2C_U - \alpha(A+2B)}{2A} \\ . \tag{3.49}$$

The following Lemma helps us to provide some theorems in order to set finite regions $\mathbf{D}$ where there exists unique solution within them which is the main results of the present paper.

Lemma 3.1. *Suppose a function $H(x)$ continuous on $[0,1]$ satisfies the inequality*

$$H(x) \leq T[k^2 H(x)], \quad x \in [0,1], \quad k^2 > 0, \tag{3.50}$$

then $H(x)$ is not positive on $[0,1]$.

Proof. Suppose that the greatest value of $H(x)$ occurs at $x = x^*$. Now, from Theorems 2.1 and 2.2, we have the following inequalities

$$H(x) \leq T[k^2 H(x)] \leq H(x^*)T[k^2] \leq \\ H(x^*)\mu(k), \quad 0 < \mu(k) < 1, \quad x \in [0,1]. \tag{3.51}$$

Therefore, setting $x = x^*$ yields $H(x^*)[1-\mu(k)] \leq 0$ and then $H(x^*) \leq 0$ hence, it should be held $H(x) \leq 0$ for all $x \in [0,1]$.

Theorem 3.2. *Consider a region $\mathbf{Q}$ as some subset of the region $[0,1] \times (-\infty, \infty)$ within which Eq. (1.1) is well-defined. Suppose there exist continuous functions $F_L(x)$ and $F_U(x)$ satisfying $F_L(x) \leq f(x,y) \leq F_U(x)$ for all $(x,y) \in \mathbf{Q}$ and let the solutions to (3.41)-(3.42) and (3.43)-(3.44) define a region $\mathbf{D} : [0,1] \times [y_L(x), y_U(x)]$. If $\mathbf{D}$ lies entirely outside of $\mathbf{Q}$ then no solution of (1.1)-(1.2) can lie entirely in $\mathbf{Q}$. If $\mathbf{Q}$ contains $\mathbf{D}$ then* *every possible solution $y_p(x)$ of (1.1)-(1.2) that lies entirely in $\mathbf{Q}$ must lie entirely in $\mathbf{D}$, and if in addition $\frac{\partial f}{\partial y}$ is continuous and $\frac{\partial f}{\partial y} > 0$ throughout $\mathbf{D}$, then a unique solution of (1.1)-(1.2) exists in $\mathbf{D}$. It is given by the limit of the sequence (3.29) for $k^2 \geq \frac{N_D}{2}$, where N_D is the maximum value of $\frac{\partial f}{\partial y}$ in $\mathbf{D}$ and $y_0(x) = \frac{1}{2}[y_L(x) + y_U(x)]$.*

Proof. The proof is based on Lemma 3.1 and Theorem 3.1. The procedure is the same as the proof of Theorem (5.1) in Ref. [9].

Theorem 3.3. *Consider a region $\mathbf{Q}$ as some subset of the region $[0,1]\times(-\infty,\infty)$ within which $\frac{\partial f}{\partial y}$ is continuous and $\frac{\partial f}{\partial y} > 0$. Suppose $F_L(x)$ and $F_U(x)$ be continuous functions such that the solutions to (3.41)-(3.42) and (3.43)-(3.44) define a region $\mathbf{D}$: $[0,1] \times [y_L(x), y_U(x)]$ which is contained in $\mathbf{Q}$. If $F_L(x) \leq f(x, y_U(x))$ and $f(x, y_L(x)) \leq F_U(x)$ for all $x \in [0,1]$ and N_D is the maximum quantity of $\frac{\partial f}{\partial y}$ in $\mathbf{D}$, then a unique solution of (1.1)-(1.2) exists in $\mathbf{D}$, and it is given by the limit sequence of (3.29) for $k^2 \geq N_D$ where $y_0(x) = \frac{1}{2}[y_L(x) + y_U(x)]$.*

Proof. Remember the Picard iteration sequence for y_{n+1} as

$$y_{n+1} = \varphi(x) + T[k^2 y_n - f_n]. \tag{3.52}$$

Also, from (3.46) we have

$$y_U(x) = \varphi_U(x) + T[k^2 y_U(x) - F_L(x)] \tag{3.53}$$

Then

$$y_U(x) - y_{n+1}(x) \geq \\ T\left[k^2 y_U - F_L - (k^2 y_n - f_n)\right], \tag{3.54}$$

because $\varphi(x) \leq \varphi_U(x)$ for all $x \in [0,1]$. From the assumptions of the theorem we have $0 \leq \frac{\partial f}{\partial y} \leq N_D$ for all $(x,y) \in \mathbf{D}$. Suppose $k^2 \geq N_D$ then $k^2 y - f(x,y)$ is an increasing function of y for all $(x,y) \in \mathbf{D}$. Therefore, if $y_n(x) \leq y_U(x)$ then $k^2 y_n - f(x, y_n) \leq k^2 y_U - f(x, y_U)$, and (3.54) is converted to

$$\begin{aligned} & y_U(x) - y_{n+1}(x) \geq \\ & T\left[k^2 y_U - F_L - (k^2 y_U - f(x, y_U))\right] \\ & = T\left[f(x, y_U) - F_L\right] \geq 0. \end{aligned} \tag{3.55}$$

then $y_{n+1}(x) \leq y_U(x)$. Also, see the proof of the theorem (5.2) in Ref. [9] to observe that if $y_L(x) \leq y_n(x)$ then $y_L(x) \leq y_{n+1}(x)$. Now, consider $y_0(x) = \frac{1}{2}[y_L(x) + y_U(x)]$ then application of Theorem 3.1 with $k^2 \geq \frac{N_D}{2}$ replaced by $k^2 \geq N_D$ completes the proof.

Theorem 3.4. *Consider a region $\mathbf{Q}$ as some subset of the region $[0,1]\times(-\infty,\infty)$ within which $\frac{\partial f}{\partial y}$ is continuous and $\frac{\partial f}{\partial y} > 0$. Suppose $F_L(x)$, $F_U(x)$ and $y_s(x)$ be continuous functions so that $F_L(x) \leq f(x, y_s(x)) \leq F_U(x)$ for all $x \in [0,1]$. If solutions to (3.41)-(3.42) and (3.43)-(3.44) define a region $\mathbf{D}$: $[0,1] \times [y_L(x), y_U(x)]$ which is contained in $\mathbf{Q}$ and $y_L(x) \leq y_s(x) \leq y_U(x)$, then a unique solution of (1.1)-(1.2) exists in $\mathbf{D}$, and it is given by the limit sequence of (3.29) for $k^2 \geq N_D$ where $y_0(x) = \frac{1}{2}[y_L(x) + y_U(x)]$ and N_D is the maximum value of $\frac{\partial f}{\partial y}$.*

Proof. Since $y_L(x) \leq y_s(x) \leq y_U(x)$ and $\frac{\partial f}{\partial y} > 0$ then $f(x, y_L(x)) \leq f(x, y_s(x)) \leq f(x, y_U(x))$. Therefore, we have $f(x, y_L(x)) \leq F_U(x)$ and $F_L(x) \leq f(x, y_U(x))$ and then the Theorem 3.3 completes the proof. The following theorem can be useful in the applications of Theorems 3.2-3.4 to discover where solutions cannot exist.

Theorem 3.5. *Supposes $y(x)$, with maximum and minimum values equal to y_{max} and y_{min} respectively, be continuous solution of (1.1)-(1.2). Further, assume $\frac{\partial f}{\partial y}$ be bounded in the region $[0,1]\times[y_{min}, y_{max}]$ and y_c denotes the ratio of the boundary value constants $\frac{C}{A}$. Then, $y_{min} \geq y_c$ if $f(x, y_{min}) \leq 0$ for all $x \in [0,1]$ and $y_{max} \leq y_c$ if $f(x, y_{max}) \geq 0$ for all $x \in [0,1]$.*

Proof. Obviously, for any $k^2 > 0$, $y(x)$ and $y_c = \frac{C}{A}$ satisfy

$$y(x) = \varphi(x) + T[k^2 y - f], \tag{3.56}$$
$$y_c = \varphi(x) + T[k^2 y_c], \tag{3.57}$$

then

$$y(x) - y_c = T\left[(k^2 y - f) - k^2 y_c\right]. \tag{3.58}$$

Consider the case $f(x, y_{min}) \leq 0$, and take k^2 large enough so that $k^2 y - f(x,y)$ be an increasing function of y in $[0,1] \times [y_{min}, y_{max}]$. Therefore, Eq. (3.58) yields

$$\begin{aligned} & y(x) - y_c \geq \\ & T\left[(k^2 y_{min} - f(x, y_{min})) - k^2 y_c\right] \\ & \geq T\left[k^2(y_{min} - y_c)\right] \\ & \geq -\mu(k)(y_{min} - y_c). \end{aligned} \tag{3.59}$$

Since this holds for all x, we may replace the left-hand side by its minimum value to obtain

$$y_{min} - y_c \geq -\mu(k)(y_{min} - y_c), \tag{3.60}$$

then, we conclude $y_{min} \geq y_c$. Also, see the proof of the theorem (5.4) in Ref. [9] to observe that $y_{max} \leq y_c$ if $f(x, y_{max}) \geq 0$ for all $x \in [0,1]$.

4 Applications

In this section, we apply the results of the previous section to give some illustrative examples. It is worth-mentioning here that our theory is applicable to all real models introduced in Ref. [9]. We now present two more examples to support our theory for general problem (1.1)-(1.2).

Example 4.1. *Consider the simple problem*

$$\begin{aligned} & y''(x) + ny'(x) + \frac{1}{x}y'(x) = e^{\beta y} - 1, \\ & \beta > 0,\ n \in \mathbb{R} \end{aligned} \tag{4.61}$$

$$y'(0) = 0, \quad y(1) + y'(1) = 1. \tag{4.62}$$

We notify that $f(x,y) < 0, y < 0$ and $f(x,y) > 0, y > 0$ for all $x \in [0,1]$. Now, setting $y_c = \frac{C}{A} = 1$ and applying Theorem 3.5, we conclude it will be contradiction if we assume $y_{min} \leq 0$ or $y_{max} > 1$ so all solutions must be contained in $\mathbf{Q} : [0,1] \times [0,1]$. *In the frame of Theorem 3.3, consider $F_L(x) = F_U(x) = 0$ then, by Example 3.1, $y_L(x) = C_L$ and $y_U(x) = C_U$ are the solutions to (3.41)-(3.42) and (3.43)-(3.44). Now, choosing $C_L = 0$ and $C_U = C = 1$ the inequalities of Theorem 3.3 i.e. $F_L(x) \leq f(x, y_U(x))$ and $f(x, y_L(x)) \leq F_U(x)$ hold for all $x \in [0,1]$. On the other hands, region* $\mathbf{D}$ *formed coincides with* $\mathbf{Q}$ *and we have $0 \leq \frac{\partial f}{\partial y} \leq \beta e^{\beta}$ in* $\mathbf{D}$*, then by Theorem 3.3, a unique solution of (1.1)-(1.2) exists in* $\mathbf{D} : [0,1] \times [0,1]$*, and it is given by the limit sequence of (3.29) for $k^2 = \beta e^{\beta}$ where $y_0(x) = \frac{1}{2}$.*

In the next example, $f(x, y(x))$ not only is allowed sign-changing but also it is singular with respect to y.

Example 4.2. *Consider the nonlinear boundary value problem*

$$\begin{aligned} & y''(x) + 2y'(x) + \frac{1}{x}y'(x) = \\ & \left(\frac{1}{2} - x\right) \exp\left(\frac{x - \frac{1}{2}}{\sin y}\right), \end{aligned} \tag{4.63}$$

$$y'(0) = 0,\ 2y(1) + y'(1) = \frac{\pi}{3} \tag{4.64}$$

In the frame of Theorem 3.4, suppose $y_s(x) = \frac{\pi}{6}$. We clearly have

$$\begin{aligned} & -\frac{1}{2}e(1+x) \leq f(x, y_s) = \left(\frac{1}{2} - x\right) e^{2x-1} \\ & \leq \frac{1}{2}e^{-1}(1+x), \end{aligned} \tag{4.65}$$

then, choosing $F_L(x) = -\frac{1}{2}e(1+x)$ and $F_U(x) = \frac{1}{2}e^{-1}(1+x)$ with $C_L = C_U = C = \frac{\pi}{3}$, by Example 3.2 we obtain

$$y_L(x) = \frac{1}{8}e^{-1}x^2 + \frac{2\pi - 3e^{-1}}{12} > 0, \tag{4.66}$$

$$y_U(x) = -\frac{1}{8}ex^2 + \frac{2\pi + 3e}{12} < \frac{\pi}{2}, \tag{4.67}$$

for all $x \in [0,1]$. Now, the singularity at $y = 0$ is avoided, and y_s is contained within the region $\mathbf{D}$ *whose boundaries are defined by (4.66)-(4.67). Since $\frac{\partial f}{\partial y}$ is continuous within the region* $\mathbf{D}$ *and further $0 \leq \frac{\partial f}{\partial y} \leq N_D$ where N_D is its maximum in the region* $\mathbf{D}$*. Consequently, a unique solution of (1.1)-(1.2) exists in* $\mathbf{D}$*, and it is given by the limit sequence of (3.29) for $k^2 \geq N_D$ where $y_0(x) = \frac{1}{2}[y_L(x) + y_U(x)]$.*

References

[1] R. P. Agarwal, D. ORegan, Nonlinear superlinear singular and nonsingular second order boundary value problems, *Journal of Differential Equations* 143 (1998) 60-95.

[2] R. P. Agarwal, D. ORegan, An upper and lower solution approach for singular boundary value problems with sign changing nonlinearities, *Mathematical Methods in the Applied Sciences* 25 (2002) 491-506.

[3] J. V. Baxley, Some singular nonlinear boundary value problems, *SIAM Journal on Mathematical Analysis* 22 (1991) 463-479.

[4] J. V. Baxley, G. S. Gersdorff, Singular reaction-diffusion boundary value problems, *Journal of Differential Equations* 115 (1995) 441-457.

[5] A. Dinmohammadi, A. Razani, E. Shivanian, Analytical solution to the nonlinear singular boundary value problem arising in biology, *Boundary Value Problems* 1 (2017) 63-74.

[6] A. Dinmohammadi, E. Shivanian, A. Razani, Existence and uniqueness of solutions for a class of singular nonlinear two-point boundary value problems with sign-changing nonlinear terms, *Numerical Functional Analysis and Optimization* 38 (2017) 344-359.

[7] A. Ebaid, A new analytical and numerical treatment for singular two-point boundary value problems via the adomian decomposition method, *Journal of computational and applied mathematics* 235 (2011) 1914-1924.

[8] A. Fink, J. A. Gatica, G. E. Hernandez, P. Waltman, Approximation of solutions of singular second order boundary value problems, *SIAM Journal on Mathematical Analysis* 22 (1991) 440-462.

[9] W. F. Ford, J. A. Pennline, Singular nonlinear two-point boundary value problems: Existence and uniqueness, *Nonlinear Analysis: Theory, Methods & Applications* 71 (2009) 1059-1072.

[10] J. Gatica, V. Oliker, P. Waltman, Singular nonlinear boundary value problems for second-order ordinary differential equations, *Journal of Differential Equations* 79 (1989) 62-78.

[11] R. Jia, J. Shao, Existence and uniqueness of periodic solutions of second-order nonlinear differential equations, *Journal of Inequalities and Applications* 1 (2013) 1-13.

[12] R. Kannan, D. O'regan, Singular and nonsingular boundary value problems with sign changing, *J. Inequal. Appl.* 5 (2000) 621-637.

[13] H. B. Keller, Numerical methods for two-point boundary-value problems, *Waltham*, 1968.

[14] H. Lu, Z. Bai, Positive radial solutions of a singular elliptic equation with sign changing nonlinearities, *Applied Mathematics Letters* 19 (2006) 555-567.

[15] J. A. Pennline, Improving convergence rate in the method of successive approximations, *Mathematics of Computation* 37 (1981) 127-134.

[16] J. A. Pennline, Constructive existence and uniqueness for some nonlinear two-point boundary value problems, *Journal of Mathematical Analysis and Applications* 96 (1983) 584-598.

[17] J. A. Pennline, Constructive existence and uniqueness for two-point boundary-value problems with a linear gradient term, *Applied Mathematics and Computation* 15 (1984) 233-260.

[18] A. Tieno, On a class of singular boundary value problems which contains the boundary conditions, *Journal of Differential Equations* 113 (1994) 1-16.

[19] H. Wang, Y. Li, Existence and uniqueness of solutions to two point boundary value problems for ordinary differential equations, *Zeitschrift fur angewandte Mathematik und Physik ZAMP* 47 (1996) 373-384.

[20] J. Wang, On positive solutions of singular nonlinear two-point boundary value prob-

lems, *Journal of Differential Equations* 107 (1994) 163-174.

[21] J. Zhou, The existence and uniqueness of the solution for nonlinear elliptic equations in hilbert spaces, *Journal of Inequalities and Applications* 1 (2015) 1-23.

Dr. Elyas Shivanian was born in Zanjan province, Iran in August 26, 1982. He started his master course in applied mathematics in 2005 at Amirkabir University of Technology and has finished MSc. thesis in the field of fuzzy linear programming in 2007. After his Ph.D. in the field of prediction of multiplicity of solutions to the boundary value problems, at Imam Khomeini International University in 2012, he became Assistant Professor at the same university. His research interests are analytical and numerical solutions of ODEs, PDEs and IEs. He has published several papers on these subjects. He also has published some papers in other fields, for more information see please http://scholar.google.com/citations?user=MFncks8AAAAJ&hl=en