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Solving Volterra Integral Equations of the Second Kind with Convolution Kernel

M. S. Barikbin [*], A. R. Vahidi [†‡], T. Damercheli [§]

Abstract

In this paper, we present an approximate method to solve the solution of the second kind Volterra integral equations. This method is based on a previous scheme, applied by Maleknejad et al., [K. Maleknejad and Aghazadeh, Numerical solution of Volterra integral equations of the second kind with convolution kernel by using Taylor-series expansion method, Appl. Math. Comput. (2005)] to gain the approximate solution of the second kind Volterra integral equations with convolution kernel and Maleknejad et al. [K. Maleknejad and T. Damercheli, Improving the accuracy of solutions of the linear second kind volterra integral equations system by using the Taylor expansion method, Indian J. Pure Appl. Math. (2014)] to gain the approximate solutions of systems of second kind Volterra integral equations with the help of Taylor expansion method. The Taylor expansion method transforms the integral equation into a linear ordinary differential equation (ODE) which, in this case, requires specified boundary conditions. Boundary conditions can be determined using the integration technique instead of differentiation technique. This method is more stable than derivative method and can be implemented to obtain an approximate solution of the Volterra integral equation with smooth and weakly singular kernels. An error analysis for the method is provided. A comparison between our obtained results and the previous results is made which shows that the suggested method is accurate enough and more stable.

Keywords : Taylor expansion; Volterra integral equation; Error analysis; Approximate solution.

1 Introduction

[*]Department of Mathematics, Yadegar-e-Emam Khomeyni (RAH) Shahr-e-Rey Branch, Islamic Azad University, Tehran, Iran.

[†]Corresponding author. alrevahidi@yahoo.com, Tel:+98(912)3599569.

[‡]Department of Mathematics, Yadegar-e-Emam Khomeyni (RAH) Shahr-e-Rey Branch, Islamic Azad University, Tehran, Iran.

[§]Department of Mathematics, Yadegar-e-Emam Khomeyni (RAH) Shahr-e-Rey Branch, Islamic Azad University, Tehran, Iran.

I N this paper, we focus on the second kind Volterra integral equations as follow

$$\varphi(t) + \lambda \int_0^t k(t,x)\varphi(x)dx = g(t), \quad 0 \leq t \leq 1. \tag{1.1}$$

In Eq. (1.1), the functions $k(t,x)$, $g(t)$ and λ are known, and $\varphi(x)$ is the unknown function to be determined. It is assumed that Eq.

(1.1) has a unique solution and for more information about the necessary and sufficient conditions for existence and uniqueness of the solution of the Eq. (1.1), the interested reader can refer to [3]. Moreover, it is assumed that the kernel $k(t, x) = k(\mid t - x \mid)$ with k continuous in $I = [0, 1]$ and decreases as $\mid t - x \mid$ increases from zero or that $k(t, x) = a(t, x)\kappa(t - x)$ with $a(t, x)$ which is continuous for $t, x \in I$ and κ is weakly singular, meaning that $\kappa(t - x) = O(\mid t - x \mid^{-\alpha})$, $0 < \alpha < 1$.

Fredholm and Volterra integral equations play an important role in many physical applications [18, 19] which include potential theory and Dirichlet problems [12, 13, 16], particle transport problems of astrophysics [20] and radioactive heat transfer problems [2, 4].

There are many approaches to obtain the solution of a usual Volterra integral equation. In general, many numerical methods such as Galerkin [5], collocation [14] and quadrature methods [1, 3, 9, 10] convert the integral equation into a linear system of algebraic equations that can be solved by direct or iterative methods.

Differing from existing methods, Volterra integral equation of the form (1.1), using the Taylor series expansion of the unknown function, can be converted into a linear ODE that can be solved. But in these methods, specified boundary conditions is needed to determine which is not possible. However, the above ODE is not directly solved. To obviate this problem, there are two approaches. We can either differentiate or integrate both sides of ordinary differential equation.

Ren et al. in [15] firstly used the derivation approach for solving Fredholm integral equations and further Maleknejad and Aghazadeh [7] used it for solving Volterra integral equations and then Maleknejad and Damercheli [8] expanded it for systems of Volterra integral equations. This method determines an approximation of the solution for Fredholm and Volterra integral equations. The accuracy of the method for Volterra integral equations is better than that of Fredholm interal equation, which is due to well-posedness of Volterra integral equations in comparison with Fredholm integral equations. However, the accuracy of the solution for both Fredholm and Volterra integral equations is not satisfying at the end of the interval (see the Examples 1, 2, 3 in [7, 15]). On the other hand, the numerical results show that the method is effective under the convolution kernel, particularly, when the rate of the decay to 0 of $k(\mid t - x \mid)$ is sufficiently large, but the technique does not perform well under the weakly singular kernel (see the Example 4 in [7, 15]) since the approximation needs derivations of the kernel which are hypersingular and would lead to inaccurate approximation.

In this paper, a new application of Taylor expansion is presented to solve the Volterra integral equations of the second kind, approximately. Taylor expansion of the unknown function $\varphi(x)$ is used to obtain the solution. Then the integration method is employed for determining specified boundary conditions, for converted linear ordinary differential equation. This method is more stable compared with derivative method and can be applied to approximate the solution of the Volterra integral equation with smooth and weakly singular kernels. Also, it is shown that when the present method is applied to the Volterra integral equation with a rapidly decaying kernel, it provides more accurate results than those obtained by the method in [7].

2 Determination of approximate solution

Consider the second kind Volterra integral equations (1.1) and without loss of generality suppose that $\lambda = 1$. In order to achieve the approximate solution, Taylors expansion of the unknown function $\varphi(x)$ at t is employed as follow

$$\varphi(x) = \varphi(t) + \varphi'(t)(x - t) + \cdots +$$
$$\frac{1}{n!}\varphi^{(n)}(t)(x - t)^n + R_n(x, t), \qquad (2.2)$$

where in relation (2.2), $R_n(t, x)$ indicates the Lagrange error bound as

$$R_n(x, t) = \frac{\varphi^{(n+1)}(\zeta)}{(n + 1)!}(x - t)^{n+1}, \qquad (2.3)$$

for some point ζ which is between t and x. In genral terms, the Lagrange error bound becomes sufficiently small as n gets great enough provided that $\varphi^{(n+1)}(t)$ is a uniformly bounded function.

It is to be noted that if we have the unknown function $\varphi(x)$ as a polynomial of the degree equal to or less than n, then the Lagrange error bound equals to zero i.e., we achieve the exact solution.

In the following, the approximate relation (2.2) is substituted for $\varphi(x)$ in the integrand into Eq. (1.1) and leads to

$$\varphi(t) + \sum_{j=0}^{n} \frac{(-1)^j}{j!} \varphi^{(j)}(t) \int_0^t k(t,x)(t-x)^j dt = g(t),$$
$$(2.4)$$

where in the above written relation, the Lagrange error bound has been eliminated for the reason that the truncated error is sufficiently small. In fact, Eq. (1.1) is converted into nth-order linear ODE (2.4) with respect to $\varphi(t)$ and its derivations up to order n. In the following, to determine $\varphi(t), ..., \varphi^{(n)}(t)$, a linear equations system has to be solved.

To materialize this aim, other n independent linear equations in terms of $\varphi(t)$ and its derivatives up to order n are needed, which is derived by integrating both sides of Eq. (1.1) n times as follow

$$\int_0^t (t-x)^{i-1}\varphi(x)dx +$$
$$\int_0^t \int_x^t (t-s)^{i-1}k(s,x)\varphi(x)dsdx = g_{(i)}(t)$$
$$i = 1, \cdots, n,$$
$$(2.5)$$

where

$$g_{(i)}(t) = \int_0^t (t-x)^{i-1}g(x)dx. \qquad (2.6)$$

Now, similarly, we insert approximate relation (2.2) for $\varphi(x)$ into Eq. (2.5) and we have

$$\int_0^t (t-x)^{i-1} \sum_{j=0}^n \frac{(-1)^j}{j!}\varphi^{(j)}(t)(t-x)^j dx + \quad (2.7)$$
$$\int_0^t k_i(t,x) \sum_{j=0}^n \frac{(-1)^j}{j!}\varphi^{(j)}(t)(t-x)^j dx = g_{(i)}(t),$$

where

$$k_i(t,x) = \int_x^t (t-s)^{i-1}k(s,x)ds, \quad i = 1, \ldots, n.$$
$$(2.8)$$

Therefore, we achieved a system of linear equations for $\varphi(x)$ as unknown function and its derivatives up to order n, composed of Eqs. (2.4) and (2.7), and we rewrite this system as

$$A(t)\phi(t) = G(t), \qquad (2.9)$$

where

$$A(t) = \begin{bmatrix} a_{00}(t) & a_{01}(t) & \ldots & a_{0n}(t) \\ a_{10}(t) & a_{11}(t) & \ldots & a_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0}(t) & a_{n1}(t) & \ldots & a_{nn}(t) \end{bmatrix}, \quad (2.10)$$

$$G(t) = \begin{bmatrix} g(t) \\ g_{(1)}(t) \\ \vdots \\ g_{(n-1)}(t) \end{bmatrix}, \phi(t) = \begin{bmatrix} \varphi(t) \\ \varphi'(t) \\ \vdots \\ \varphi^{(n)}(t) \end{bmatrix}.$$
$$(2.11)$$

We remember, in coefficient matrix (2.10), the first row denotes to coefficients of $\varphi^{(j)}(t)$ in Eq. (2.4) for $j = 0, \ldots, n$ and the other rows denote to coefficients of $\varphi^{(j)}(t)$ in Eq. (2.7) for $j = 0, \ldots, n$. Finally, a standard method is used to the above system which yields an nth-order approximate solution of Eq. (1.1).

3 Error analysis

Theorem 3.1 *If we have the solvability conditions of system (2.9) then the unique solution of system (2.9) is proposed as $\phi(t) = A(t)^{-1}G(T)$. At present, we show the components of $A(t)^{-1}$ of $(n+1) \times (n+1)$ dimension by $a_{kl}^*(t)$, $0 \leq k, l \leq n$. Besides, we assume that $a_{jl}^*(t)$ are uniformly bounded means there exists a positive constant like C independent of n as $| a_{jl}^*(t) | \leq C$ for $t \in I, j, l \leq n$, then we have*

$$| \varphi_n(t) - \varphi(t) | \leq CKH_n | I | \left(e^{|I|} - \sum_{j=0}^n \frac{| I |^j}{j!} \right),$$
$$(3.12)$$

where φ_n denotes the nth-order approximation of the exact solution $\varphi(t)$,
$H_n = max_{j>n, t \in I} |\varphi^{(j)}(t)|$, $K = max_{I \times I} k(t,x), t, x \in I$. As we mentioned earlier, φ_n equals to $\varphi(t)$ if $\varphi(t)$ be as a polynomial function of degree equal to or less than n.

Proof. See [6, 17].

Table 1: Comparison of the absolute errors for approximate solutions in Example 4.1.

t	n=0 for two methods	n= 2 method in [7]	Our approx.	n= 4 method in [7]	Our approx.	n= 6 method in [7]	Our approx.
0.1	0.1078×10^{-3}	2.7031×10^{-7}	2.2779×10^{-12}	2.5536×10^{-8}	1.1102×10^{-15}	1.2401×10^{-9}	6.6613×10^{-16}
0.2	0.8814×10^{-3}	0.1749×10^{-4}	5.8522×10^{-10}	9.3845×10^{-7}	6.8833×10^{-15}	7.9976×10^{-8}	8.8817×10^{-16}
0.3	0.3077×10^{-2}	0.2007×10^{-3}	1.4894×10^{-8}	0.1625×10^{-4}	3.4239×10^{-13}	9.2362×10^{-7}	1.1102×10^{-15}
0.4	0.7689×10^{-2}	0.1133×10^{-2}	1.4630×10^{-7}	0.1024×10^{-3}	5.7105×10^{-12}	5.3223×10^{-6}	1.1102×10^{-15}
0.5	0.0161	0.4330×10^{-2}	8.4963×10^{-7}	0.4104×10^{-3}	4.8663×10^{-11}	2.1147×10^{-5}	2.2204×10^{-16}
0.6	0.0308	0.0129	3.5277×10^{-6}	0.1254×10^{-2}	2.6865×10^{-10}	6.6936×10^{-5}	1.1102×10^{-15}
0.7	0.0553	0.0324	1.1593×10^{-5}	0.3197×10^{-2}	1.0880×10^{-9}	1.8206×10^{-4}	6.6613×10^{-16}
0.8	0.0950	0.0717	3.2044×10^{-5}	0.7150×10^{-2}	3.4732×10^{-9}	4.4439×10^{-4}	3.1086×10^{-15}
0.9	0.1578	0.1441	7.7487×10^{-5}	0.0144	9.1148×10^{-9}	9.9894×10^{-4}	3.1086×10^{-15}
1.0	0.2546	0.2681	1.6840×10^{-4}	0.2712	2.0098×10^{-8}	2.1016×10^{-3}	1.7763×10^{-15}

Table 2: Comparison of the absolute errors for approximate solutions in Example 4.2

t	n=0 for two methods	n= 2 method in [7]	Our approx.	n= 4 method in [7]	Our approx.	n= 6 method in [7]	Our approx.
0.1	0.0059	7.8437×10^{-8}	1.1102×10^{-16}	4.6960×10^{-8}	1.1102×10^{-16}	7.8478×10^{-8}	3.6637×10^{-15}
0.2	0.0241	2.5267×10^{-6}	2.2204×10^{-16}	1.4961×10^{-6}	0	2.5120×10^{-6}	2.2204×10^{-16}
0.3	0.0549	0.1932×10^{-4}	4.4408×10^{-16}	0.1121×10^{-4}	4.4408×10^{-16}	1.8968×10^{-5}	4.4408×10^{-16}
0.4	0.0984	0.8200×10^{-4}	8.8817×10^{-16}	0.4623×10^{-4}	8.8817×10^{-16}	7.9031×10^{-5}	8.8817×10^{-16}
0.5	0.1546	0.2517×10^{-3}	8.8817×10^{-16}	0.1366×10^{-3}	8.8817×10^{-16}	2.3715×10^{-4}	8.8817×10^{-16}
0.6	0.2234	0.6287×10^{-3}	0	0.3260×10^{-3}	8.8817×10^{-16}	5.7726×10^{-4}	8.8817×10^{-16}
0.7	0.3044	0.1360×10^{-2}	8.8817×10^{-16}	0.6683×10^{-3}	8.8817×10^{-16}	1.2147×10^{-3}	0
0.8	0.3971	0.2643×10^{-2}	0	0.1222×10^{-2}	0	2.2956×10^{-3}	0
0.9	0.5011	0.4726×10^{-2}	0	0.2046×10^{-2}	0	3.9936×10^{-3}	0
1.0	0.6157	0.7905×10^{-2}	1.7763×10^{-15}	0.3187×10^{-2}	8.8817×10^{-16}	6.5040×10^{-3}	8.8817×10^{-16}

Table 3: Comparison of the absolute errors for approximate solutions in Example 4.3.

| t	n=0 for two methods	n= 2 method in [7]	Our approx.	n= 4 method in [7]	Our approx.	n= 6 method in [7]	Our approx.
0.1	0.2071×10^{-4}	0.1229×10^{-4}	4.4553×10^{-13}	0.1229×10^{-4}	1.1102×10^{-16}	0.1229×10^{-4}	0
0.2	0.3255×10^{-3}	0.1867×10^{-3}	1.1360×10^{-10}	0.1867×10^{-3}	1.1102×10^{-16}	0.1867×10^{-3}	0
0.3	0.1601×10^{-2}	0.8647×10^{-3}	2.8898×10^{-9}	0.8647×10^{-3}	4.4408×10^{-16}	0.8647×10^{-3}	1.1102×10^{-16}
0.4	0.4864×10^{-2}	0.2393×10^{-2}	2.8564×10^{-8}	0.2393×10^{-2}	1.0547×10^{-14}	0.2393×10^{-2}	1.1102×10^{-16}
0.5	0.0113	0.4855×10^{-2}	1.6797×10^{-7}	0.4855×10^{-2}	1.5298×10^{-13}	0.4855×10^{-2}	1.1102×10^{-16}
0.6	0.0221	0.7807×10^{-2}	7.1040×10^{-7}	0.7807×10^{-2}	1.3454×10^{-12}	0.7807×10^{-2}	1.1102×10^{-16}
0.7	0.0383	0.0101	2.3909×10^{-6}	0.0101	8.4176×10^{-12}	0.0101	1.1102×10^{-16}
0.8	0.0608	0.0101	6.8019×10^{-6}	0.0101	4.1018×10^{-11}	0.0101	1.1102×10^{-16}
0.9	0.0901	0.5732×10^{-2}	1.7007×10^{-5}	0.5732×10^{-2}	1.6503×10^{-10}	0.5732×10^{-2}	4.4408×10^{-16}
1.0	0.1263	0.5152×10^{-2}	3.8378×10^{-5}	0.5152×10^{-2}	5.7060×10^{-10}	0.5152×10^{-2}	1.5543×10^{-15}

4 Numerical examples

In this section, the method proposed in Section (2) is implemented to some examples which the obtained results show the efficiency and the accuracy of the method. All computations are carried out using Mathematica 5.

Example 4.1 *Consider the second kind Volterra integral equation (1.1) with $\lambda = -\frac{2}{\pi}$ and $k(t, x) = \frac{1}{4+(t-x)^2}$. To evaluate the approximate solution of the problem using the method, $g(t)$ is chosen so that the exact solution is $\varphi(t) = 1 + t^2 + t^5$. The error between the exact solution and the approximate solution of $\varphi(t)$ with $n = 0, 2, 4, 6$ are tabled*

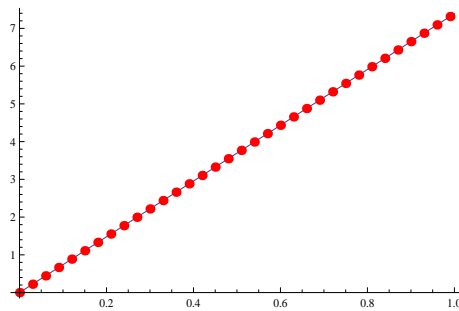**Table 4:** Comparison of the absolute errors for approximate solutions in Example 4.4.

t	n=0	n=2	n=4	.9 n=6
0.1	1.4376×10^{-4}	1.0174×10^{-7}	1.7347×10^{-18}	3.4694×10^{-18}
0.2	7.0219×10^{-4}	1.1037×10^{-6}	6.9388×10^{-18}	2.0816×10^{-17}
0.3	1.5080×10^{-3}	4.0671×10^{-6}	6.9388×10^{-18}	4.1633×10^{-17}
0.4	2.1851×10^{-3}	9.3796×10^{-6}	6.9388×10^{-18}	6.9388×10^{-18}
0.5	2.3293×10^{-3}	1.6119×10^{-5}	1.3877×10^{-17}	6.9388×10^{-18}
0.6	1.6430×10^{-3}	2.1545×10^{-5}	3.4694×10^{-17}	5.5511×10^{-17}
0.7	5.3768×10^{-5}	2.0667×10^{-5}	2.7755×10^{-17}	6.2450×10^{-17}
0.8	2.1741×10^{-3}	5.8556×10^{-6}	3.8163×10^{-17}	5.8980×10^{-17}
0.9	4.3314×10^{-3}	3.3506×10^{-5}	1.1102×10^{-16}	2.9316×10^{-16}
1.0	5.1566×10^{-3}	1.1134×10^{-4}	1.4922×10^{-16}	1.0052×10^{-15}

in Table 1. Fig 1 also shows the graph of the exact and the approximate solution for $n = 6$ in Taylor expansion.
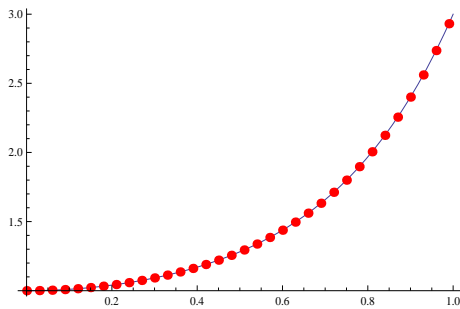
Figure 1: Approximate solution for Example 4.1 for $n = 6$

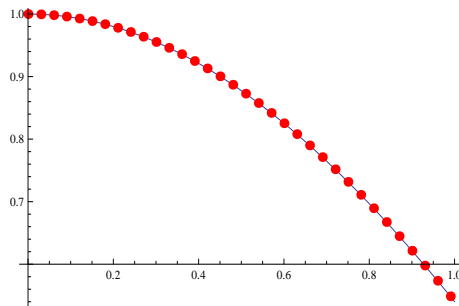**Figure 2:** Approximate solution for Example 4.2 for $n = 6$.

Figure 3: Approximate solution for Example 4.3 for $n = 6$.

Example 4.2 *In this problem, the kernel $k(t, x)$ and the parameter λ are exactly the same as in Example 4.1. $g(t)$ is selected such that the exact solution is $\varphi(t) = e^2 t$. The absolute errors of the approximate values with $n = 0, 2, 4, 6$ are shown in Table 2. Fig. 2 also shows the graph of the exact and the the approximate solution with $n = 6$.*

Example 4.3 *In this example, consider (1.1) with $\lambda = -1$, $k(t, x) = t - x$ and $g(t) = 1$ with the exact solution $\varphi(t) = \cos(t)$. Numerical results for $n = 0, 2, 4, 6$ are presented in Table 3. Fig. 3 is the graph of the exact and the approximate solution for $n = 6$.*

Example 4.4 *In this problem, we consider the integral equation (1.1) with a weakly singular kernel $k(t, x) = (t - x)^{\frac{-1}{3}}$ and $\lambda = -\frac{1}{10}$. $g(t)$ is chosen such that Eq. (1.1) has the exact solution $\varphi(t) = (t(1 - t))^2$. Numerical results for $n = 0, 2, 4$ are proposed in Table 4. Fig. 4 is the graph of the exact solution and the approximate solution with $n = 4$.*
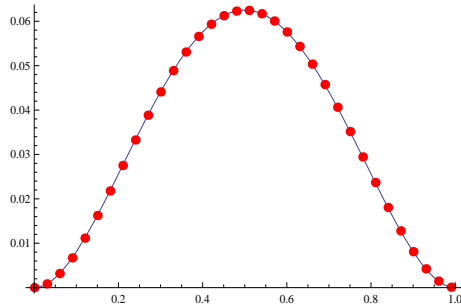
Figure 4: Approximate solution for Example 4.4 for $n = 6$.

5 Conclusion

In this paper, we proposed a novel application of Taylor expansion to obtain the approximate solution of the Volterra integral equations of the second kind. With the help of Taylor expansion method the integral equation converted into a linear ODE which needed specified boundary conditions. To obtain boundary conditions, the integration technique was used. As results showed, the method proposed here is more stable than derivative method and can be easily implemented to approximate the solution of the Volterra integral equation with smooth and weakly singular kernels. Also, as the present method is implemented to the Volterra integral equation with a rapidly decaying kernel, it leads to the more accurate results than those of the method in [7].

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Tayebeh Damercheli is Assistance Professor of Applied Mathematics at the Department of mathematic, Islamic Azad University. His main research interest is including numerical solution of deterministic and stochastic integral equations, differential equations and dynamic systems arise from optimal control problems.

He was born in the city of Qazvin-Iran in 1970. He received B.Sc and M.Sc degrees in applied mathematics. He started the Ph.D course on numerical analysis field in Islamic Azad University Shahre-Rey branch in 2015. Currently, he teaches in the Islamic Azad University Takestan Branch. His main research interest is including numerical solution of deterministic and stochastic integral equations and differential equations.

Alireza Vahidi is Associate Professor of Applied Mathematics at the Department of mathematic, Islamic Azad University. His main researches interest is including numerical solution of integral equations and differential equations.