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Note on Power Graphs of Certain Semigroups

P. Soltani *†, A. Tehranian ‡, H. Doostie §, H. Rasouli ¶

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Abstract

The finite semigroups are applicable in several branches of sciences. The power graph of such semigroups is also an strategic tool in demonstrating the properties of semigroups. The completeness and the Eulerianity of power graphs associated to finite commutative semigroups and finite non-commutative epigroups are studied in this paper. We show that these graphs may be non-complete for the monogenic semigroups and we gives a necessary and sufficient condition for such graphs to be complete when finite regular epigroups are considered. This study answers in part the natural question "Is there any non-isomorphic non-group semigroups with the same complete power graph?"

Keywords : Graphs; Power graphs; Semigroups; Non-group semigroups.

1 Introduction

The interesting properties of power graphs of finite groups studied during the years and completeness or Eulerianity of constructed graphs discussed inquisitively. For a finite semigroup, the study of such graph is along with natural questions because of the variety of semigroups. Some of the questions will be considered in this paper concerning the finite commutative semigroups and finite non-commutative epigroups (quasi-commutative or non quasi-commutative). It is well-known that the power graph of a finite group is a complete graph iff it is cyclic of order of a power of a prime or the product of two primes. In this paper we prove that this is not the case in general, for finite monogenic semigroups. Also, we will give required conditions for a finite regular epigroup to have complete power graph.

2 Preliminaries

Following [2, 3, 4, 5, 10] we recall the definition of undirected power graph $P(S)$, for an algebraic structure S. Any element of S considered as a vertex of $P(S)$ and two corresponding vertices to the elements a and b are adjacent iff one of the relators $a = b^m$ or $b = a^m$ holds in S, for some integer $m > 1$. We may recall the definitions of *Eulerian* and *semi-Eulerian* graphs as well, i.e, a connected graph is Eulerian or semi-Elerian if every vertex is of even degree, either exactly two vertices are of odd degrees.

*Corresponding author. parvinsoltani12@yahoo.co, Tel:+98(938)7480603.
†Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.
‡Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.
§Department of Mathematics, Kharazmi University, Tehran, Iran.
¶Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.

The non-group semigroup S is called *epigroup* if for each $x \in S$ a power of x belongs to a subgroup of S [14]. An special type of epigroups which will be studied here for the completeness of its power graph, is the *quasi-commutative* semigroup (this type of semigroups studied by many authors since 1972, and one may consult [11, 12], for detailed information.)

In decomposition of non-group non-commutative finite semigroups into the union of groups, one may consult Clifford [6] and Mukherjee [11]. The former studies the inverse semigroups and the later investigates the quasi-commutative semigroups to prove that "a regular quasi-commutative semigroup may be decomposed into a union of groups". By these definitions and the results of Mukherjee [11] we deduce that a regular quasi-commutative semigroup is an epigroup. Also, we recall the results of Hosseinzadeh et al. [8]. This article presents infinite classes of non-regular epigroups which are not quasi-commutative.

Another notion which we use in this paper is the notion of presentation of a semigroup. As a simple explanation we may say that a semigroup could be presented by a set of generators and a set of relations (or in other words, relators). We are not interested in to explain the construction of a presentation which is related to free semigroups. The requisite prefaces of it may be found in [13]. Our notation in the semigroup theory is fairly standard and for the general preliminaries we follow [6, 9]. Moreover, on the power graphs of abelian groups we will need in the sequel, the following two lemmas:

Lemma 2.1 ([4] Theorem 2.12)**.** *The power graph of a finite group G is complete iff G is the cyclic group of order of a power of a prime integer.*

Lemma 2.2 ([5] Theorem 5)**.** *For every different primes p_1 and q_1 where $p_1 > q_1$, suppose that G is a finite group of order p_1q_1. Then,*

(i). $P(G) \simeq (K_{p_1-1} \cup K_{q_1-1}) + K_{\phi(p_1q_1)+1}$ iff G is cyclic. (ϕ is the well-known Eulerian function).

(ii). If G is non-cyclic then $P(G) \simeq K_1 + (pK_{q_1-1} \cup K_{p_1-1})$.

In connection with the quasi-commutative semigroups the following result should be recalled as well.

Lemma 2.3 ([15] Theorem A)**.** *If a finite epigroup S is quasi-commutative then all of its subgroups are Hamiltonian groups. (A group is called Hamiltonian if all of its subgroups are normal)*

Our first results concerning the power graphs of semigroups are the following propositions in which $T_n = \langle a \mid a^{n+1} = a^r \rangle$ is the non-group monogenic semigroup of order n and of index $r \geqslant 2$.

3 Completeness of power graphs

Proposition 3.1. *The relators $a^{kn} = a^{k(r-1)}$ and $a^{n^k} = a^{(r-1)^k}$ hold in the semigroup T_n, for each positive integer k.*

Proof. By considering the relator $a^{n+1} = a^r$ we clearly have $a^n = a^{n+1-1} = a^{r-1}$. So both of the relators hold for $k = 1$. Assume the relators are true for k. Since

$$\begin{aligned} a^{(k+1)n} &= a^{kn} \cdot a^n = a^{k(r-1)} \cdot a^n \\ &= a^{k(r-1)} \cdot a^{r-1} = a^{(k+1)(r-1)}, \end{aligned}$$

and

$$\begin{aligned} a^{n^{(k+1)}} &= (a^{n^k})^n = a^{n(r-1)^k} \\ &= (a^n)^{(r-1)^k} = (a^{r-1})^{(r-1)^k} \\ &= a^{(r-1)^{k+1}}, \text{ (for, } a^{kn} = a^{k(r-1)}), \end{aligned}$$

so, proof will be completed by using an induction method on k.

□

Proposition 3.2. *Consider the semigroup $T_n = \langle a \mid a^{n+1} = a^2 \rangle$ where $n > 3$ is an integer. Then the power graph of T_n is the complete graph K_n iff n satisfies one of the following conditions:*

(i) n is in the form $p^m + 1$, for some prime p and positive integer m,

(ii) n is in the form $pq + 1$, for some different odd primes p and q.

Proof. Since $T_n = \{a, a^2, \ldots, a^n\}$ so every element of the set $G = \{a^2, \ldots, a^n\}$ is a power of the element a, which makes a to be a vertex of $P(T_n)$ being adjacent to all other vertices. By using Proposition 3.1, we get the relator $a^{kn} = a^n$ for every positive integer k and so $c = a^n$ generates G. Moreover, $e = a^{n-1}$ is the identity element of G. This implies that G is a cyclic group. Hence, $P(T_n)$ is a complete graph if and only if $P(G)$ is complete and therefore, the results follows at once by using the Lemmas 2.1 and 2.2. □

The next result concerns the power graph of the semigroup with the finite presentation

$$\pi = \langle a, b \mid a^{m+1} = a, b^{n+1} = b, ab = ba, a^m = b^n \rangle,$$

where, m and n are positive integers greater than 3. This interesting presentation was given by K. Ahmadidelir [1] where he proved that S is indeed the direct product of monogenic semigroups of orders m and n.

Proposition 3.3. *Let S be the semigroup as above where $m = n$. Then, the power graph of S is Eulerian iff one of the following conditions holds:*

(i) n is in the form $p^\alpha + 1$, for some prime p and positive integer α,

(ii) n is in the form $pq + 1$, for some different odd primes p and q.

Proof. By letting $m = n$ in π the semigroup S may be decomposed into the union of groups:

$$\begin{aligned} S =& \langle a^i \mid 1 \leqslant i \leqslant m \rangle \cup \langle b^j \mid 1 \leqslant j \leqslant m \rangle \\ & \cup \langle ab \rangle \cup \langle a^i b^j \mid 1 \leqslant i \neq j \leqslant m \rangle. \end{aligned}$$

Each group is of order m and a^m is the identity element of each group because of the relator $a^m = b^m$. Now, each of the conditions (i) and (ii) is a necessary and sufficient condition for each of these groups to have the associated complete graph K_m (Lemmas 2.1 and 2.2.) The vertex a^m is the unique coincided vertex of these graphs and so $P(S)$ is Eulerian. □

4 Conclusions

Certain concrete examples of non-isomorphic semigroups of the same order and of the same complete power graphs will be given here. Consider the cyclic group $\mathbb{Z}_n = \langle a \mid a^n = 1 \rangle$ and the non-group monogenic semigroup $T_n = \langle a \mid a^{n+1} = a^2 \rangle$ are of order n. By the results of Section 3, the graph $P(T_n)$ is complete iff $n = p^\alpha + 1$ (for a prime p), or $n = pq + 1$, (for some odd primes p and q). Also, the graph $P(\mathbb{Z}_n)$ is complete iff $n = p'^\beta$ or $n = p'q'$, for some primes p, p' and q' where, p' and q' are odd. So, there are three possible cases to consider which are $p^\alpha + 1 = p'^\beta$, $p^\alpha + 1 = p'q'$ and $pq + 1 = p'^\beta$.

The following example examines certain possible cases for n where $P(T_n) = P(\mathbb{Z}_n) \simeq K_n$.

Example 4.1. *For every integer n where $3 \leqslant n \leqslant 16$ we have:*

$$P(T_n) = P(\mathbb{Z}_n) \simeq K_n,$$

if and only if $n = 3, 4, 5, 8, 9, 10, 14, 16$.

Indeed, we may apply the above mentioned three cases for n to gather the results in the following table. The capital alphabets **C** and **E** stand for completeness and Eulerianity of the power graphs in this table.

n	$P(\mathbb{Z}_n)$	$P(T_n)$	n	$P(\mathbb{Z}_n)$	$P(T_n)$
3	**C**	**C**	10	**C**	**C**
4	**C**	**C**	11	**C**	**E**
5	**C**	**C**	12	**E**	**C**
6	**C**	**E**	13	**C**	**E**
7	**C**	**E**	14	**C**	**C**
8	**C**	**C**	15	**C**	**E**
9	**C**	**C**	16	**C**	**C**

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