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Research Article

$*$-Fusion Frames in Hilbert Modules Over Locally C^*-Algebras

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Abstract

The main purpose of this paper is to introduce the notion of $*$-fusion frames in Hilbert modules over locally C^*-algebras to study some properties about these frames. We present some results of frames in the view of $*$-fusion frames in Hilbert modules over locally C^*-algebras, inparticular we give the reconstruction formula for these frames.

Keywords : C^*-algebra; Locally C^*-algebra; Hilbert C^*-module; Frame; $*$-fusion frame.

1 Introduction

Frames for Hilbert spaces were introduced in 1952 by Duffin and Schaeffer [5] to study some problems in nonharmonic Fourier series. Then Daubecheies, Grassman and Mayer [4] reintroduced and developed them. Various generalizations of frames e.g. frames of subspaces and g-frames were developed [3, 12, 13]. Frank and Larson [6] presented a general approach to the frame theory in Hilbert C^*-modules. A. Khosravi and B. Khosravi [9] generalized the concept of fusion frames and g-frames to Hilbert C^*-modules. A. Alijani and M.A. Dehghan [1] introduced $*$-frames and studied the properties of them in Hilbert C^*-modules. Finally, M. Azhini and N. Haddadzadeh [2] generalized the theory of fusion frames to Hilbert modules over locally C^*-algebras.

It is well known that Hilbert C^*-modules are generalizations of Hilbert spaces which the inner product takes values in a C^*-algebra. The theory of Hilbert C^*-modules has applications in the study of locally compact quantum groups, complete maps between C^*-algebras, non-commutative geometry, and KK-theory. There are some differences between Hilbert C^*-modules and Hilbert spaces. For example, there exist closed subspaces in Hilbert C^*-modules that have no orthogonal complement [10]. Moreover, every bounded operator on a Hilbert space has an adjoint such that there are bounded operators on Hilbert C^*-modules which have not this property [11]. So, problems about frames and $*$-frames for Hilbert C^*-modules are more complicated than those for Hilbert spaces. This makes the topic of the frames for Hilbert C^*-modules important and absorbing. In this paper, we introduce $*$-fusion frames for Hilbert modules over locally C^*-algebras and give some results about them.

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2 Preliminaries

In this section, locally C^*-algebras and Hilbert modules over them are defined. Recall that a C^*-seminorm on a topological $*$-algebra $\mathcal{A}$ is a seminorm p such that $p(ab) \leq p(a)p(b)$ and $p(aa^*) = (p(a))^2$ for all $a, b \in \mathcal{A}$.

Definition 2.1. *A locally C^*-algebra is a Hausdorff complete complex topological $*$-algebra $\mathcal{A}$ whose topology is determined by its continuous C^*-seminorms.*

In the sense that a net $\{a_\alpha\}_{\alpha \in I}$ converges to 0 if and only if the net $\{p(a_\alpha)\}_{\alpha \in I}$ converges to 0, for all continuous C^*seminorm p on $\mathcal{A}$.
Note that, each C^*-algebra is a locally C^*-algebra.

The set of all continuous C^*-seminorms on $\mathcal{A}$ is denoted by $S(\mathcal{A})$. Now, let $\mathcal{A}$ be a unital locally C^*-algebra with unit $1_{\mathcal{A}}$ and $a \in \mathcal{A}$. Then a is called positive if $a^* = a$ and $sp(a) = \{\lambda \in \mathbb{C} : \lambda 1_{\mathcal{A}} - a \text{ is not invertible}\} \subseteq \mathbb{R}^+$. The set of all positive elements of $\mathcal{A}$ denotes by $\mathcal{A}^+$. If $a, b \in \mathcal{A}$, then $a \leq b$ means that $b - a \in \mathcal{A}^+$.

Proposition 2.1. *([7]) Let $\mathcal{A}$ be a unital locally C^*-algebra with unit $1_{\mathcal{A}}$. Then for any $p \in S(\mathcal{A})$ and $a, b \in \mathcal{A}$, the followings hold:*
(1) $p(a) = p(a^*)$
(2) $p(1_{\mathcal{A}}) = 1$
(3) *If $a, b \in \mathcal{A}^+$ and $a \leq b$, then $p(a) \leq p(b)$*
(4) *If $1_{\mathcal{A}} \leq b$, then b is invertible and $b^{-1} \leq 1_{\mathcal{A}}$*
(5) *If $a, b \in \mathcal{A}^+$ are invertible and $0 \leq a \leq b$, then $0 \leq b^{-1} \leq a^{-1}$*
(6) *If $a \leq b$ and $c \in \mathcal{A}$, then $c^*ac \leq c^*bc$*
(7) *If $a, b \in \mathcal{A}^+$ and $a^2 \leq b^2$, then $0 \leq a \leq b$.*

Now, we recall some definitions and basic properties of Hilbert modules over locally C^*-algebras, for more detailes see [8].

Definition 2.2. *A pre-Hilbert module over locally C^*-algebra $\mathcal{A}$ is a complex vector space E which is also a left $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : E \times E \to \mathcal{A}$ which is $\mathbb{C}$-linear and $\mathcal{A}$-linear in its first variable and satisfies the following conditions:*
(i) $\langle x, x \rangle \geq 0,$
(ii) $\langle x, x \rangle = 0$ *iff* $x = 0,$
(iii) $\langle x, y \rangle^* = \langle y, x \rangle$, *for all* $x, y \in E.$

A pre-Hilbert $\mathcal{A}$-module E is called Hilbert $\mathcal{A}$-module if E is complete with respect to the topology determined by the family of seminorms

$$\bar{p}_E(x) = \sqrt{p(\langle x, x \rangle)} \quad (x \in E, p \in S(\mathcal{A})).$$

If $\mathcal{A}$ is a locally C^*-algebra, then it is a Hilbert $\mathcal{A}$-module with respect to the inner product $\langle a, b \rangle = ab^* \quad (a, b \in \mathcal{A})$.

Lemma 2.1. *[8, Lemma 2.1] For every $p \in S(\mathcal{A})$ and for all $x, y \in E$, the Cauchy-Bunyakovskii inequality holds*

$$p(\langle x, y \rangle)^2 \leq p(\langle x, x \rangle) p(\langle y, y \rangle).$$

Example 2.1. *Let $l^2(\mathcal{A})$ be the set of all sequences $\{a_n\}_{n \in \mathbb{N}}$ of elements of a locally C^*-algebra $\mathcal{A}$ such that the series $\sum_{n=1}^\infty a_n a_n^*$ is convergent in $\mathcal{A}$. Then $l^2(\mathcal{A})$ is a Hilbert $\mathcal{A}$-module with respect to the pointwise operations and inner product defined by*

$$\langle \{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}} \rangle = \sum_{n=1}^\infty a_n b_n^*.$$

Definition 2.3. *Let M be a closed submodule of a Hilbert $\mathcal{A}$-module E. Define*

$$M^\perp = \{y \in E : \langle x, y \rangle = 0, \textit{ for all } x \in M\}.$$

Then $M^\perp$ is a closed submodule of E. A closed submodule M in a Hilbert $\mathcal{A}$-module E is called orthogonally complemented if $E = M \oplus M^\perp$.

Let E and F be two locally Hilbert $\mathcal{A}$-modules. An $\mathcal{A}$-module map $T : E \to F$ is said to be bounded if for each $p \in S(\mathcal{A})$, there exists $C_p > 0$ such that

$$\bar{p}_E(Tx) \leq C_p \bar{p}_E(x) \quad (x \in E).$$

The set of all bounded $\mathcal{A}$-module maps from E to F is denoted by $Hom_{\mathcal{A}}(E, F)$ and we set $Hom_{\mathcal{A}}(E, E) = End_{\mathcal{A}}(E)$.
Let $T \in Hom_{\mathcal{A}}(E, F)$, T is called adjointable if there exists a map $T^* \in Hom_{\mathcal{A}}(F, E)$ such that

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

for all $x \in E, y \in F$. The set of all adjointable operators from E to F is denoted by $Hom^*_{\mathcal{A}}(E, F)$ and we set $Hom^*_{\mathcal{A}}(E, E) = End^*_{\mathcal{A}}(E)$.

3 $*$-Fusion frames in Hilbert modules over locally C^*-algebras

In this section, we assume that $\mathcal{A}$ is a unital locally C^*-algebra and E is a Hilbert $\mathcal{A}$-module. We introduce $*$-fusion frames in Hilbert modules over locally C^*-algebras, and then we give some results about them.

Definition 3.1. *Let $\{v_i \in \mathcal{A} : i \in I\}$ be a sequence of weights in $\mathcal{A}$, that is each v_i is a positive invertible element from the center of $\mathcal{A}$, and let $\{M_i : i \in I\}$ be a sequence of orthogonally complemented submodules of E. Then $\{(M_i, v_i) : i \in I\}$ is called a $*$-fusion frame if there are two strictly nonzero elements $C, D \in \mathcal{A}$ such that*

$$C\langle x, x\rangle C^* \leq \sum_{i\in I} v_i^2 \langle P_{M_i}(x), (3.1)$$

$$P_{M_i}(x) \leq D\langle x, x\rangle D^*, \quad (x \in E),$$

where P_{M_i} is the orthogonal projection of E onto M_i.

We call C and D the lower and upper bounds of the $*$-fusion frame. Since $\mathcal{A}$ is not a partial ordered set, lower and upper $*$-frame bounds may not have order and the optimal bounds may not exist. If $C = D = \lambda$, the family $\{(M_i, v_i) : i \in I\}$ is called a λ-tight $*$-fusion frame and if $C = D = 1_{\mathcal{A}}$, it is called a Parseval $*$-fusion frame. If in (3.1), we only have the upper bound, then $\{(M_i, v_i) : i \in I\}$ is called a $*$-Bessel fusion sequence with $*$-Bessel bound D. Now, we give some results about $*$-fusion frames.

Remark 3.1. *Note that each fusion frame is a $*$-fusion frame. For this, let $\{(M_i, v_i) : i \in I\}$ be a fusion frame for the Hilbert $\mathcal{A}$-module E with real frame bounds C and D. Then for $x \in E$, we have*

$$(\sqrt{C})1_{\mathcal{A}}\langle x, x\rangle(\sqrt{C})1_{\mathcal{A}} \leq \sum_{i\in I} v_i^2 \langle P_{M_i}(x), (3.2)$$

$$P_{M_i}(x) \leq (\sqrt{D})1_{\mathcal{A}}\langle x, x\rangle(\sqrt{D})1_{\mathcal{A}}.$$

Hence, $\{(M_i, v_i) : i \in I\}$ is a $*$-fusion frame with C^*-algebra valued bounds $(\sqrt{C})1_{\mathcal{A}}$ and $(\sqrt{D})1_{\mathcal{A}}$, where $1_{\mathcal{A}}$ is the identity element of $\mathcal{A}$.

Example 3.1. *Let $\{M_i : i \in I\}$ be a sequence of Hilbert $\mathcal{A}$-modules and*

$$\mathcal{X} = \oplus_{i\in\mathbb{N}} M_i = \{\{x_i\}_{i\in\mathbb{N}} : x_i \in M_i$$

and

$$\sum_{i\in\mathbb{N}} \langle x_i, x_i\rangle \text{ is norm convergent in } \mathcal{A}\}.$$

Then $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module with inner product $\langle\{x_i\}, \{y_i\}\rangle = \sum_{i\in\mathbb{N}}\langle x_i, y_i\rangle$, point wise operations and the norm defined by $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. Then $\{M_i\}_{i\in\mathbb{N}}$ is a Parseval $*$-fusion frame with respect to $\{v_i : i \in I\}$, where $v_i = 1$ for all $i \in I$ ([9]).

Proposition 3.1. *Let E be a Hilbert $\mathcal{A}$-module and let $\{v_i : i \in I\}$ be a family of weights in $\mathcal{A}$. Let for each $i \in I$, M_i be an orthogonally complemented submodule of E and $\{x_{ij} : j \in J_i\}$ a frame for M_i with positive bounds C_i and D_i in the center of $\mathcal{A}$. Suppose that $C_i^2 \geq 1_{\mathcal{A}}$ for each $i \in I$ and $D_p = \sup_i p(D_i) < \infty$, for some $p \in S(\mathcal{A})$. Then the following conditions are equivalent.*
(i) $\{v_i x_{ij} : i \in I; j \in J_i\}$ is a $*$-frame for E.
(ii) $\{(M_i, v_i) : i \in I\}$ is a $*$-fusion frame for E.

Proof. Since C_i and D_i in the center of $\mathcal{A}$ and for each $i \in I$, $\{x_{ij} : j \in J_i\}$ is a frame for M_i with positive bounds C_i and D_i, hence for any $x \in M_i$ we have

$$C_i^2\langle x, x\rangle \leq \sum_{i\in J_i}\langle x, x_{ij}\rangle\langle x_{ij}, x\rangle \leq D_i^2\langle x, x\rangle.$$

Since $C_i^2 \geq 1_{\mathcal{A}}$, thus for each $a \in \mathcal{A}$, $aa^*C_i^2 = aC_i^2a^* \geq aa^*$, therefore for $x \in E$, we get

$$\begin{aligned}
&\textstyle\sum_{i\in I} v_i^2\langle P_{M_i}(x), P_{M_i}(x)\rangle \\
&\leq \sum_{i\in I} C_i^2 v_i^2\langle P_{M_i}(x), P_{M_i}(x)\rangle \\
&\leq \sum_{i\in I}\sum_{i\in J_i}\langle v_i P_{M_i}(x), x_{ij}\rangle\langle x_{ij}, v_i P_{M_i}(x)\rangle \\
&= \sum_{i\in I}\sum_{i\in J_i} v_i^2\langle P_{M_i}(x), x_{ij}\rangle\langle x_{ij}, P_{M_i}(x)\rangle \\
&\leq \sum_{i\in I} D_i^2 v_i^2\langle P_{M_i}(x), P_{M_i}(x)\rangle \\
&\leq \sum_{i\in I} p(D_i^2) v_i^2\langle P_{M_i}(x), P_{M_i}(x)\rangle \\
&\leq D_p^2\sum_{i\in I} v_i^2\langle P_{M_i}(x), P_{M_i}(x)\rangle.
\end{aligned}$$

Hence, we can write

$$\sum_{i\in I} v_i^2 \langle P_{M_i}(x), P_{M_i}(x)\rangle$$
$$\leq \sum_{i\in I}\sum_{i\in J_i} \langle x, v_i x_{ij}\rangle \langle v_i x_{ij}, x\rangle$$
$$\leq D_p^2 \sum_{i\in I} v_i^2 \langle P_{M_i}(x), P_{M_i}(x)\rangle .$$

This shows that if $\{v_i x_{ij} : i \in I; j \in J_i\}$ is a $*$-frame for E with frame bounds A and B, then $\{(M_i, v_i) : i \in I\}$ is a $*$-fusion frame for E with frame bounds $\frac{A}{D_p^2}$ and B. Conversely if $\{(M_i, v_i) : i \in I\}$ is a $*$-fusion frame for E with frame bounds A and B, then $\{v_i x_{ij} : i \in I; j \in J_i\}$ is a $*$-frame for E with frame bounds A and BD_p^2. This completes the proof.

Now, We generalize [6, Theorem 4.1] to $*$-Bessel fusion sequences. First, by a little modification in the proof of [2, Lemma 4.4], we get the following lemma.

Lemma 3.1. *Let $\{(M_i, v_i) : i \in I\}$ be a $*$-Bessel fusion sequence for a Hilbert $\mathcal{A}$-module E with $*$-Bessel bound D. Then for each $x = (x_i)_{i\in I}$ in the Hilbert $\mathcal{A}$-module $M = \bigoplus_{i\in I} M_i$, the series $\sum_{i\in I} v_i x_i$ converges unconditionally and for each $p \in S(\mathcal{A})$, we have*

$$\bar{p}_E(\sum_{i\in I} v_i x_i) \leq \sqrt{p(D)}\bar{p}_M(x).$$

We need the following proposition in the proof of the next theorem.

Proposition 3.2. *[2, Proposition 3.1] Let $T : E \to F$ and $T^* : F \to E$ be two maps such that the equality $\langle x, T^*y\rangle = \langle Tx, y\rangle$ holds for all $x \in E, y \in F$. Then $T \in Hom^*_{\mathcal{A}}(E, F)$.*

Theorem 3.1. *Let $\{(M_i, v_i) : i \in I\}$ be a $*$-Bessel fusion sequence for a Hilbert $\mathcal{A}$-module E with $*$-Bessel bound D. Then, the corresponding frame transform $\theta : E \to l^2(E)$ defined by $\theta(x) = (v_i P_{M_i}(x))_{i\in I}$ for $x \in E$, is also bounded and its adjoint operator $\theta^* : l^2(E) \to E$ defined as $\theta^*(y) = \sum_{i\in I} v_i P_{M_i}(y_i)$ for each $y = (y_i)_{i\in I} \in l^2(E)$, is bounded.*

Proof. Since $\{(M_i, v_i) : i \in I\}$ is a $*$-Bessel fusion sequence, we have

$$\sum_{i\in I} v_i^2 \langle P_{M_i}(x), P_{M_i}(x)\rangle \leq D\langle x, x\rangle D^*,$$

hence θ is well-defined and for each $p \in S(\mathcal{A})$ and $x \in E$, we get

$$p\langle \theta(x), \theta(x)\rangle \leq p(D\langle x, x\rangle D^*) \leq (p(D))^2 p\langle x, x\rangle .$$

Hence, if $\bar{p}_E$ and $\bar{p}_{l^2(E)}$ are continuous seminorms on E and $l^2(E)$, respectively, we obtain

$$\bar{p}_{l^2(E)}(\theta(x)) \leq \sqrt{(p(D))^2}\bar{p}_E(x) = p(D)\bar{p}_E(x).$$

Therefore θ is bounded. Now for each $y = (y_i)_{i\in I} \in l^2(E)$ define $\theta^*(y) = \sum_{i\in I} v_i P_{M_i}(y_i)$, by [2, Proposition 2.2], the series $\sum_{i\in I}\langle y_i, y_i\rangle$ converges unconditionally. Moreover

$$\sum_{i\in I}\langle P_{M_i}(y_i), P_{M_i}(y_i)\rangle \leq \sum_{i\in I}\langle y_i, y_i\rangle .$$

Therefore $(P_{M_i}(y_i))_{i\in I}$ is in $\bigoplus_{i\in I} M_i$. Hence by Lemma 3.1, $\sum_{i\in I} v_i P_{M_i}(y_i)$ converges unconditionally and θ^* is well-defined. On the other hand, for each $x \in E$ and $y = (y_i)_{i\in I} \in l^2(E)$, we have

$$\langle x, \theta^*(y)\rangle =$$
$$\left\langle x, \sum_{i\in I} v_i P_{M_i}(y_i)\right\rangle = \sum_{i\in I}\langle v_i P_{M_i}(x), y_i\rangle = \langle \theta(x), y\rangle ,$$

and so by Proposition 3.2, θ^* is bounded. This completes the proof.

If $\{(M_i, v_i) : i \in I\}$ is a $*$-fusion frame for E with frame bounds C and D and M is an orthogonally complemented submodule of E, for each $i \in I$ and $x \in M$ we have

$$\sum_{i\in I} v_i^2 \langle P_{M_i\cap M}(x), P_{M_i\cap M}(x)\rangle$$
$$= \sum_{i\in I} v_i^2 \langle P_{M_i}(P_M(x)), P_{M_i}(P_M(x))\rangle$$
$$= \sum_{i\in I} v_i^2 \langle P_{M_i}(x), P_{M_i}(x)\rangle .$$

Hence we have the following result.

Proposition 3.3. *Let E be a Hilbert $\mathcal{A}$-module and let $\{(M_i, v_i) : i \in I\}$ be a $*$-fusion frame for E with frame bounds C and D. If M is an orthogonally complemented submodule of E. Then $\{(M_i \cap M, v_i) : i \in I\}$ is a $*$-fusion frame for M with frame bounds C and D.*

By the polariation identity

$$\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \left\langle x + i^k y, x + i^k y\right\rangle$$

we get the next proposition.

Proposition 3.4. *Let $\{(M_i, v_i) : i \in I\}$ be a Parseval $*$-fusion frame for a Hilbert $\mathcal{A}$-module E. Then, the corresponding frame transform θ preserves the inner product.*

Definition 3.2. *Let $\{(M_i, v_i) : i \in I\}$ be a $*$-fusion frame for a Hilbert $\mathcal{A}$-module E. Then the fusion frame operator S for $\{(M_i, v_i) : i \in I\}$ is defined by*

$$S(x) = \theta^*\theta(x) = \sum_{i\in I} v_i^2 P_{M_i}(x), \quad (x \in E).$$

Our next result is a generalization of [9, Theorem 2.11] for $*$-fusion frames with invertible $*$-fusion frame bounds.

Theorem 3.2. *(Reconstruction formula) Let $\{(M_i, v_i) : i \in I\}$ be a $*$-fusion frame for a Hilbert $\mathcal{A}$-module E with $*$-fusion frame operator S and strictly nonzero $*$-fusion frame bounds C and D in the center of unital locally C^*-algebra $\mathcal{A}$. Then, S is a positive, self-adjoint and invertible operator on E such that for each $x \in E$ and $p \in S(\mathcal{A})$*

$$(p(C^{-1}))^{-2}\bar{p}_E(x) \le \bar{p}_E(S^{\frac{1}{2}}) \le (p(D))^2\bar{p}_E(x)$$

and

$$x = \sum_{i\in I} v_i^2 S^{-1} P_{M_i}(x).$$

Proof. It is clear that $\langle S(x), y\rangle = \langle x, S(y)\rangle$, for each $x, y \in E$. Thus by Proposition 3.2, $S \in End^*_{\mathcal{A}}$ and $S^* = S$. Also for each $x \in E$, we have

$$\langle S(x), x\rangle = \sum_{i\in I} v_i^2 \langle P_{M_i}(x), x\rangle = \sum_{i\in I} v_i^2 \langle P_{M_i}(x), P_{M_i}(x)\rangle .$$

Hence S is a positive operator. Hence there is a positive element T in $End^*_{\mathcal{A}}(E)$ such that $S = T^*T$. We show that T has the closed range. Let $\{Tx_n\}$ be a sequence in R_T such that $Tx_n \to y$ as $n \to \infty$. Then for each $p \in S(\mathcal{A})$ we have

$$\begin{aligned} &p(C\langle x_n - x_m, x_n - x_m\rangle C^*) \\ &\le p(\langle S(x_n - x_m), x_n - x_m\rangle) \\ &= p(\langle T(x_n - x_m), T(x_n - x_m)\rangle), \end{aligned}$$

for $n, m \in \mathbb{N}$. Since $\{Tx_n\}$ is a cauchy sequence in E, so $p(C\langle x_n - x_m, x_n - x_m\rangle C^*) \to 0$, for $n, m \in \mathbb{N}$. Moreover

$$\begin{aligned} &p(\langle x_n - x_m, x_n - x_m\rangle) \\ &= p(C^{-1}C\langle x_n - x_m, x_n - x_m\rangle C^*(C^*)^{-1}) \\ &\le (p(C^{-1}))^2 p(C\langle x_n - x_m, x_n - x_m\rangle C^*). \end{aligned}$$

Hence the sequence $\{x_n\}$ is cauchy and so there is $x \in E$ such that $x_n \to x$ as $n \to \infty$. By the definition of $*$-fusion frames, we get

$$\begin{aligned} &p(\langle T(x_n - x), T(x_n - x)\rangle) \\ &\le (p(D))^2 p(\langle x_n - x, x_n - x\rangle). \end{aligned}$$

Therefore $\bar{p_E}(Tx_n - Tx) \to 0$ as $n \to \infty$ implies that $Tx = y$. Consequently R_T is closed. Similarly one can see that T is injective. Therefore $S = T^*T$ is invertible. Furthermore we have $\langle x, x\rangle \le C^{-1}\langle Sx, x\rangle (C^*)^{-1}$ and $\langle Sx, x\rangle \le D\langle x, x\rangle D^*$ and so for each $p \in S(\mathcal{A})$

$$\begin{aligned} &(p(C^{-1}))^{-2} p(\langle x, x\rangle) \\ &\le p(\langle Sx, x\rangle) \le (p(D))^2 p(\langle x, x\rangle), \end{aligned}$$

for each $x \in E$. Therefore

$$(p(C^{-1}))^{-2}\bar{p}_E(x) \le \bar{p}_E(S^{\frac{1}{2}}) \le (p(D))^2\bar{p}_E(x).$$

Also for each $x \in E$, we have

$$x = S^{-1}S(x) = \sum_{i\in I} v_i^2 S^{-1} P_{M_i}(x).$$

This completes the proof.

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