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Research Article

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An Explicit Numerical Technique for Nonlinear Nonlocal Time-Delay Dynamical Systems via Quadratic Spline Approach

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Abstract

Dynamical systems with delay are widespread in nature. The study of time-delay induced changes in the collective behavior of systems of coupled nonlinear oscillators is a subject of great interest, both because of its fundamental importance from the point of view of dynamical systems and because of its practical applications. In this paper, an explicit technique is proposed for numerical solution of nonlocal dynamical systems with time delay. The proposed method is adopted quadratic spline interpolation. Then, the error analysis of the developed method is discussed. It is exploited in the discussion of nonlocal delay Ikeda and Hutchinson models. Finally, the performance of the presented approach is verified by applying the error and convergence study for different values of fractional order parameters.

Keywords : Fractional calculus; Fractional delay differential equation; Numerical method; Chaotic attractor; Quadratic spline interpolation; Ikeda model; Hutchinson model.

1 Introduction

FRactional calculus (FC) is related to integrals and derivatives of desired orders [11]. Over past decades, FC has interested significant attention due to its extensive usages in a various of fields such as bioengineering [14], signal processing [21], mechanics [29], ecology [7] and physics [26]. Furthermore, in recent years, remarkable contribution has been created in fractional integro or differential equations in the modeling and numerical solutions very phenomenon for instance studying of finance [19], dynamical systems [5], electrical circuit [20] and epidemic [12].

Fractional delay differential equations (FDDEs) are useful mathematical tools for modelling phenomenons in very diverse fields in applied sciences. The FDDEs are investigated in several field including epidemic [13], astrophysics [23], engineering [33]. The existence and uniqueness theorems for FDDEs are studied in [15]. Moreover, various analysis and numerical methods for solving different classes of FDDEs are presented in many papers such as finite difference [16], Chebyshev polynomials [3, 4], Hermite wavelet

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[24], Laguerre wavelet [27], reproducing kernel [32], Jacobi collocation [1], Dickson polynomials [22], and spline interpolations [17, 18] methods.

In this study, we inspect the following FDDE

$$\begin{cases} {}^{C}D^{\gamma}_{0,t}u(t) = G(t, u(t), u(t-\lambda)), & t \in (0, T] \\ u(t) = \Lambda(t), & [-\lambda, 0], \end{cases} \tag{1.1}$$

where ${}^{C}D^{\gamma}_{0,t}u(t)$ demonstrates Caputo fractional derivative of $\gamma \in \mathbb{R}^+$ that is formularized as follows, [25],

$${}^{C}D^{\gamma}_{0,t}u(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{u'(\phi)}{(t-\phi)^{\gamma}} d\phi, \tag{1.2}$$

where $0 \le \gamma \le 1,\ \gamma \in \mathbb{N}$, the unknown function, $u(t)$, is a continuously differentiable and a smooth function determined on $\Psi = [0, T]$. In addition, λ represents the delay time, and the history function is stated by $\Lambda(t)$ on the interval $t \in [-\lambda, 0]$.

The structure of the rest of this discussion is collocated as follows. In Section 2, we proposes an explicit approximation manner via the quadratic interpolation for discretizing and solving the FDDE (1.1). Moreover, we check the error analysis of the developed approach in Section 2. We examine the accuracy of this method considering the nonlocal Ikeda, Hutchinson and unified chaotic systems with time delays in section 3. Finally, in Section 4, we summary the concluding attentions.

2 Computational algorithm for FDDE

In this section, we exhibit an explicit scheme to solve FDDE (1.1). Furthermore, we study the error and convergence analysis of the developed manner. Thought the study, we presume $t_l = l\Delta$, where $l \in \{-k, -k+1, \dots, -1, 0, 1, \dots, \varrho\}$, and $\Delta = \lfloor \frac{T}{\varrho} \rfloor$ means the uniform step size, $k = \lfloor \frac{\lambda}{\Delta} \rfloor$ and $k, \varrho \in \mathbb{N}$.

Definition 2.1 ([25]). *The left Riemann-Liouville fractional integral of order $\gamma \in \mathbb{R}^+$, for a function $u(t)$ is stated as*

$$\mathcal{J}^{\gamma}_{0,t}u(t) = \frac{1}{\Gamma(\gamma)} \int_0^t u(\phi)(t-\phi)^{\gamma-1} d\phi, \tag{2.1}$$

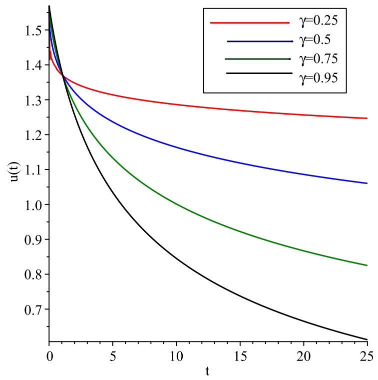

Figure 1: Numerical results for (3.3) by the presented scheme $u(t)$ versus t: for diverse values of γ and $\delta = \tau = 0.4$, $\lambda = 0.5$ and $\Delta = 0.05$.

where t and $\phi \in \mathbb{R}^+$ and $\Gamma(\cdot)$ explains the Gamma function.

According to definitions (1.2) and (2.1), we obtain:

$$\begin{aligned} \left(\mathcal{J}^{\gamma}_{0,t}\,{}^{C}D^{\gamma}_{0,t}\right)u(t) &= u(t) - \sum_{v=0}^{b-1} \frac{u^{(v)}(0)t^{v}}{v!}, \\ \left({}^{C}D^{\gamma}_{0,t}\mathcal{J}^{\gamma}_{0,t}\right)u(t) &= u(t), \end{aligned} \tag{2.2}$$

for $\gamma \in (b-1, b]$.

Proposition 2.1. *[28] Presume that $u(t) \in C^3(\Psi)$ be a function, $\gamma > 0$, $|u''(t)| \le A$ and $|u'''(t)| \le B$, where $A, B > 0$. The approximation of the fractional-order integral, $\left(\mathcal{J}^{\gamma}_{0,t_\varrho}[u(t)]\right)_{approx}$, can be stated as*

$$\mathcal{J}^{\gamma}_{0,t_\varrho}[u(t)] \approx \left(\mathcal{J}^{\gamma}_{0,t_\varrho}[u(t)]\right)_{approx} \equiv \frac{\Delta^{\gamma}}{\Gamma(\gamma+2)} \sum_{l=0}^{\varrho} \sigma_{l,\varrho} u_l, \tag{2.3}$$

where

$$\sigma_{l,\varrho} = \begin{cases} \beta_{l,\varrho} - \delta_{l+1,\varrho}, & l = 0 \\ \beta_{l,\varrho} + \delta_{l,\varrho} - \delta_{l+1,\varrho}, & 1 \le l \le \varrho - 1 \\ \beta_{l,\varrho} + \delta_{l,\varrho}, & l = \varrho \end{cases}, \tag{2.4}$$

such that

$$\beta_{l,\varrho} = \begin{cases} (\varrho-1)^{\gamma+1} - (\varrho)^{\gamma}(\varrho-\gamma-1), & l = 0 \\ (\varrho-l+1)^{\gamma+1} - 2(\varrho-l)^{\gamma+1} \\ \qquad +(\varrho-l-1)^{\gamma+1}, & 1 \le l \le \varrho-1 \\ 1, & l = \varrho \end{cases}, \tag{2.5}$$

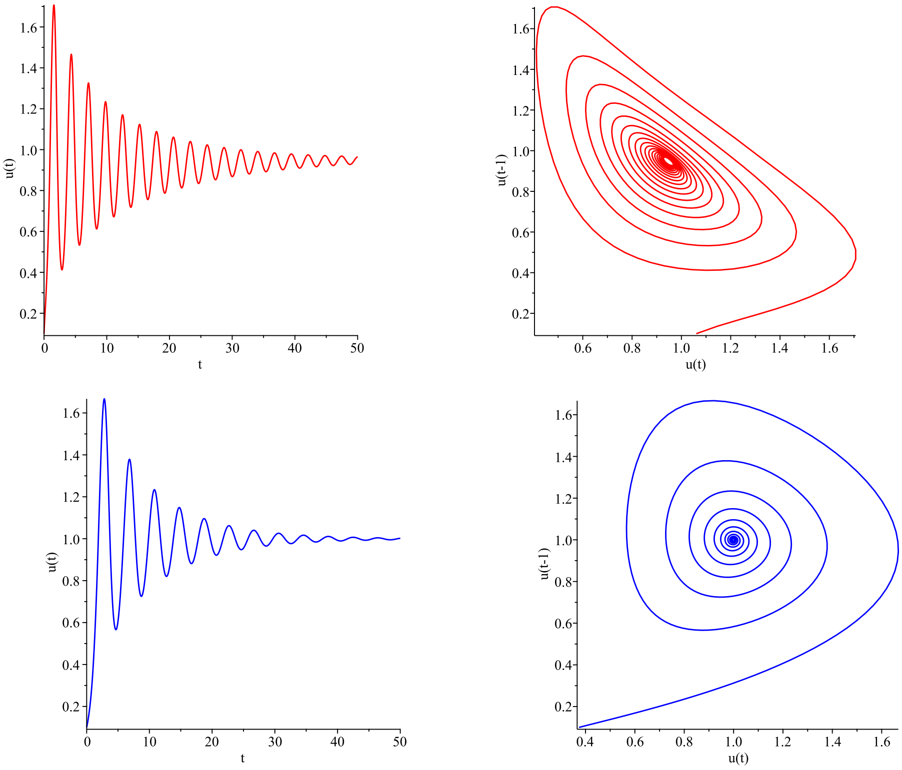

Figure 2: Time response (left) and phase-space solution (right) for Eq. (3.4), with $\gamma = 0.5$ (top) and $\gamma = 0.95$ (bottom), for $\alpha = 1.4$ and $\Delta = 0.05$.

and

$$\delta_{l,\varrho} = (\varrho - l + 1)^{\gamma+1} - (\varrho - l)^{\gamma+1} - \tfrac{\gamma+1}{2}\Big((\varrho - l + 1)^{\gamma} + (\varrho - l)^{\gamma}\Big), \quad 1 \le l \le \varrho. \tag{2.6}$$

Further, the truncated error of relation (2.3), $\mathcal{R}_{\varrho} = \mathcal{J}^{\gamma}_{0,t_{\varrho}}[u(t)] - \Big(\mathcal{J}^{\gamma}_{0,t_{\varrho}}[u(t)]\Big)_{approx}$, *is bounded, such as*

$$|\mathcal{R}_{\varrho}| \le \frac{A}{8\Gamma(\gamma+1)}\Delta^{\gamma+2} + \frac{\sqrt{3}B}{9\Gamma(\gamma+1)}\Delta^{\gamma+3}. \tag{2.7}$$

The Eq. (1.1), can be stated as

$$u(t) = T_{b-1}[u;0](t) + \mathcal{J}^{\gamma}_{0,t}[G(t, u(t), u(t-\lambda)], \tag{2.8}$$

where

$$T_{b-1}[u;0](t) = \sum_{m=0}^{b-1} \Lambda^{(m)}(0)\frac{t^m}{m!},$$

and

$$\mathcal{J}^{\gamma}_{0,t}[G(t, u(t), u(t-\lambda)] = \int_0^t \frac{(t-\phi)^{\gamma-1}}{\Gamma(\gamma)} G(\phi, u(\phi), u(\phi-\lambda)d\phi.$$

The discretized of (2.8) is obtained as

$$u_{\varrho} = T_{b-1}[u;0](t_{\varrho}) + \mathcal{J}^{\gamma}_{0,t_{\varrho}}[G(t, u(t), u(t-\lambda)]. \tag{2.9}$$

Applying the formula (2.3), we get

$$u_{\varrho} = T_{b-1}[u;0](t_{\varrho}) + \frac{\Delta^{\gamma}}{\Gamma(\gamma+2)}\sum_{l=0}^{\varrho} \sigma_{l,\varrho} G(t_l, u_l, u_{l-k}), \tag{2.10}$$

where $\sigma_{l,\varrho}$ is described by (2.4). Furthermore, the nonlinear source term $G(t,\cdot,\cdot)$ is discretized as:

$$|G(t_{\varrho}, u_{\varrho}, u_{\varrho-k}) - G(t_{\varrho}, u_{\varrho-1}, u_{\varrho-k})| \le \kappa h = \mathcal{O}(h), \tag{2.11}$$

where $\kappa \in \mathbb{R}^+$ is Lipschitz constants for $G(t,\cdot,\cdot)$. Hence

$$u_{\varrho} = T_{b-1}[u;0](t_{\varrho}) + \frac{\Delta^{\gamma}}{\Gamma(\gamma+2)}\sigma_{\varrho,\varrho} G(t_{\varrho}, u_{\varrho-1}, u_{\varrho-k}) + \frac{\Delta^{\gamma}}{\Gamma(\gamma+2)}\sum_{l=0}^{\varrho-1} \sigma_{l,\varrho} G(t_l, u_l, u_{l-k}). \tag{2.12}$$

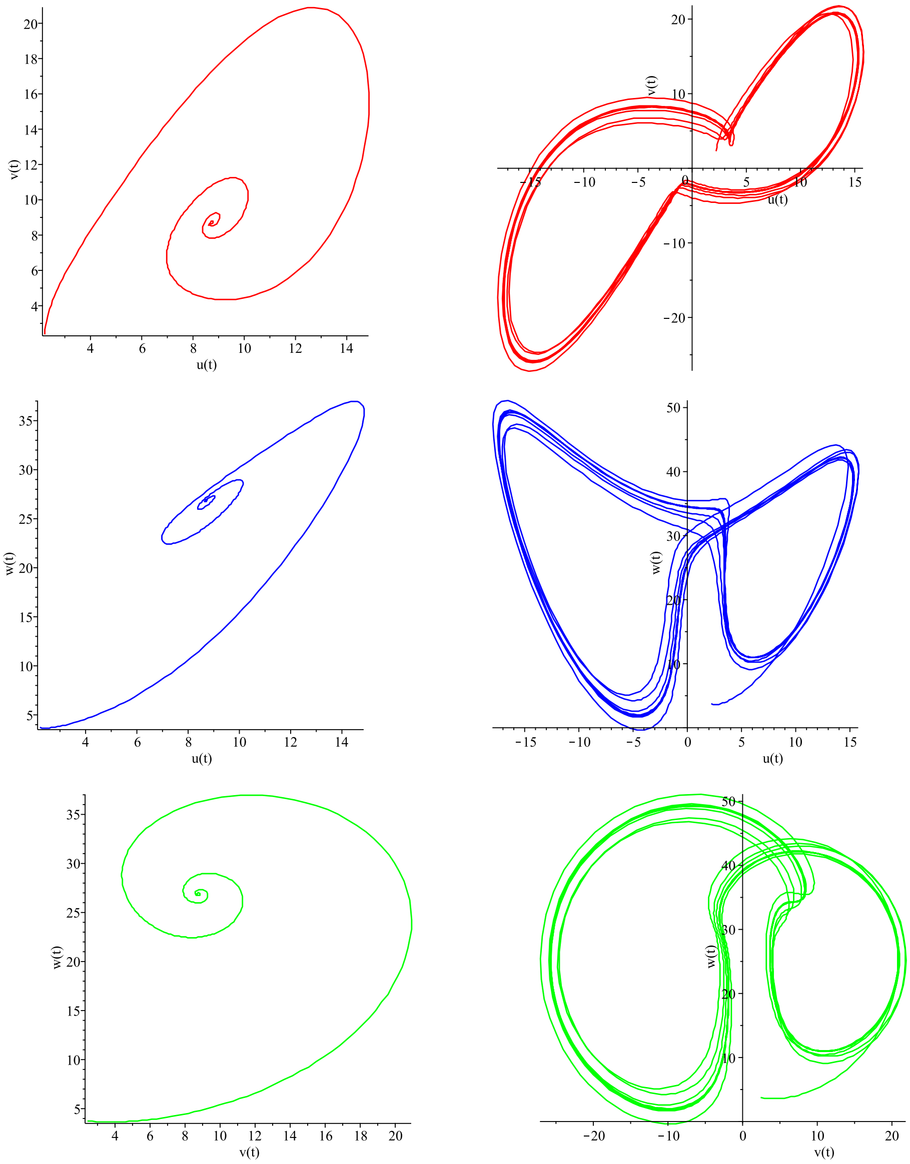

Figure 3: Phase curves of nonlocal Lorenz system without (left) and with (right) delay-time for Eq. (3.5), with $\delta = 0$ and $\lambda_u = 0.005,\ \lambda_v = 0.025$ and $\lambda_w = 0.25$, for $\gamma_1 = 0.005,\ \gamma_2 = 0.025,\ \gamma_3 = 0.25$, and $\Delta = 0.005$ in $t \in [0, 10]$.

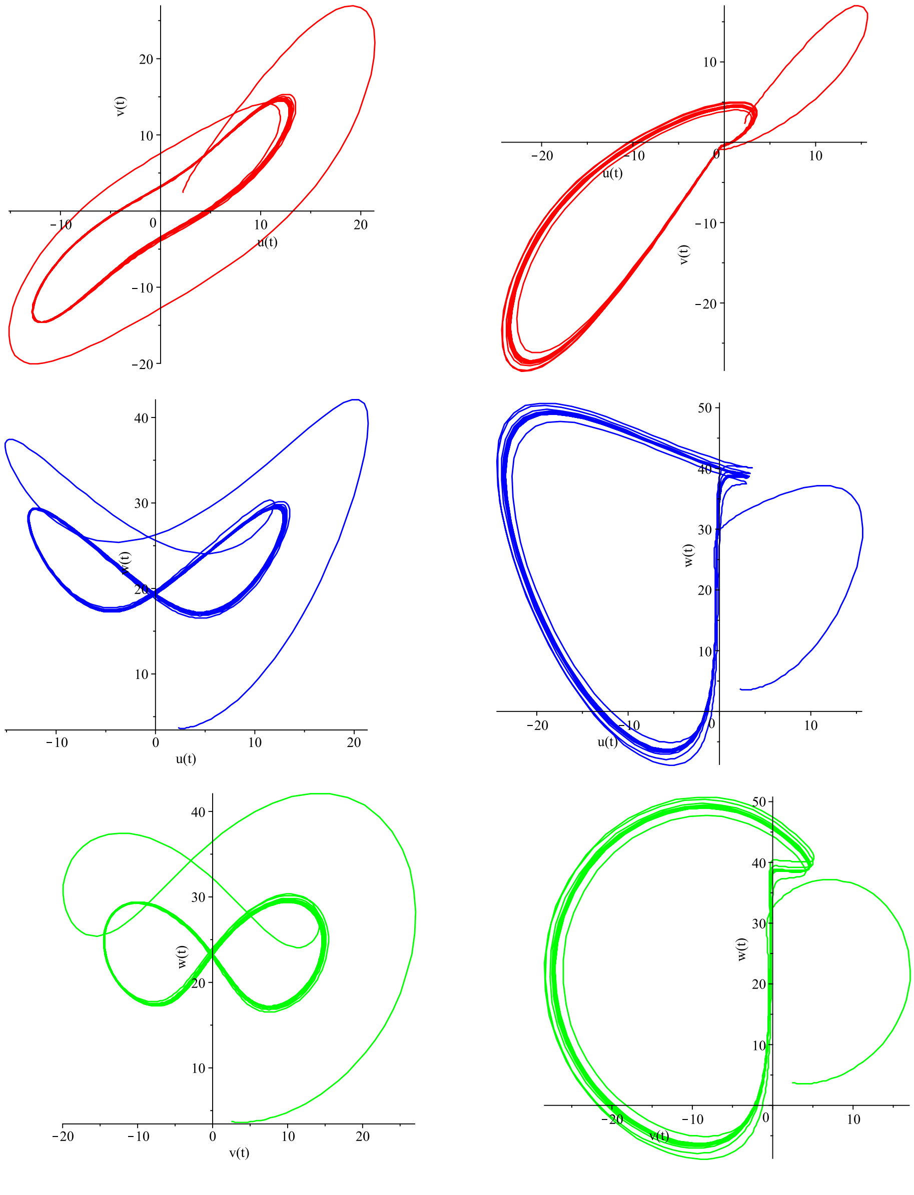

Figure 4: Phase curves of nonlocal Lü system without (left) and with (right) delay-time for Eq. (3.5), with $\delta = 0.8$ and $\lambda_u = 0.005,\ \lambda_v = 0.025$ and $\lambda_w = 0.25$, for $\gamma_1 = 0.005,\ \gamma_2 = 0.025,\ \gamma_3 = 0.25$, and $\Delta = 0.005$ in $t \in [0, 10]$.

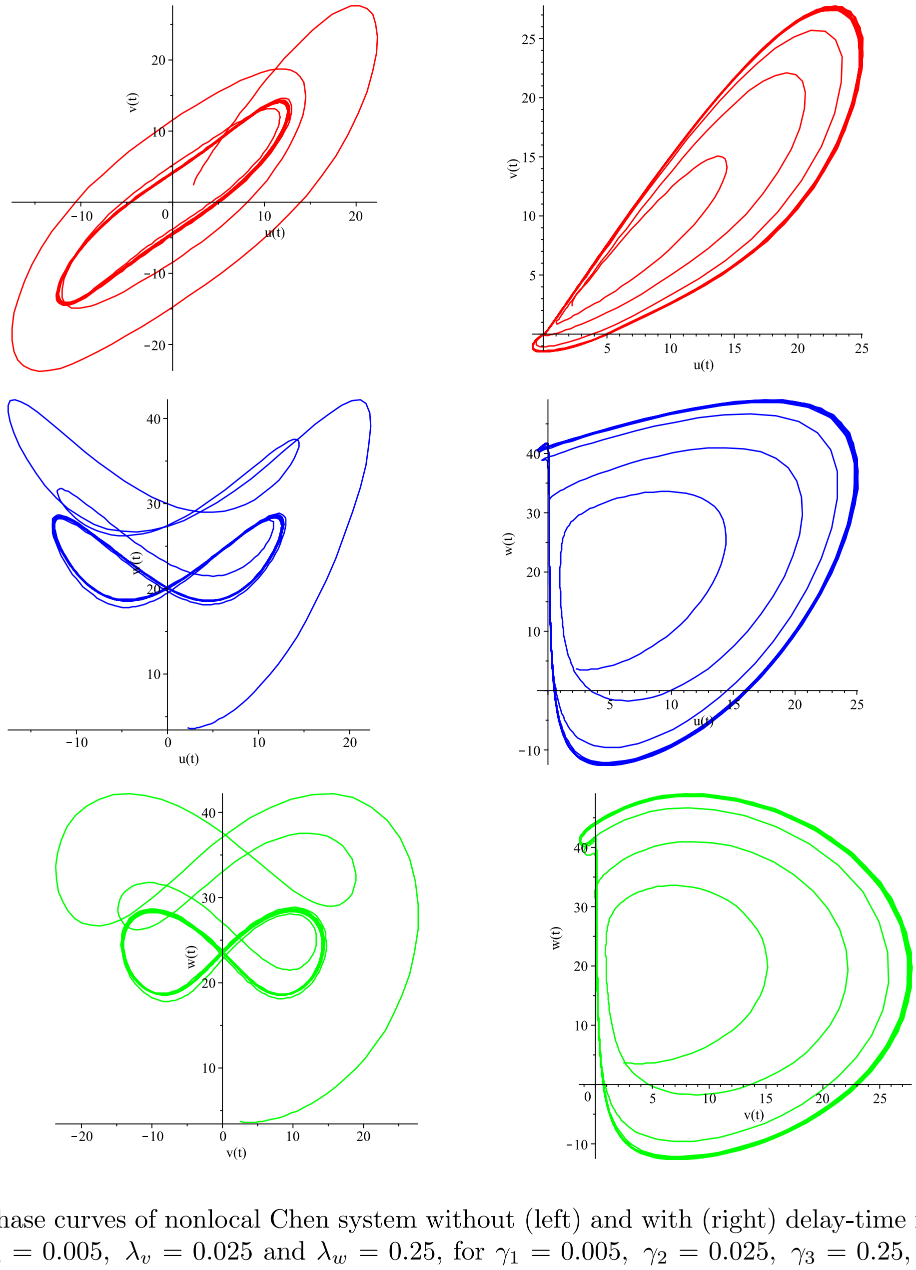

Figure 5: Phase curves of nonlocal Chen system without (left) and with (right) delay-time for Eq. (3.5), with $\delta = 1$ and $\lambda_u = 0.005$, $\lambda_v = 0.025$ and $\lambda_w = 0.25$, for $\gamma_1 = 0.005$, $\gamma_2 = 0.025$, $\gamma_3 = 0.25$, and $\Delta = 0.005$ in $t \in [0, 10]$.

Table 1: Example 3.1: The collation of E_ϱ, ECO and CPU time (sec) results for Eq. (3.3) by the cubic spline [30] and presented schemes for diverse values of γ and Δ, and $\delta = \tau = 0.4$, $\lambda = 0.5$, and $T = 25$.

		Cubic spline scheme [30]			Presented scheme		
γ	Δ	E_ϱ	*ECO*	*CPU time*	E_ϱ	*ECO*	*CPU time*
0.25	0.1	1.80×10^{-2}	–	8.265	3.68×10^{-3}	–	6.203
	0.05	8.50×10^{-3}	1.08	32.515	1.61×10^{-3}	1.20	24.578
	0.025	3.80×10^{-3}	1.16	144.672	6.98×10^{-4}	1.21	103.984
0.5	0.1	1.34×10^{-2}	–	8.204	1.81×10^{-3}	–	6.047
	0.05	4.99×10^{-3}	1.43	32.688	6.72×10^{-4}	1.43	24.796
	0.025	1.84×10^{-3}	1.44	141.344	2.46×10^{-4}	1.45	104.172
0.75	0.1	7.65×10^{-3}	–	8.234	1.72×10^{-3}	–	6.172
	0.05	2.39×10^{-3}	1.68	32.641	5.34×10^{-4}	1.69	24.812
	0.025	7.52×10^{-4}	1.67	143.125	1.64×10^{-4}	1.70	105.781
0.95	0.1	6.29×10^{-3}	–	8.515	3.71×10^{-3}	–	5.969
	0.05	1.74×10^{-3}	1.85	34.282	9.89×10^{-4}	1.91	24.891
	0.025	4.75×10^{-4}	1.87	141.953	2.60×10^{-4}	1.93	104.297

Table 2: Example 3.2: The collation of E_ϱ and ECO results for Eq. (3.4) by the presented scheme for diverse values of γ and Δ, $\lambda = 1$, and $T = 50$.

		$\gamma = 0.5$		$\gamma = 0.95$	
α	Δ	E_ϱ	*ECO*	E_ϱ	*ECO*
0.3	0.1	2.12×10^{-3}	–	7.21×10^{-5}	–
	0.05	7.61×10^{-4}	1.48	1.96×10^{-6}	1.88
	0.025	2.78×10^{-4}	1.45	5.10×10^{-6}	1.94
1.4	0.1	3.17×10^{-3}	–	7.85×10^{-3}	–
	0.05	2.20×10^{-3}	1.45	2.11×10^{-3}	1.90
	0.025	4.59×10^{-4}	1.38	5.54×10^{-4}	1.93

At present, we peruse the error analysis of the presented scheme for approximating the solution of the Eq. (1.1).

Theorem 2.1. *Suppose that $u(t) \in C^3(\Psi)$ and u_l, $l = 0, \dots, \varrho$, are the exact and approximate, respectively, solutions of the Eq. (1.1). Moreover, let the function $G(t,\cdot,\cdot)$ Eq. in (1.1) satisfies the Lipschitz condition with respect to its variables,*

$$\begin{aligned}&|G(t,\omega_1,\hat{\omega}_1) - G(t,\omega_2,\hat{\omega}_2)| \\ &\qquad \le S_1|\omega_1 - \omega_2| + S_2|\hat{\omega}_1 - \hat{\omega}_2|,\end{aligned} \tag{2.13}$$

where S_1 and $S_2 \in \mathbb{R}^+$ are constants. Then

$$|E(t_\varrho)| \le C\Delta^{\gamma+1}, \tag{2.14}$$

where $E(t_\varrho) = u(t_\varrho) - u_\varrho$ and constant $C \in \mathbb{R}^+$ is independent of γ and Δ.

Proof. Let $E_0 = 0$. From (2.12), we get

$$\begin{aligned}|E(t_\varrho)| &= \Bigg|\frac{1}{\Gamma(\gamma)}\int_0^{t_\varrho}(t_\varrho - \phi)^{\gamma-1}G(\phi, u(\phi), u(\phi-\lambda))d\phi \\ &\quad - \frac{\Delta^\gamma}{\Gamma(\gamma+2)}\Bigg(\sigma_{\varrho,\varrho}G(t_\varrho, u_{\varrho-1}, u_{\varrho-k}) \\ &\quad + \sum_{l=0}^{\varrho-1}\sigma_{l,\varrho}G(t_l, u_l, u_{l-k})\Bigg)\Bigg|,\end{aligned}$$

and hence,

$$\begin{aligned}|E(t_\varrho)| &\le \Bigg|\frac{1}{\Gamma(\gamma)}\int_0^{t_\varrho}(t_\varrho - \phi)^{\gamma-1}G(\phi, u(\phi), u(\phi-\lambda))d\phi \\ &\quad - \frac{\Delta^\gamma}{\Gamma(\gamma+2)}\sum_{l=0}^{\varrho}\sigma_{l,\varrho}G(t_l, u_l, u_{l-k})\Bigg| \\ &\quad + \frac{\Delta^\gamma}{\Gamma(\gamma+2)}\sum_{l=0}^{\varrho-1}\sigma_{l,\varrho}\Big|G(t_l, u(t_l), u(t_l-\lambda)) - G(t_l, u_l, u_{l-k})\Big| \\ &\quad + \frac{\Delta^\gamma}{\Gamma(\gamma+2)}\sigma_{\varrho,\varrho}\Big|G(t_\varrho, u(t_\varrho), u(t_\varrho-\lambda)) - G(t_\varrho, u_{\varrho-1}, u_{\varrho-k})\Big|.\end{aligned}$$

Accordingly, after some reductions, we gain

$$
\begin{aligned}
|E(t_\varrho)| \leq & \frac{1}{\Gamma(\gamma+1)}\left(\frac{A}{8}\Delta^{\gamma+2}+\frac{\sqrt{3}B}{9}\Delta^{\gamma+3}\right) \\
&+\frac{(S_1+S_2)}{\Gamma(\gamma+2)}\Delta^{\gamma}\sum_{l=0}^{\varrho-1}\sigma_{l,\varrho}\left(\frac{A\Delta^2}{8}+\frac{\sqrt{3}B\Delta^3}{9}\right) \\
&+\frac{(3-\gamma)\kappa}{2\Gamma(\gamma+2)}\Delta^{\gamma+1},
\end{aligned}
$$

where

$$
\sum_{l=0}^{\varrho-1}\sigma_{l,\varrho} \leq (2-\gamma)\varrho^{1-\varrho}-1 \equiv \tau.
$$

Thus, we have

$$
\begin{aligned}
|E(t_\varrho)| \leq & \left\{\frac{1}{\Gamma(\gamma+1)}\left(\frac{A}{8}+\frac{\sqrt{3}B}{9}\right)\right. \\
&+\frac{1}{\Gamma(\gamma+2)}\left((S_1+S_2)\tau\left(\frac{A}{8}+\frac{\sqrt{3}B}{9}\right)\right. \\
&\left.\left.+\frac{(3-\gamma)\kappa}{2}\right)\right\}\max\{\Delta^{\gamma+3},\Delta^{\gamma+2},\Delta^{1+\gamma}\} \\
=& C\Delta^{\gamma+1}
\end{aligned}
\tag{2.15}
$$

where constant

$$
\begin{aligned}
C = & \left\{\frac{1}{\Gamma(\gamma+1)}\left(\frac{A}{8}+\frac{\sqrt{3}B}{9}\right)\right. \\
&\left.+\frac{(S_1+S_2)}{\Gamma(\gamma+2)}\tau\left(\frac{A}{8}+\frac{\sqrt{3}B}{9}\right)+\frac{(3-\gamma)\kappa}{2\Gamma(\gamma+2)}\right\}
\end{aligned}
$$

□

3 Numerical examples

In this section, we investigate the performance and validate of presented scheme with some examples. In order to analyse the accuracy and computational efficiency of the presented technique, we consider the *mean absolute error* (MAE), E_ϱ, and the *experimental convergence order* (ECO), where stated as

$$
E_\varrho = \frac{1}{\varrho}\sum_{l=1}^{\varrho}|u_l^\varrho - u_{2l}^{2\varrho}|, \tag{3.1}
$$

and

$$
ECO = \log_2\left(\frac{E_{2\varrho}}{E_\varrho}\right), \tag{3.2}
$$

where the approximate values of $u(t_l)$ are indicated by u_l^ϱ and $u_{2l}^{2\varrho}$, and the number of interior mesh points is represented by ϱ. All the numerical results are performed with Maple v2019 running in an Intel (R) Core (TM) i7-7500U CPU @ 2.70 GHz machine.

Example 3.1. *The nonlinear fractional Ikeda system with time delay is defined as*

$$
\begin{cases} {}^{C}D_{0,t}^{\gamma}u(t) = \delta\sin(u(t-\lambda)) - \tau u(t) \\ u(t) = \frac{\pi}{2}, \qquad t\in[-\lambda,0] \end{cases}, \tag{3.3}
$$

where $0<\gamma\leq 1$, $u(t)$ *indicates the phase delay of the electric field during the amplifier,* $\delta>0$ *represents the light intensity injected in the system,* λ *is the feedback lag time in the booster and* $\tau>0$ *shows the relaxation factor. The Eq. (3.1) with non-fractional and fractional terms, i.e., with* $\gamma=1$ *and* $\gamma\in(0,1]$, *was studied in [8, 9, 10, 30].*

Fig. 1 demonstrates the approximation results of Eq. (3.3) for $\delta=\tau=0.4$, and $\lambda=0.5$, in $t\in[0,25]$ diverse values of γ and $\Delta=0.05$. Moreover, Table 1 contrasts E_ϱ, ECO and CPU time (sec) of expression (3.3) by applying the cubic spline [30] and presented schemes with various step sizes of Δ for $\gamma=\{0.25,0.5,0.75,0.95\}$, in the interval $t\in[0,25]$. The numerical results show that for all amounts of γ, the approximation errors of presented technique reduce by decreasing Δ.

Example 3.2. *The nonlinear fractional delay Hutchinson equation is defined as*

$$
\begin{cases} {}^{C}D_{0,t}^{\gamma}u(t) = \alpha u(t)(1-u(t-\lambda)) \\ u(t) = 0.1, \qquad t\in[-\lambda,0] \end{cases}, \tag{3.4}
$$

where $0<\gamma\leq 1$, $\alpha>0$ *is the parameter and the delay time* λ *expresses topological variations in the society scale. The Eq. (3.1) with non-fractional and fractional terms, i.e., with* $\gamma=1$ *and* $\gamma\in(0,1]$, *was investigated in [2, 6, 30].*

Fig. 2 illustrates the time history of oscillatory and phase-space solutions of the Eq. (3.4) for $\gamma=0.5$ and $\gamma=0.95$ with $\alpha=1.4$ and $\Delta=0.05$. Table 2 studies the performance indices E_ϱ and ECO of expression (3.4) for $\gamma=\{0.5,0.95\}$ and

$\alpha = \{0.3, 1.4\}$, with $\Delta = \{0.1, 0.05, 0.025\}$ and $\lambda = 1$ in the interval $t \in [0, 50]$. We confirm that by reducing the step size, we get a high precision approximation.

Example 3.3. *Consider the following nonlocal unified chaotic system ([31])*

$$\begin{cases} {}^{C}D_{0,t}^{\gamma_1} u(t) = (25\delta + 10)(v(t) - u(t - \lambda_u)) \\ {}^{C}D_{0,t}^{\gamma_2} v(t) = (28 - 35\delta)u(t) - u(t)w(t) \\ \qquad\qquad + (29\delta - 1)v(t - \lambda_v) \\ {}^{C}D_{0,t}^{\gamma_3} w(t) = u(t)v(t) - \frac{(8+\delta)}{3} w(t - \lambda_w) \\ u(t) = 2.2, \qquad t \in [-\lambda_u, 0] \\ v(t) = 2.4, \qquad t \in [-\lambda_v, 0] \\ w(t) = 3.8, \qquad t \in [-\lambda_w, 0] \end{cases}, \tag{3.5}$$

where $0 < \gamma_1, \gamma_2, \gamma_3 \leq 1$, *the delay parameters for the* u, v *and* w *variables are represent* $\lambda_u, \lambda_v, \lambda_w \geq 0$, *respectively. Moreover, we express*

$$\begin{cases} \textit{Nonlocal Lorenz system}, & \delta = 0 \\ \textit{Nonlocal Lü system}, & \delta = 0.8 \\ \textit{Nonlocal Chen system}, & \delta = 1 \end{cases}.$$

In Figs. 3, 4 and 5, we plot the phase curves of the system (3.5) by means of presented scheme with $\Delta = 0.005$ and $\lambda_u = 0.005$, $\lambda_v = 0.025$ and $\lambda_w = 0.25$, for $\gamma_1 = 0.96$, $\gamma_2 = 0.9$ and $\gamma_3 = 0.8$ in $t \in [0, 10]$. We can observe delay effect on the chaotic system (3.5) in these figures.

4 Conclusion

In this study, an explicit numerical method for numerical solution of nonlocal differential equations with time delay based on quadratic interpolation was presented. The efficiency of the presented numerical technique in solving the nonlocal delay models of Ikeda and Hutchinson was investigated. Moreover, the nonlocal unified chaotic systems with time delay were solved by the proposed algorithms and the results were analyzed using phase portraits. The results reveal the feasibility of the presented algorithm for nonlinear systems with time delay.

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