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Research Article

Cross-inefficiency with the Variable Returns to Scale in DEA

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Abstract

The cross-efficiency ranking method is a well-known method in DEA which is frequently used under the constant returns to scale assumption; while various applications exist based on the variable returns to scale (VRS). This is due to the presence of negative input-oriented VRS cross-efficiencies. In this paper, each cross-efficiency is replaced by an equivalent distance measure as inefficiency measure. Then, the cross-inefficiency method is developed under the VRS assumption.

Keywords : Data Envelopment Analysis; Negative cross-efficiency; Variable returns to scale; Cross-inefficiency; VRS production possibility set.

1 Introduction

Data envelopment analysis (DEA) is a technique to measure the relative efficiency of the homogenous decision making units (DMUs). Since this technique was introduced by Charnes et al. [5], extensive researches were conducted in DEA and many concepts have been introduced. One of these concepts being considered as an important factor in efficiency evaluation from the beginning is the returns to scale (RTS). This concept was discussed for the first time by Banker [3] and also Banker et al. [2] in DEA. By deleting the "Ray Unboundedness" postulate from the postulates of constructing the production possibility set (PPS), instead of the CCR model [5] which deals with the constant returns to scale (CRS), they achieved the BCC model [3] which assumes variable returns to scale (VRS). An important point about these classical models is that both of them divide the DMUs into two efficient and inefficient groups, while, there is often a need to fully rank them. For this reason, many ranking methods have been proposed based on various concepts. An important one of these concepts is cross-efficiency which was first developed by sexton et al. [19].

The classical DEA models evaluate the efficiency of each DMU in its best situation. For this purpose, each DMU is allowed to use its most favorable weights that are generally different from the best weights of the other DMUs. This is while all of the DMUs are experiencing similar circumstances. To deal with this issue, cross-efficiency ranking method uses the cross-evaluation efficiency scores to rank the DMUs.

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One of the advantages of this method is ranking the DMUs in a unique order; moreover, the unrealistic weight schemes are eliminated without predetermining any weight restrictions (Wu et al. [22]). This ranking method has been used in a variety of applications. For example, Ganji et al. [12] proposed a double frontier cross-efficiency method for measuring road safety performance. Liu et al. [16] presented an application of state key laboratories in China using the cross-efficiency prospect method. Huang et al. [13] proposed a coastal urban disaster vulnerability assessment method based on the cross-efficiency models. Chen et al. [8] used a cross efficiency model considering the game relationship of DMUs to evaluate and analyze the provincial electric energy efficiency of China. Yang and Wei [24] applied game cross-efficiency DEA to analyze the urban total factor energy efficiency of 26 Chinese prefectural-level cities from 2005 to 2015 under environmental constraints.

Despite the many advantages of cross-efficiency ranking method, there is an important point about this method that undermines its validity. That is the presence of negative cross-efficiencies in the conventional input-oriented cross-efficiency method under the VRS assumption. While, in the DEA literature, all the efficiency scores of DMUs must have non-negative values. This issue can be considered as the main reason to use the cross-efficiency method almost exclusively with CCR model, while, VRS is one of the most common assumptions in efficiency evaluation done by DEA. There are no many studies to address this issue. For instance, when this issue occurred in the Soares de Mello et al. [11], the DMUs that generated the negative efficiencies in the cross-evaluation matrix were not taken into account for the ranking. While, it must be discussed that why did these negative efficiencies appear and how should this issue were interpreted? Angulo-Meza et al. [1] and Wu et al. [23] to avoid the negative efficiencies, without further analyses, added a set of constraints in the BCC multipliers model. However, their small intuitive change in the BCC multipliers model changes the original frontier of the production possibility set under the variable returns to scale assumption (T_v). In the following, Soares de Mello et al. [10] showed why the aforementioned constraints were added and they compared the modified BCC multipliers model with Non-Decreasing Returns to Scale (NDRS) model, proposed by Charnes et al. [6] and Cooper et al. [9], which avoids negative efficiencies, too. They also graphically analyzed the addition of those constraints through the concept of non-observed DMUs which have been used previously by Thanassoulis et al. [21], Jahanshahloo and Soleimani- Damaneh [14], among others.

A remarkable point about the aforementioned methods is that all of them change the efficiency frontier of T_v (i.e., production possibility set under the VRS assumption) to avoid the negative efficiencies. Therefore, it is reasonable that the created change may reduce the validity of their ranking results.

In the following, Lim and Zhu [15] showed that negative VRS cross-efficiency is related to free production of outputs. In fact, they concluded that cross-efficiency evaluation under the VRS assumption is not proper in its conventional model regardless of whether the problem of negative cross-efficiency actually arises or not. Therefore, they developed some change in the framework of cross-efficiency evaluation based on a geometric view of the relationship between the VRS and CRS models. They proposed that VRS cross-efficiency evaluation should be done via a series of CRS models under translated Cartesian coordinate systems. In the better words, each DMU, via solving the VRS model, seeks an optimal bundle of weights with which its CRS-efficiency score, measured under a translated Cartesian coordinate system, is maximized. This approach also clearly changes the T_v frontier and it is logical that it cannot be considered as a ranking method under the VRS assumption.

In the current paper, similar to Lim and Zhu [15], we use a geometric interpretation of the cross-efficiency and try to address the negative cross-efficiency problem. However, our interpretation is from a totally different viewpoint which does not make any changes in production possibility set T_V. Indeed, this paper replaces the cross-efficiencies by the equivalent geometric quantities which include a particular distance measure from the supporting hyper-planes of T_V. This distance measure can be considered as an inefficiency measure. Therefore, based on the proposed distance measure, the cross-efficiency method is developed and transformed into the cross-inefficiency method.

The rest of the paper is organized as follows: Section 2 presents the conventional cross-efficiency method under the VRS assumption and shows that negative efficiencies may appear only in the input orientation of this method. Moreover, in this Section, the approach of Lim and Zhu [15] is briefly reviewed and discussed. In Section 3, a new development of the cross-efficiency method under the VRS assumption is proposed to avoid the negative efficiencies. The developed method is illustrated using a numerical example in Section 4. Finally, conclusions are provided in Section 5.

2 Cross-efficiency method with variable returns to scale

In this section, necessary preliminaries are reviewed and discussed including conventional cross-efficiency method under the VRS assumption along with Lim and Zhu's [15] approach to avoid the negative cross-efficiency.

2.1 Conventional VRS cross-efficiency

Consider n DMUs where each DMU_j uses the input vector $x_j = (x_{1j}, x_{2j}, ..., x_{mj})^t$ to produce the output vector $y_j = (y_{1j}, y_{2j}, ..., y_{sj})^t$. The input-oriented linear fractional model to evaluate the DMU_o, considering the variable returns to scale, is the model (2.1):

$$\begin{aligned} Max\ & \frac{u^t y_o + w}{v^t x_o} \\ s.t.\ & \frac{u^t y_j + w}{v^t x_j} \leq 1, \quad j = 1, \ldots, n, \\ & u \geq 0, v \geq 0. \end{aligned} \tag{2.1}$$

The value of $\frac{u^t y_o + w}{v^t x_o}$, which belongs to the interval $[0, 1]$, is considered as efficiency measure of DMU_o. In fact, model (2.1) obtains the possible maximum efficiency of DMU_o while the efficiency of the other DMUs cannot be greater than 1. According to DEA literature, model (2.1) can be linearized as the model (2.2):

$$\begin{aligned} Max\ & u^t y_o + w \\ s.t.\ & v^t x_o = 1, \\ & u^t y_j - v^t x_j + w \leq 0, \quad j = 1, \ldots, n, \\ & u \geq 0, v \geq 0. \end{aligned} \tag{2.2}$$

Model (2.2) is called the input-oriented multipliers BCC model. After obtaining the efficiency of DMU_o, cross-efficiency method uses the optimal solution of the model (2.2) to obtain the efficiencies of other DMUs. This means that if (u_o^*, v_o^*, w_o^*) $(o \in \{1, ..., n\})$ is an optimal set of weights for DMU_o evaluated by the model (2.2), then efficiency score of the DMUs corresponding to this set of weights denoted by θ_{oj}, is calculated by the relation (2.3) as follows:

$$\theta_{oj} = \frac{u_o^{t*} y_j + w_o^*}{v_o^{t*} x_j} \quad j = 1, \ldots, n \tag{2.3}$$

At the end, by solving the model (2.2) for all DMU_i $(i \in \{1, ..., n\})$, the efficiency index $\theta_j = \frac{1}{n}\sum_{i=1}^{n} \theta_{ij}$ $(j = 1, \ldots, n)$ corresponding to DMU_j can be obtained. As can be seen, there is no guarantee that the efficiency score of DMU_j is non-negative when it is evaluated by DMU_o, but this is meaningless. This may occur when the unrestricted variable "w" is negative enough. To avoid the negative efficiencies, one can consider the "w" as a positive variable in the model (2.2). In this way, model (2.2) is converted to NDRS model (Charnes et al. [6]; Cooper et al. [9]). Another way is adding the constraints $u^t y_j + w \geq 0$ $(j = 1, ..., n)$ to the model (2.2) (see [10]). However, as has been shown in Soares de Mello et al. [10], both ways probably change the efficiency frontier of T_V and this does not seem desirable.

The output oriented linear fractional model to evaluate the DMU_o considering variable returns to scale is considered as the model (2.4):

$$\begin{aligned} Min\ & \frac{v^t x_o - w}{u^t y_o} \\ s.t.\ & \frac{v^t x_j - w}{u^t y_j} \geq 1, \quad j = 1, \ldots, n, \\ & u \geq 0, v \geq 0. \end{aligned} \tag{2.4}$$

In this orientation, the efficiency score of DMU_j corresponding to DMU_o is defined as $\phi_{oj} =$

$\frac{u_o^{t*}y_j}{v_o^{t*}x_j - w_o^*}$, where (u_o^*, v_o^*, w_o^*) is an optimal solution of the model (2.4). According to the constraints of the model (2.4), it can be seen that none of the efficiency scores can be negative.

Remark 2.1 *Since the negative efficiency issue may only appear in the input orientation under the VRS assumption, the current paper discusses this orientation.*

2.2 Lim and Zhu's [15] approach to avoid the negative cross-efficiency

Lim and Zhu [15] claimed that the negative efficiency issue is caused by situations where weights chosen by some DMUs are invalid for cross-evaluating other DMUs. In better words, the optimal weights chosen by a VRS-efficient DMU exhibiting IRS or DRS are not valid for cross-evaluating other DMUs. In fact, they believe that it is related to free production of outputs and some kind of adjustment is required for those invalid weights to be properly used for cross-efficiency evaluation. Here, we briefly review their approach to deal with the free production of outputs and overcome the negative efficiency problem. Moreover, some notation about this approach are presented at the end.

To better understand, similar to Lim and Zhu [15], a one-input one-output simple graphical example is used. Consider 5 DMUs under the VRS assumption along with their corresponding T_v as shown in Fig. 1.

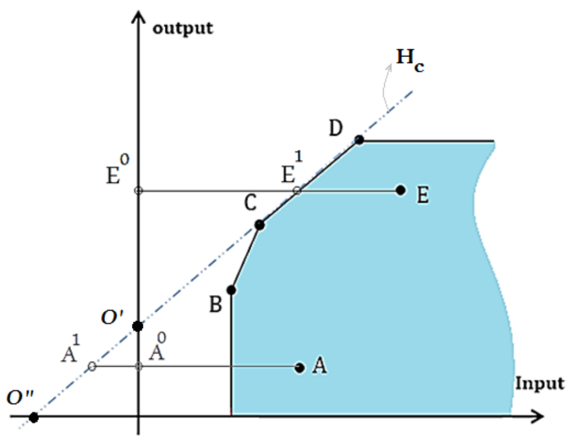

Figure 1: 5 DMUs with single input and single output.

According to the DEA literature, each of the optimal set of weights related to the DMU under evaluation corresponds to a supporting hyperplane of T_v. The supporting hyperplane H_c associated with an optimal set of weights chosen by DMU C is described through dashed line in Fig. 1. Now, cross-efficiencies of other DMUs evaluated by DMU C can be determined with reference to the hyperplane H_c. For example, a cross-efficiency of DMU E is $\frac{E^0E^1}{E^0E}$. There is no problem in calculating the cross-efficiency of DMUs, except for DMU A. In fact, model (2.2) forces the cross-efficiency of DMU A to be determined in reference to the negative-input segment of hyperplane H_c. In better words, the negative sign of DMU cross-efficiency A i.e. $\frac{A^0A^1}{A^0A}$ is due to the position of A_1. Lim and Zhu [15] claimed that the optimal set of weights chosen by DMU C is not valid for determining cross-efficiency of DMU A, because, the efficient frontier associated with the optimal weights chosen by DMU C extends to induce the unacceptable point O', which represents a free production of outputs in the underlying technology. According to Podinovski and Bouzdine-Chameeva [18], when a technology allows producing positive outputs with zero inputs, it is said to allow the free production of outputs. However, Lim and Zhu [15] extended this definition to include the case of negative outputs with zero inputs. Therefore, they named 'positive outputs with zero inputs' and 'negative outputs with zero inputs' as 'type I' and 'type II' of free production of outputs, respectively. Type II of free production of outputs arises when the supporting hyperplane associated with an optimal set of weights chosen by the DMU under evaluation collides with the output-axis below the input-axis. This can be seen in related to DMU B in Fig. 2. Here, O'' is a point that represents the type II of free production of outputs.

Although the negative cross-efficiency problem does not occur in type II free production of outputs, Lim and Zhu [15] believed their claim made in type I is still applied (to provide a more general framework).

According to the above observations and considering that any supporting hyperplane of the efficient frontier in the CRS model does not extend to induce the free production of outputs along with its perpetual validity for cross-efficiency

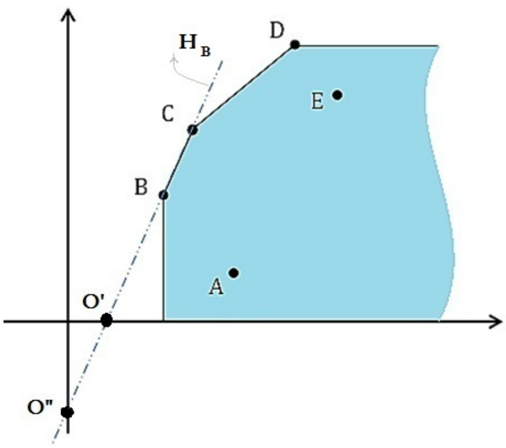

Figure 2: Type II of free production of outputs.

evaluation, Lim and Zhu [15] proposed to do the VRS cross-efficiency evaluation using a series of CRS models under translated Cartesian coordinate systems. For example, related to Fig. 1 and 2, it is sufficient that the origin point in Fig. 1 and Fig. 2 are transferred to the point O'' and O', respectively and the new production possibility sets are considered under the CRS assumption. The scientific basis that creates a link between the VRS and CRS models is provided in Theorem 2.1.

Theorem 2.1 *Given any optimal solution* (u_o^*, v_o^*, w_o^*) *from the model* (2.2) *chosen by a VRS-efficient* DMU_o, *a CRS-efficiency score of* DMU_o, *measured under the translated Cartesian coordinate system defined by an adjusted origin* $O^* = (\frac{\beta_1 w_o^*}{v_{o1}^*}, ..., \frac{\beta_m w_o^*}{v_{om}^*}, \frac{-\beta_{m+1} w_o^*}{u_{o1}^*}, ..., \frac{-\beta_{m+s} w_o^*}{u_{os}^*})$, *is unity, for any* $\beta_k \in R^+ (k = 1, ..., m+s)$ *such that* $\sum_{k=1}^{m+s} \beta_k = 1$. *[15]*

Corollaries 2.1 and 2.2 result from Theorem 2.1 (their proofs are available in [15]).

Corollary 2.1 *The supporting hyperplane of the efficient frontier associated with an optimal set of weights in model* (2.2) *chosen by a VRS-efficient DMU exhibiting DRS extends to induce type I free production of outputs in the underlying technology.*

Corollary 2.2 *The supporting hyperplane of the efficient frontier associated with an optimal set of weights in the model* (2.2) *chosen by a VRS-efficient DMU exhibiting IRS extends to induce type II free production of outputs in the underlying technology.*

According to Theorem 2.1 and its corollaries, the optimal weights chosen by a VRS- efficient DMU exhibiting IRS or DRS are not valid for cross-evaluating other DMUs. Moreover, it is concluded that the VRS model for any DMU can be casted as the CRS model for the same DMU under a translated Cartesian coordinate system. To this end, it is sufficient when DMU_o cross-evaluates DMU_j using its optimal solution (u_o^*, v_o^*, w_o^*) from the model (2.2) , the translation of the coordinate system is considered defined by an adjusted origin $O^* = (\beta_1 w_o^*/v_{o1}^*, ..., \beta_m w_o^*/v_{om}^*, 0, ..., 0)$ where 0 repeats s times for the output associated coordinates, $\sum_{k=1}^{m} \beta_k = 1$ and $\beta_k \in R^+ (k = 1, ..., m)$. With this translation, a CRS cross-efficiency θ'_{oj} of DMU_j is determined as the relation (2.5):

$$\theta'_{oj} = \frac{u_o^{t*} y_j}{v_o^{t*} x_j - w_o^*} \quad j = 1, \ldots, n \tag{2.5}$$

It can easily be proved that the cross-efficiencies θ'_{ij} $(i, j = 1, ..., n)$ obtained from the relation (2.5), are positive and less than or equal to unity. Note that the relation (2.5) is used for a VRS cross-efficiency of DMU_j (evaluated by DMU_o) under the original coordinate system. It means that (u_o^*, v_o^*, w_o^*) in the relation (2.5) is an optimal set of weights obtained from the conventional VRS model (2.2).

At the end, the attention to some points related to Lim and Zhu's [15] approach seems to be necessary. Firstly, it should be noted that this approach changes the VRS production possibility set (T_v) with the CRS one (T_c), while, the evaluation is considered on the VRS assumption. On the other hand, by translating the coordinate system, the original size of the DMUs will be changed. In this regard, the authors believe that this change likely has the same effect on all the DMUs and does not make a change in the ranking of DMUs. However, as they themselves have pointed out, this is when the shape of the PPS is not changed. Actually, this translation in the VRS frontiers shape seems to be able to cause ambiguity in calculating the efficiency of the DMUs. This issue is visible in Fig. (3).

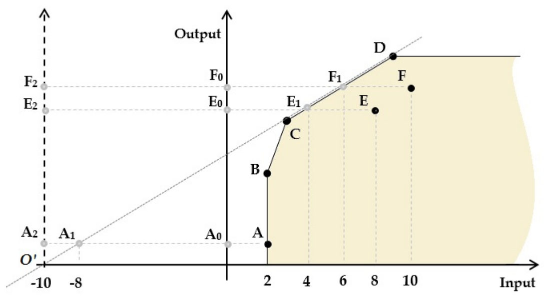

Figure 3: DMUs with unusual production technology.

Based on Fig. 3, the cross-efficiency of DMU F and DMU E (evaluated by DMU D) according to the original coordinate system are $\frac{F_0F_1}{F_0F} = 0.6$ and $\frac{E_0E_1}{E_0E} = 0.5$, respectively. These cross-efficiencies according to the translated coordinate system (with the origin point O') change to $\frac{F_2F_1}{F_2F} = 0.8$ and $\frac{E_2E_1}{E_2E} = 0.78$, respectively. In other words, according to original coordinate system, DMU F and DMU E to reach their project on the hyperplane H_D had to move as much as $\frac{F_1F}{F_0F} = 0.4$ and $\frac{E_1E}{E_0E} = 0.5$ of their input vectors, respectively; while according to translated coordinate system, these amounts change to $\frac{F_1F}{F_2F} = 0.2$ and $\frac{E_1E}{E_2E} = 0.22$, respectively. Here, it is well seen that the translation has a same effect on both DMU F and DMU E.Indeed, the size of the route for both DMUs has almost halved. However, that is not true about the DMU A. According to original coordinate system, DMU A had to move as much as 5 $(= \frac{A_1A}{A_0A})$ times that of its input vector; while this amount according to translated system is $\frac{A_1A}{A_2A} = 0.83$. The DMUs like DMU A often earn large negative cross-efficiencies in the conventional VRS cross-evaluation. Inspired by the work of Sexton et al. [19], we call such DMUs as a "DMUs with unusual production technology". Now, by considering the possibility of existence of the DMUs with unusual production technology, the question arises: which coordinate systems should be used to calculate the efficiency? In this paper, we would prefer to use the original coordinate system. Because, in this situation, each DMU is evaluated based on its original size.

According to the mentioned points, in the next section, a development of the conventional VRS cross-efficiency, is proposed to change neither the T_v nor the coordinate system, and lead to the similar results with Lim and Zhu's [15] approach. Of course, as noted, it is clear that the results obtained from an approach may have significant differences with the results obtained from another one for some units (such as DMU A with unusual production technology in Fig. 3).

3 Cross-inefficiency method under the VRS assumption

The concept of the cross-efficiency was first introduced by Sexton et al. [19] as a way to identify the DMUs which have an unusual production technology in the production possibility set under the CRS assumption (T_c). Accordingly, since the ratio formulation of the DEA model places weights directly on the individual inputs and outputs, they used that version (the multiplier CCR model). Since then, many papers have been published in which concept of the cross-efficiency is used for the other purposes, such as ranking DMUs, target setting, project selection, etc (e.g. Oral et al. [17]; Chen and Wang [7], Zhou et al. [25]).

Use of the cross-efficiency concept for ranking the DMUs has no problem while the production technology is considered with CRS assumption. The problem (i.e. negative cross-efficiency) appears when the concept is used with VRS assumption. Here, an interesting question to ask is that: why should θ_{oj} presented in the relation (2.3) be considered as an efficiency measure for DMU_j? In the following, to answer this question, we show that each of the cross-efficiencies is equal to a geometric quantity which includes a distance measure according to the supporting hyper-planes of T_v. Then, this distance measure which can be considered as an inefficiency measure, is used as basis to develop the conventional input-oriented cross-efficiency method under the VRS assumption.

Again, we use a graphical example including 5 DMUs (plotted by black points) under the VRS assumption with single input and single output as shown in Fig. 4.

Assume (u_o^*, v_o^*, w_o^*) is an optimal set of weights chosen by DMU_o from the model (2.2). There-

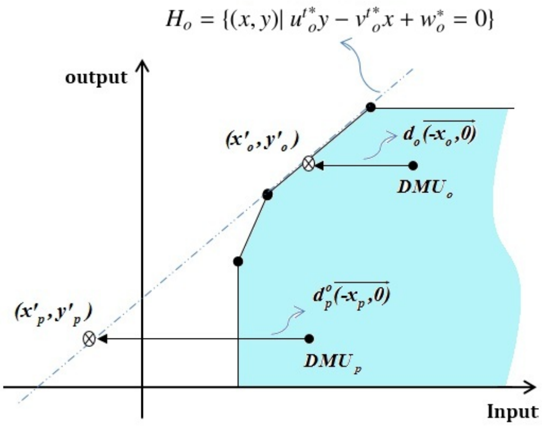

Figure 4: 5 DMUs with single input and single output.

fore, $H_o = \{(x,y)|\ u^{t*}_o y - v^{t*}_o x + w^*_o = 0\}$ is a supporting hyper-plane of T_v such that DMU_o is projected onto it. This hyper-plane is described with dashed line in Fig. 4.

Now, consider the distance measure d_{oo} as defined in relation (3.6) to quantify the distance between the DMU_o and its project on the hyper-plane H_o, with its related equation $u^{t*}_o y - v^{t*}_o x + w^*_o = 0$:

$$(x'_o, y'_o) = (x_o, y_o) + d_{oo}(-x_o, 0) \tag{3.6}$$

where (x'_o, y'_o) is the project of DMU_o on the hyper-plane H_o. More precisely, by moving in the direction of $(-x_o, 0)$ as much as d_{oo}, DMU_o is projected onto the (x'_o, y'_o) on the hyper-plane H_o. According to this definition, d_{oo} can be considered as inefficiency measure of DMU_o such that the larger its value, the more inefficiency is expected for DMU_o. From Fig. 4, it is obvious that $0 \le d_{oo} \le 1$. The more d_{oo} value is closer to 0, the more efficient the DMU_o and conversely, the more d_{oo} value is closer to 1, the more inefficient the DMU_o. The exact relationship between the cross-efficiency θ_{oj} and the distance measure d_{oj} (for all $j = 1, ..., n$) is expressed in Theorem 3.1.

Theorem 3.1 *Suppose θ_{oj} is a cross-efficiency of DMU_j $(j = 1, ..., n)$ evaluated by DMU_o, according to the optimal set of weights (u^*_o, v^*_o, w^*_o). Moreover, let d_{oj} be the distance measure that satisfies the equality $(x'_j, y'_j) = (x_j, y_j) + d_{oj}(-x_j, 0)$; where, (x'_j, y'_j) is the project of the DMU_j on the hyper-plane $H_o = \{(x,y)|u^{t*}_o y - v^{t*}_o x + w^*_o = 0\}$. Then, $\theta_{oj} = 1 - d_{oj}$ for all $j = 1, ..., n$.*

Proof. Since $(x'_j, y'_j) = (x_j - d_{oj}x_j, y_j)$ is a projection on the hyper-plane H_o then the relation (3.7) should be satisfied as follows:

$$u^{t*}_o y_j - v^{t*}_o (x_j - d_{oj}x_j) + w^*_o = 0 \tag{3.7}$$

or equivalently as the relation (3.8):

$$u^{t*}_o y_j - v^{t*}_o x_j (1 - d_{oj}) + w^*_o = 0 \tag{3.8}$$

now, relation (3.9) is concluded from the relation (3.8) as follows:

$$\frac{u^{t*}_o y_j + w^*_o}{v^{t*}_o x_j} = 1 - d_{oj} \tag{3.9}$$

The left side of the relation (3.9) is the same cross-efficiency θ_{oj} and the proof is complete. □

Here, the reason of the existence of the negative cross-efficiency is determined. In the better words, although d_{oj} is non-negative for all $j = 1, ..., n$, it is not necessarily less than 1 for all of them. Thus, θ_{oj} may be negative for some $j \in \{1, ..., n\}$. For example, in the Fig. 4, the distance measure d_{op} corresponding to DMU_p is greater than 1, thus θ_{op} must be negative.

So far, providing Theorem (3.1), it is shown that the cross-efficiency of a DMU is actually based on the particular distance measure. Because this distance measure can be considered as an inefficiency index, it is logical that this measure itself is directly used as a ranking measure. Accordingly, we name the d_{oj} as a "cross-inefficiency of the DMU_j evaluated by DMU_o" and based on that develop the conventional VRS cross-efficiency method[1] to the cross-inefficiency method as follows:

VRS cross-inefficiency method:

Step 0. Solve the linear programming model (3.10) for all $i = 1, ..., n$:

$$\begin{aligned} &Max\ u^t y_i + w \\ &s.t.\ v^t x_i = 1, \\ &\quad u^t y_j - v^t x_j + w \le 0,\ j = 1, ..., n, \\ &\quad u \ge 0, v \ge 0. \end{aligned} \tag{3.10}$$

[1]Here, we develop the VRS cross-efficiency method in the input orientation but all of the mentioned statements and relations can be generalized to the output orientation.

Suppose that (u_i^*, v_i^*, w_i^*) is an optimal set of weights corresponding to DMU_i. Then, go to step 1.

Step 1. Obtain the cross-inefficiency $\hat{\theta}_{ij}$ from the relation (3.11) as follows:

$$\theta_{ij} = \frac{u_i^{t*} y_j + w_i^*}{v_i^{t*} x_j} \Longrightarrow \quad \hat{\theta}_{ij} = 1 - \theta_{ij} \quad (3.11)$$

Step 2. Calculate the cross-inefficiency score $\hat{\theta}_j = \frac{1}{n} \sum_{i=1}^{n} \hat{\theta}_{ij}$ of DMU_j for all $j = 1, ..., n$.

Step 3. Rank the DMUs using the inefficiency indices $\hat{\theta}_j$s $(j = 1, ..., n)$ such that the more larger $\hat{\theta}_j$, the worse rank DMU_j has.

■

Remark 3.1 *In the cases of non-uniqueness of the optimal set of weights in model (3.10), a set of secondary goals can be added to this model (as the existing secondary goal models like the neutral models [20] and [4]).*

Remark 3.2 *The proposed VRS cross-inefficiency method ranks all of the DMUs without applying any changes in the T_v. However, a major objection raised against the developed method may be that the ranking index $\hat{\theta}_j$ represents the distance between DMU_j and its project on a supporting hyper-plane of T_v, not necessary on the efficiency frontier of T_v. In other words, when the project of DMU_j is most probably outside of the T_v, this index cannot be appropriate as a ranking criterion.* [2] *In response to the aforementioned objection, according to Lim and Zhu [15], it can be stated that "in the circumstance of benchmarking, the efficient DMUs as defined by DEA may not necessarily form a production frontier, but rather lead to a best-practice frontier". Moreover, since the general concept of cross-efficiency is to look at the performance of a DMU by using other DMUs weights or facets, it is reasonable to apply the facets to all DMUs and to generate VRS cross-efficiency [15].*

[2] That is the same issue which also exists in all versions of the cross-efficiency method under the both VRS and CRS assumptions. Perhaps for this reason, Sexton et al. [19] uses the concept of cross-efficiency just to identify the DMUs which have an unusual production technology, not for ranking DMUs (contrary to what has been known in the DEA literature).

4 Numerical example

In this section, similar to Lim and Zhu [15], the proposed cross-inefficiency method is applied to the set of 37 project proposals relating to the Turkish iron and steel industry studied in Oral et al. [17]. The related data are listed in Table 1. Direct economic contribution, indirect economic contribution, technological contribution, scientific contribution, and social contribution are the outputs of each project, along with the budget as its single input. The purpose is to select the projects by the decreasing order of their cross-efficiency scores until the allowance of considered budget for the program (given 1000).

Table 1: Data set of 37 project proposals with 5 outputs and single input.

Project	Direct eco. contribution	Indirect eco. contribution	Technological contribution	Scientific contribution	Social contribution	Budgete
1	67.53	70.82	62.64	44.91	46.28	84.2
2	58.94	62.86	57.47	42.84	45.64	90
3	22.27	9.68	6.73	10.99	5.92	50.2
4	47.32	47.05	21.75	20.82	19.64	67.5
5	48.96	48.48	34.9	32.73	26.21	75.4
6	58.88	77.16	35.42	29.11	26.08	90
7	50.1	58.2	36.12	32.46	18.9	87.4
8	47.46	49.54	46.89	24.54	36.35	88.8
9	55.26	61.09	38.93	47.71	29.47	95.9
10	52.4	55.09	53.45	19.52	46.57	77.5
11	55.13	55.54	55.13	23.36	46.31	76.5
12	32.09	34.04	33.57	10.6	29.36	47.5
13	27.49	39	34.51	21.25	25.74	58.5
14	77.17	83.35	60.01	41.37	51.91	95
15	72	68.32	25.84	36.64	25.84	83.8
16	39.74	34.54	38.01	15.79	33.06	35.4
17	38.5	28.65	51.18	59.59	48.82	32.1
18	41.23	47.18	40.01	10.18	38.86	46.7
19	53.02	51.34	42.48	17.42	46.3	78.6
20	19.91	18.98	25.49	8.66	27.04	54.1
21	50.96	53.56	55.47	30.23	54.72	74.4
22	53.36	46.47	49.72	36.53	50.44	82.1
23	61.6	66.59	64.54	39.1	51.12	75.6
24	52.56	55.11	57.58	39.69	56.49	92.3
25	31.22	29.84	33.08	13.27	36.75	68.5
26	54.64	58.05	60.03	31.16	46.71	69.3
27	50.4	53.58	53.06	26.68	48.85	57.1
28	30.76	32.45	36.63	25.45	34.79	80
29	48.97	54.97	51.52	23.02	45.75	72
30	59.68	63.78	54.8	15.94	44.04	82.9
31	48.28	55.58	53.3	7.61	36.74	44.6
32	39.78	51.69	35.1	5.3	29.57	54.5
33	24.93	29.72	28.72	8.38	23.45	52.7
34	22.32	33.12	18.94	4.03	9.58	28
35	48.83	53.41	40.82	10.45	33.72	36
36	61.45	70.22	58.26	19.53	49.33	64.1
37	57.78	72.1	43.83	16.14	31.32	66.4

Since some of the conventional cross-efficiencies will be negative, they are not valid to be used in ranking and selecting the project. To solve the problem, it is sufficient to calculate the cross-inefficiencies of the projects and then obtain the cross-inefficiency scores based on the relation

(3.11). These scores can be seen in the second column of Table 2 in increasing order. The corresponding cross-efficiency scores obtained from Lim and Zhu's [15] approach along with their relevant ranks are listed in columns 3 and 4. For comparison between the existing methods, the same results related to the NDRS cross-efficiency method [6, 9] and Soares de Mello et al.'s approach [10] are listed in columns 5 to 8. In addition, the cross- and simple VRS efficiency scores can be seen in columns 9 and 10, respectively. At the end, the project selection results based on the cross-inefficiency scores and the cross-efficiency scores obtained from Lim and Zhu's [15] approach are shown in columns 11 and 12, respectively. As

Table 2: Project selection results in increasing order with respect to the cross-inefficiency scores.

Projects	Cross-inefficiency scores	Lim and Zhu's [15] cross-efficiency scores	Ranks	NDRS [6, 9] cross-efficiency scores	Ranks	Soares et al.'s approach [10] cross-efficiency scores	Ranks	Conventional cross-efficiency scores	Simple VRS efficiency scores	Selection with cross-inefficiency	Selection with Lim and Zhu's [15] approach	Budget
17	0.13084	0.942	1	0.98848	1	0.99161	1	0.9237	1	Yes	Yes	32.1
36	0.14751	0.7699	3	0.68919	6	0.75497	4	0.8297	1	Yes	Yes	64.1
23	0.22557	0.7178	5	0.59871	10	0.66433	7	0.7533	1	Yes	Yes	75.6
14	0.24301	0.7066	7	0.54665	14	0.64262	10	0.7303	1	Yes	Yes	95
27	0.27869	0.711	6	0.6672	8	0.69797	6	0.7214	0.8935	Yes	Yes	57.1
1	0.28759	0.6892	8	0.5588	13	0.6378	11	0.6895	1	Yes	Yes	84.2
31	0.29031	0.7635	4	0.80412	3	0.80874	3	0.7018	1	Yes	Yes	44.6
35	0.29863	0.8411	2	0.95321	2	0.96746	2	0.7161	1	Yes	Yes	36
26	0.32913	0.6715	12	0.58529	11	0.62732	12	0.6588	0.8955	Yes	Yes	69.3
21	0.37931	0.6335	14	0.52615	16	0.55529	14	0.6199	1	Yes	Yes	74.4
37	0.41981	0.6719	11	0.62406	9	0.65997	8	0.5708	1	Yes	Yes	66.4
2	0.43736	0.5991	17	0.47935	22	0.52675	19	0.5525	0.7629	Yes	-	90
11	0.44594	0.5996	15	0.50744	18	0.54717	15	0.547	0.71	Yes	Yes	76.5
30	0.44788	0.5974	18	0.49442	20	0.54036	16	0.5389	0.7232	Yes	-	82.9
24	0.4518	0.5807	19	0.4427	29	0.47363	25	0.5459	1	-	-	92.3
29	0.46377	0.5995	16	0.52353	17	0.53807	17	0.5372	0.6809	-	Yes	72
10	0.47911	0.58	20	0.49035	21	0.51743	20	0.5171	0.6452	-	-	77.5
22	0.51842	0.551	24	0.44947	27	0.491	22	0.4861	0.7091	-	-	82.1
18	0.52247	0.652	13	0.68578	7	0.65436	9	0.5046	0.7781	-	Yes	46.7
19	0.56001	0.5399	25	0.45717	25	0.48872	23	0.4465	0.6412	-	-	78.6
6	0.58227	0.5612	23	0.47869	23	0.50768	21	0.4158	0.8866	-	-	90
15	0.585	0.5706	21	0.49602	19	0.58651	13	0.4154	1	-	-	83.8
9	0.65064	0.5013	26	0.41521	31	0.4453	29	0.3538	0.7169	-	-	95.9
16	0.67264	0.6831	10	0.78385	4	0.74929	5	0.3767	0.9238	-	Yes	35.4
8	0.68223	0.4676	29	0.39183	33	0.39943	31	0.323	0.4468	-	-	88.8
32	0.69882	0.5614	22	0.58123	12	0.53443	18	0.3264	0.6481	-	-	54.5
5	0.74401	0.4815	28	0.44424	28	0.45967	27	0.2713	0.5667	-	-	75.4
7	0.76135	0.4628	30	0.41195	32	0.42069	30	0.246	0.6368	-	-	87.4
4	0.87892	0.4487	32	0.45023	26	0.45453	28	0.1504	0.5384	-	-	67.5
28	0.87996	0.3784	35	0.33812	36	0.29108	35	0.1518	0.3895	-	-	80
25	0.90249	0.3915	33	0.36836	34	0.31466	33	0.1375	0.4502	-	-	68.5
12	0.90462	0.4927	27	0.54222	15	0.46024	26	0.1458	0.6489	-	-	47.5
13	0.90799	0.4508	31	0.46879	24	0.37406	32	0.1351	0.5453	-	-	58.5
33	1.10449	0.3858	34	0.42958	30	0.31253	34	-0.0452	0.5588	-	-	52.7
20	1.21684	0.3157	36	0.35015	35	0.21955	36	-0.1435	0.5513	-	-	54.1
34	1.5933	0.6851	9	0.75384	5	0.4801	24	-0.4647	1	-	Yes	28
3	1.68723	0.1739	37	0.26332	37	0.13952	37	0.5870	0.568	-	-	50.2
Budget sum										948.2	957.4	

can be seen in Table 2, there are some differences in ranking orders between the two methods and thus they choose different set of projects. Actually, the cross-inefficiency method leads to select a smaller number of the projects (just two projects). However, the selection of both methods are largely identical.

It should be noted that among the projects, there is also a project that its rank obtained by cross-inefficiency method is very different from its rank obtained by Lim and Zhu's [15] approach. That is Project 37 which is ranked 9th by Lim and Zhu's [15] approach; while it is ranked 36th by the cross-inefficiency method. In fact, according to the negativity of its simple cross-efficiency, this project may be from the same DMUs which have unusual production technologies.

5 Conclusions

This paper tried to address one of the main disadvantages of the conventional cross-efficiency method, namely the existence of negative efficiency. To this end, it was shown that the cross-efficiency was based on the particular distance measure with respect to the input vector of each DMU. Then, this distance measure was used to construct the new ranking index. In this way, a new development of the conventional cross-efficiency method under the VRS assumption was proposed as cross-inefficiency method. It should be mentioned that when there is no DMU with unusual production technology, there is no difference between the ranks obtained from the conventional cross-efficiency method and cross-inefficiency method. Moreover, it should be noted that the development idea (i.e. using the distance measure) has been presented in the input orientation, but it can be generalized to the output orientation.

As a main result of this paper, from the DEA point of view, we believe that when a number of DMUs with unusual production technology exist, it seems better not to use the VRS cross-evaluation (neither cross-efficiency or cross-inefficiency); even when they do not exist, not because of inducing free production of outputs (as pointed out by Lim and Zhu [15]) but because of comparing some DMUs with the projects which are not in the production possibility set (as pointed out in Remark 3.2). Unless, that is used from the other points of view such as benchmarking in operations management.

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