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Research Article

Pseudo-spectral Matrix and Normalized Grunwald Approximation for Numerical Solution of Time Fractional Fokker-Planck Equation

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Abstract

This paper presents a new numerical method to solve time fractional Fokker-Planck equation. The space dimension is discretized to the Gauss-Lobatto points, then we apply pseudo-spectral successive integration matrix for this dimension. This approach shows that with less number of points, we can approximate the solution with more accuracy. The numerical results of the examples are displayed.

Keywords : Grunwald-Letnikov Derivative; Pseudo-Spectral Integration Matrix; Gauss-Lobatto Points; Fractional Fokker-Planck Equation.

1 Introduction

In [35] the fractional kinetic of the diffusion, diffusion-advection and Fokker-Planck type were presented which derived asymptotically random walk models and the generalization of the master and the Langevin equations. In [46] the concepts of fractional kinetic were discussed in cases like, particle dynamics in different potentials, particle advection in fluids, plasma physics and fusion devices, quantum optics and etc. Chechkin with co-workers in [3] proposed the fractional Fokker-Planck equation (FFPE) for the kinetic description of relaxation and superdiffusion processes in constant magnetic and random electric fields. R. Friedrich in [14] presented the FFPE for the joint position-velocity probability distribution of a single fluid particle in a turbulent flow. Meerschaert, et al. in [29] applied a generalization of fractional diffusion equation including multidimensional advection and fractional dispersion. They extended the fractional diffusion equation to two and three dimensions. In [34] illustrated how FFPE for description of anomalous diffusion in external fields can be derived from a generalization of a master equation. In [18] presented a modification of FFPE and the authors of [43] presented a new modeling of subdiffusion in space-time-dependent force fields such that without having to referring to FFPE. Also, Ralf Metzler with co-workers have many papers in study on fractional diffusion equations which FFPE is playing important role in some of them [2, 32, 33, 34, 36].

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As it is clear, the fractional kinetics equations are developing. But, because of the complex structure of these equations, the analytical solutions for these equations are very rare. Hence, it is expected that the numerical methods to solve these equations will spread more quickly. Currently, the numerical solution for FDEs are known but for the FPDEs are not known sufficiently and requires more works. The authors of [27] applied explicit and semi-implicit schemes to solve diffusion-reaction equations. Yuste [45] considered an extension of weighted average method for the ordinary (non-fractional) diffusion equations and used Grunwald-Letnikov approximation for the Riemann-Liouville time derivative. Scherer et al. [41] by modification of the Grunwald-Letnikov approximation for Caputo time derivative in fractional diffusion equation with non-zero initial conditions presented a new numerical approach. Meerchaert and Tadjeran in [30] developed practical numerical method for solution of one dimensional space fractional advection dispersion equation with variable coefficients based on shifted Grunwald-Letnikov approximation. Also they applied shifted Grunwald-Letnikov approximation for two sided FPDE in [31]. Ervin and Roop [12] presented a theoretical framework for the Galerkin finite element approximation to the steady state fractional advection dispersion equation and in [13] extended this work to the variational solution of this equation on bounded domains in R^d. Valko and Abate [42] to solve the time fractional diffusion equation on a semi-infinite domain applied numerical inversion of 2-D Laplace transforms. Liang and Chen [22] solved the fractional wave-diffusion equation by using a combination of symbolic mathematics and numerical inversion of Laplace transform. For solution of the time fractional diffusion equation proposed a numerical approach based on FDM in time and Legendre spectral method in space by Lin and Xu [24]. Zhang, Lin and Anh [48] considered the Levy-Feller fractional diffusion equation and presented a numerical approximation to it on its probability interpretation. Igor Podlubny and co-workers in [40] proposed a general method for the numerical solution of FPDEs based on the matrix form representation of discretized fractional operators which had been introduced in [39]. William McLean and Kassem Mustapha [28] applied a piecewice-constant, discontinuous Galerkin method for the time discretization of a subdiffusion equation. The analytical solutions of time fractional Benny-Lin equation and time fractional telegraph equation approximated in [17] and [6], respectively. Jiang and Ma in [21] developed high-order method based on high-order finite elements method for space and FDM for time to solve time FPDEs.

With regard to the importance of FFPE, the numerical method to solve this equation is interested for researchers. Liu, Anh and Turner [25] presented a numerical scheme to solve space FFPE with instantaneous source such that the equation transformed into a system of ODEs which is solved by a method of lines. Weihua Deng [7] solve the time FFPE such that firstly transformed it into time fractional ODE in the sence of Caputo derivative then used the combination of predictor-corrector and method of lines. Weihua Deng [8] developed the finite element method for the solution of space and time FFPE which was an effective tool for describing a process with both trap and flights. In [4] some practical numerical methods to solve time FFPE are used and also the solvability, stability, consistency and convergence of these methods are discussed. The authors of [23] used finite difference method to solve time FFPE and in [47] finite difference/element methods are presented to solve time FFPE with Dirichlet boundary conditions.

Our main work in this paper is development of used technique in [16] to solve the time FFPE. We use the Gauss-Lobatto points to discretize the space dimension and for discretization of time dimension in this equation we use the Grunwald-Letnikov approximation. Hence, we mention very briefly the history of the main method which is Pseudo-spectral successive integration matrix as follows. El-Gendi [10] developed a new numerical method based on the Clenshaw and Curtis quadrature scheme [5] to present a new method for the numerical solution of linear integral equations of Fredholm and Volterra types, then this method was extended to the linear integro-differential and ordinary differential equations. In this method an operational matrix for integration was presented. El-Gendi with co-workers [11] for successive integration of a function generalized the El-Gendi operational matrix to present a new matrix. Elsayed M. E. Elbarbary [9] using some properties of derivatives and integrals of Chebyshev polynomials, derived an operational matrix for n-fold integrations (pseudo-spectral integration matrix) of a function. In fact, this matrix was a modification of El-Gendi successive integration matrix which was more accurate. Gholami [15] for the first time, applied this matrix with FDM to solve a PDE then in [1] with co-authors used this matrix to solve a PDE alone. Now, we apply the pseudo-spectral successive integration matrix to discretization of the space dimension and Grunwald-Letnikov approximation for the time dimension for numerical solution of time FFPE.

2 Preliminaries

2.1 Concepts of Fractional Derivatives

In this subsection we present the most important definitions for the fractional derivatives.

Definition 2.1 *The Riemann-Liouville fractional derivative of order $m-1<\alpha<m$ is*

$$ {}_aD_x^\alpha f(x) = \left[\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{d\xi^m}\int_a^\xi \frac{f(\eta)}{(\xi-\eta)^{\alpha-m+1}}\,d\eta\right]_{\xi=x}, \tag{2.1}$$

$$ {}_xD_b^\alpha f(x) = \left[\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{d\xi^m}\int_\xi^b \frac{f(\eta)}{(\eta-\xi)^{\alpha-m+1}}\,d\eta\right]_{\xi=x}. \tag{2.2}$$

Definition 2.2 *The Caputo fractional derivative of order $m-1<\alpha<m$ is*

$$ {}_a^CD_x^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)}\int_a^x \frac{f^{(m)}(\eta)}{(x-\eta)^{\alpha-m+1}}\,d\eta, \tag{2.3}$$

$$ {}_x^CD_b^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)}\int_x^b \frac{f(\eta)}{(\eta-x)^{\alpha-m+1}}\,d\eta. \tag{2.4}$$

Definition 2.3 *[26] The Grunwald-Letnikov fractional derivative of order $m-1<\alpha<m$ is*

$$D_{a^+}^\alpha f(x) = \lim_{h\to 0, nh=x-a} h^{-\alpha}\sum_{j=0}^{n}(-1)^j\begin{pmatrix}\alpha\\ j\end{pmatrix} f(x-jh), \tag{2.5}$$

$$D_{b^-}^\alpha f(x) = \lim_{h\to 0, nh=b-x} h^{-\alpha}\sum_{j=0}^{n}(-1)^j\begin{pmatrix}\alpha\\ j\end{pmatrix} f(x+jh). \tag{2.6}$$

From [38] we can write

$$D_{a^+}^\alpha f(x) = \sum_{j=0}^{m-1}\frac{f^{(j)}(a)(x-a)^{j-\alpha}}{\Gamma(j-\alpha+1)} + \frac{1}{\Gamma(m-\alpha)}\int_a^x \frac{f^{(m)}(\eta)}{(x-\eta)^{\alpha-m+1}}\,d\eta, \tag{2.7}$$

$$D_{b^-}^\alpha f(x) = \sum_{j=0}^{m-1}\frac{(-1)^j f^{(j)}(b)(b-x)^{j-\alpha}}{\Gamma(j-\alpha+1)} + \frac{(-1)^m}{\Gamma(m-\alpha)}\int_x^b \frac{f^{(m)}(\eta)}{(\eta-x)^{\alpha-m+1}}\,d\eta, \tag{2.8}$$

for $m-1<\alpha<m$. Using repeated integration by parts then differentiation of Riemann-Liouville fractional derivative we have

$$\frac{d^m}{d\xi^m}\int_a^\xi \frac{f(\eta)}{(\xi-\eta)^{\alpha-m+1}}\,d\eta = \Gamma(m-\alpha)$$
$$\sum_{j=0}^{m-1}\frac{f^{(j)}(a)(\xi-a)^{j-\alpha}}{\Gamma(j-\alpha+1)} + \int_a^\xi \frac{f^{(m)}(\eta)}{(\xi-\eta)^{\alpha-m+1}}\,d\eta. \tag{2.9}$$

Similarly

$$\frac{d^m}{d\xi^m}\int_\xi^b \frac{f(\eta)}{(\eta-\xi)^{\alpha-m+1}}\,d\eta = \Gamma(m-\alpha)$$
$$\sum_{j=0}^{m-1}\frac{(-1)^j f^{(j)}(b)(b-\xi)^{j-\alpha}}{\Gamma(j-\alpha+1)}$$
$$+(-1)^m\int_\xi^b \frac{f^{(m)}(\eta)}{(\eta-\xi)^{\alpha-m+1}}\,d\eta, \tag{2.10}$$

These equations show that

$${}_aD_x^\alpha f(x) = D_{a^+}^\alpha f(x), \quad {}_bD_x^\alpha f(x) = D_{b^-}^\alpha f(x). \tag{2.11}$$

Indeed, the Grunwald-Letnikov derivative and the Riemann-Liouville derivative are equivalent if the function $f(x)$ has $m-1$ continuous derivatives and $f^{(m)}(x)$ is integrable on closed interval $[a,b]$. Using this fact [25], by the relationship between Riemann-Liouville fractional derivative and Grunwald-Letnikov fractional derivative we will derive a numerical solution such that we use the Riemann-Liouville definition during problem formulation and then the Grunwald-Letnikov definition for achieving the numerical solution. From the standard Grunwald definition we have

Definition 2.4 *[49] The standard Grunwald formula for $w(x,t)$ which $a \le x \le b$ is*

$$D^{\alpha}_{a^+} w(x,t) =$$

$$\lim_{M_1\to\infty} h_1^{-\alpha} \sum_{j=0}^{M_1} (-1)^j \left(\begin{array}{c} \alpha \\ j \end{array} \right) w(x - jh_1, t), \quad (2.12)$$

$$D^{\alpha}_{b^-} w(x,t) =$$

$$\lim_{M_2\to\infty} h_2^{-\alpha} \sum_{j=0}^{M_2} (-1)^j \left(\begin{array}{c} \alpha \\ j \end{array} \right) w(x + jh_2, t), \quad (2.13)$$

where $M_1, M_2 \in N, h_1 = \frac{x-a}{M_1}, h_2 = \frac{b-x}{M_2}$and $g_{\alpha}^{(j)}$ are the normalized Grunwald weights functions defined as

$$g_{\alpha}^{(j)} = -\frac{\alpha - j + 1}{j}\, g_{\alpha}^{(j-1)}, \quad j = 1, 2, ... \quad (2.14)$$

with $g_{\alpha}^{(0)} = 1$.

Let $\Omega = [a,b] \times [0,T], (x,t) \in \Omega, t_k = k\tau, k = 0(1)n, x_i = a + ih, i = 0(1)m$,with $\tau = \frac{T}{n}$ and $h = \frac{b-a}{m}$ be time and space steps, respectively. From [30], for $w(x,t) \in L^1(\Omega), D^{\alpha}_{a^+} w(x,t) \in \ell(\Omega)$ and $D^{\alpha}_{b^-} w(x,t) \in \ell(\Omega)$, we obtain

$$D^{\alpha}_{a^+} w(x_i, t_k) =$$

$$h^{-\alpha} \sum_{j=0}^{i} (-1)^j \left(\begin{array}{c} \alpha \\ j \end{array} \right) w(x_{i-j}, t_k) + O(h), \quad (2.15)$$

$$D^{\alpha}_{b^-} w(x_i, t_k) =$$

$$h^{-\alpha} \sum_{j=0}^{m-i} (-1)^j \left(\begin{array}{c} \alpha \\ j \end{array} \right) w(x_{i+j}, t_k) + O(h). \quad (2.16)$$

2.2 Pseudo-spectral integration matrix

We assume that $(P_N f)(x)$ is the N^{th} order Chebyshev interpolating polynomial of the function $f(x)$ at the points $(x_k, f(x_k))$ where

$$(P_N f)(x) = \sum_{j=0}^{N} f_j\, \varphi_j(x), \quad (2.17)$$

with

$$\varphi_j(x) = \frac{2\alpha_j}{N} \sum_{r=0}^{N} \alpha_r\, T_r(x)\, T_r(x_j), \quad (2.18)$$

where $\varphi_j(x_k) = \delta_{j,k}$, ($\delta_{j,k}$ is the Kronecker delta) and $\alpha_0 = \alpha_N = 1/2$, $\alpha_j = 1$ for $j = 1(1)N - 1$. Since $(P_N f)(x)$ is a unique interpolating polynomial of order N, it can be expressed in terms of a series expansion of the classical Chebyshev polynomials, hence we have

$$(P_N f)(x) = \sum_{r=0}^{N} a_r\, T_r(x), \quad (2.19)$$

where

$$a_r = \frac{2\alpha_r}{N} \sum_{j=0}^{N} \alpha_j\, f(x_j)\, T_r(x_j). \quad (2.20)$$

The successive integration of $f(x)$ in the interval $[-1, x_k]$ can be estimated by successive integration of $(P_N f)(x)$. Thus we have

$$I_n(f) \simeq \sum_{r=0}^{N} a_r$$

$$\int_{-1}^{x} \int_{-1}^{t_{n-1}} ... \int_{-1}^{t_2} \int_{-1}^{t_1} T_r(t_0)\, dt_0\, dt_1 ... dt_{n-2}\, dt_{n-1}. \quad (2.21)$$

Theorem 2.1 *[19] The exact relation between Chebyshev functions and its derivatives is expressed as*

$$T_r(x) = \sum_{m=0}^{n} \frac{(-1)^m \binom{n}{m}}{2^n \chi_m}\, T^{(n)}_{r+n-2m}, \qquad r > n,$$

where

$$\chi_m = \prod_{\substack{j=0 \\ j\ne n-m}}^{n} (r + n - m - j).$$

Theorem 2.2 *[9] The successive integration of Chebyshev polynomials is expressed in terms of Chebyshev polynomials as*

$$\int_{-1}^{x} \int_{-1}^{t_{n-1}} ... \int_{-1}^{t_2} \int_{-1}^{t_1} T_r(t_0)\, dt_0\, dt_1 ... dt_{n-2}\, dt_{n-1}$$

$$= \sum_{m=0}^{n-\gamma_r} \beta_r\, \frac{(-1)^m \binom{n}{m}}{2^n \chi_m}\, \xi_{n,m,r}(x),$$

where

$$\xi_{n,m,r}(x) = T_{r+n-2m}(x) - \sum_{i=0}^{n-1} \eta_i\, T^{(i)}_{r+n-2m}(-1),$$

$$\eta_i = \sum_{j=0}^{i} \frac{x^j}{(i-j)!j!},$$

$$\beta_i = \begin{cases} 2 & i = 0, \\ 1 & i > 0, \end{cases}$$

$$\chi_m = \prod_{\substack{j=0 \\ j \neq n-m}}^{n} (r+n-m-j),$$

$$\gamma_i = \begin{cases} n & i = 0, \\ n-i+1 & 1 \leq i \leq n, \\ 0 & i > n, \end{cases}$$

Thus, from Theorem (2.2) and relations (2.20) and (2.21), we have

$$I_n(f) \simeq \sum_{j=0}^{N} \left(\frac{2\alpha_j}{N} \sum_{r=0}^{N} \alpha_r\, T_r(x_j) \right.$$

$$\left. \sum_{m=0}^{n-\gamma_r} \beta_r \frac{(-1)^m \binom{n}{m}}{2^n \chi_m} \xi_{n,m,r}(x) \right) f(x_j).$$

The matrix form of the successive integration of the function $f(x)$ at the Gauss-Lobatto points x_k is

$$[I_n(f)] = \left[\sum_{j=0}^{N} \left(\frac{2\alpha_j}{N} \sum_{r=0}^{N} \alpha_r\, T_r(x_j) \right. \right.$$

$$\left. \left. \sum_{m=0}^{n-\gamma_r} \beta_r \frac{(-1)^m \binom{n}{m}}{2^n \chi_m} \xi_{n,m,r}(x) \right) f(x_j) \right] = \Theta^{(n)}[f]. \tag{2.22}$$

The elements of the matrix $\Theta^{(n)}$ are

$$\vartheta_{k,j}^{(n)} = \frac{2\alpha_j}{N} \sum_{r=0}^{N} \alpha_r\, T_r(x_j)$$

$$\sum_{m=0}^{n-\gamma_r} \beta_r \frac{(-1)^m \binom{n}{m}}{2^n \chi_m} \xi_{n,m,r}(x_k). \tag{2.23}$$

The matrix $\Theta^{(n)}$ in (2.22), presented in [9], is called the pseudo-spectral integration matrix.

3 Fractional Fokker-Planck Equation

We consider the time fractional Fokker-Planck equation $(FFPE)$

$$\frac{\partial w(x,t)}{\partial t} = {}_0D_t^{1-\alpha} \left[\frac{\partial f(x)}{\partial x} + K_\alpha \frac{\partial^2}{\partial x^2} \right] w(x,t), \tag{3.24}$$

$(x,t) \in [a,b] \times [0,t]$. With initial condition

$$w(x,0) = p(x), \qquad a \leq x \leq b,$$

and boundary conditions

$$w(a,t) = g_1(t), \quad w(b,t) = g_2(t), \quad 0 \leq t \leq T,$$

where g_1, g_2 and p are known functions and w is unknown, also ${}_0D_t^{1-\alpha}$ is the Riemann-Liouville derivative of order $1-\alpha$ $(0 \leq \alpha \leq 1)$, i.e.

$${}_0D_t^{1-\alpha} w(x,t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t \frac{w(x,\tau)}{(t-\tau)^{1-\alpha}} d\tau.$$

We can rewrite (3.24) as in [4]

$${}_0D_t^{\alpha} w(x,t) - \frac{w(x,0)}{t^\alpha \Gamma(1-\alpha)} =$$

$$\left[\frac{\partial f(x)}{\partial x} + K_\alpha \frac{\partial^2}{\partial x^2} \right] w(x,t). \tag{3.25}$$

$(x,t) \in [a,b] \times [0,t]$. But, by the relationship between Caputo fractional derivative $\frac{\partial^\alpha}{\partial t^\alpha}$ and Riemann-Liouville fractional derivative ${}_0D_t^\alpha$, i.e.

$$\frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = {}_0D_t^\alpha w(x,t) - \frac{w(x,0)}{t^\alpha \Gamma(1-\alpha)},$$

we can write the $(FFPE)$ in (3.24) as

$$\frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = \left[\frac{\partial f(x)}{\partial x} + K_\alpha \frac{\partial^2}{\partial x^2} \right] w(x,t). \tag{3.26}$$

Let $x_i = -\cos \frac{i\pi}{N}$ for $N \in \mathbb{N}$ be the Gauss-Lobatto points. Now, we apply pseudo-spectral successive integration matrix to solve $(FFPE)$. Assume that

$$\left. \frac{\partial^2 w(x,t)}{\partial x^2} \right|_{x_i} = \varphi(x_i,t), \tag{3.27}$$

$$\left. \frac{\partial w(x,t)}{\partial x} \right|_{x_i} = \sum_{j=0}^{N} \vartheta_{i,j}^{(1)} \varphi(x_j,t) + c_1, \tag{3.28}$$

Table 1: Max error for example (4.1) when $N = 4$ and $m = 4$.

t	$\alpha = 0.1$	$\alpha = 0.4$	$\alpha = 0.7$	$\alpha = 0.9$
0.25	$7.49E-5$	$3.60E-4$	$7.58E-4$	$1.10E-3$
0.5	$1.58E-4$	$8.09E-4$	$1.85E-3$	$2.87E-3$
0.75	$2.42E-4$	$1.27E-3$	$2.99E-3$	$4.56E-3$
1	$3.26E-4$	$1.73E-3$	$4.14E-3$	$6.67E-3$

Table 2: Max errors for example (4.1) when $N = 4, 8$ and $m = 10$.

t	$\alpha = 0.1$	$\alpha = 0.4$	$\alpha = 0.7$	$\alpha = 0.9$
0.1	$1.19E-5$	$5.52E-5$	$1.08E-4$	$1.44E-4$
0.2	$2.52E-5$	$1.25E-4$	$2.70E-4$	$3.95E-4$
0.3	$3.85E-5$	$1.97E-4$	$4.45E-4$	$6.79E-4$
0.4	$5.19E-5$	$2.69E-4$	$6.24E-4$	$9.76E-4$
0.5	$6.53E-5$	$3.42E-4$	$8.05E-4$	$1.28E-3$
0.6	$7.87E-5$	$4.15E-4$	$9.87E-4$	$1.58E-3$
0.7	$9.21E-5$	$4.89E-4$	$1.17E-3$	$1.89E-3$
0.8	$1.05E-4$	$5.63E-4$	$1.35E-3$	$2.19E-3$
0.9	$1.19E-4$	$6.37E-4$	$1.54E-3$	$2.50E-3$
1	$1.32E-4$	$7.11E-4$	$1.72E-3$	$2.81E-3$

Table 3: Max errors for example (4.1) when $N = 4, 8$ and $m = 20$.

t	$\alpha = 0.1$	$\alpha = 0.4$	$\alpha = 0.7$	$\alpha = 0.9$
0.05	$2.96E-6$	$1.32E-5$	$2.35E-5$	$2.81E-5$
0.1	$6.26E-6$	$3.02E-5$	$6.06E-5$	$8.08E-5$
0.15	$9.58E-6$	$4.76E-5$	$1.02E-4$	$1.45E-4$
0.2	$1.29E-5$	$6.54E-5$	$1.45E-4$	$2.14E-4$
0.25	$1.62E-5$	$8.33E-5$	$1.88E-4$	$2.86E-4$
0.3	$1.96E-5$	$1.01E-4$	$2.33E-4$	$3.60E-4$
0.35	$2.29E-5$	$1.19E-4$	$2.77E-4$	$4.34E-4$
0.4	$2.63E-5$	$1.38E-4$	$3.23E-4$	$5.09E-4$
0.45	$2.96E-5$	$1.56E-4$	$3.68E-4$	$5.85E-4$
0.5	$3.29E-5$	$1.74E-4$	$4.13E-4$	$6.61E-4$
0.55	$3.63E-5$	$1.92E-4$	$4.59E-4$	$7.37E-4$
0.6	$3.96E-5$	$2.11E-4$	$5.05E-4$	$8.13E-4$
0.65	$4.30E-5$	$2.29E-4$	$5.51E-4$	$8.90E-4$
0.7	$4.63E-5$	$2.48E-4$	$5.96E-4$	$9.66E-4$
0.75	$4.96E-5$	$2.66E-4$	$6.42E-4$	$1.04E-3$
0.8	$5.30E-5$	$2.84E-4$	$6.89E-4$	$1.12E-3$
0.85	$5.63E-5$	$3.03E-4$	$7.35E-4$	$1.19E-3$
0.9	$5.96E-5$	$3.21E-4$	$7.81E-4$	$1.27E-3$
0.95	$6.30E-5$	$3.40E-4$	$8.27E-4$	$1.34E-3$
1	$6.63E-5$	$3.58E-4$	$8.73E-4$	$1.41E-3$

Table 4: The comparison of our method and INAS in [44].

Our method				INAS		
N and m	Number of points	$Maxerrors$	α	$Maxerrors$	Number of points	N and m
N=m=4	25	$1.73E-3$	0.4	$3.00E-3$	2601	N=m=50
N=m=4	25	$4.14E-3$	0.7	$4.69E-3$	2601	N=m=50
N=m=4	25	$6.67E-3$	0.9	$6.07E-3$	2601	N=m=50

Table 5: The comparison of our method and INAS in [44].

Our method				INAS		
N and m	Number of points	$Maxerrors$	α	$Maxerrors$	Number of points	N and m
N=4, m=10	55	$7.11E-4$	0.4	$1.51E-3$	10201	N=m=100
N=4, m=10	55	$1.72E-3$	0.7	$2.35E-3$	10201	N=m=100
N=4, m=10	55	$2.81E-3$	0.9	$3.04E-3$	10201	N=m=100

Table 6: The comparison of our method and INAS in [44].

Our method				INAS		
N and m	Number of points	$Maxerrors$	α	$Maxerrors$	Number of points	N and m
N=4, m=20	105	$3.58E-4$	0.4	$6.00E-4$	40401	N=m=200
N=4, m=20	105	$8.73E-4$	0.7	$1.23E-3$	40401	N=m=200
N=4, m=20	105	$1.41E-3$	0.9	$1.50E-3$	40401	N=m=200

Table 7: Max error for example (4.2) when $N = 4$ and $m = 4$.

t	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.8$
0.25	$1.42E-3$	$5.25E-3$	$1.18E-2$
0.5	$2.69E-3$	$8.35E-3$	$1.63E-2$
0.75	$3.91E-3$	$1.08E-2$	$1.88E-2$
1	$5.10E-3$	$1.29E-2$	$2.06E-2$

Table 8: Max errors for example (4.2) when $N = 4, 8$ and $m = 10$.

t	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.8$
0.1	$2.66E-4$	$1.21E-3$	$3.15E-3$
0.2	$5.04E-4$	$1.96E-3$	$4.72E-3$
0.3	$7.29E-4$	$2.55E-3$	$5.67E-3$
0.4	$9.50E-4$	$3.06E-3$	$6.34E-3$
0.5	$1.17E-3$	$3.52E-3$	$6.85E-3$
0.6	$1.39E-3$	$3.94E-3$	$7.27E-3$
0.7	$1.61E-3$	$4.34E-3$	$7.64E-3$
0.8	$1.84E-3$	$4.73E-3$	$7.97E-3$
0.9	$2.07E-3$	$5.10E-3$	$8.28E-3$
1	$2.31E-3$	$5.46E-3$	$8.57E-3$

$$w(x_i,t) = \sum_{j=0}^{N} \vartheta_{i,j}^{(2)}\, \varphi(x_j,t) + c_1(x_i+1) + c_2, \tag{3.29}$$

for $i = 0(1)N$. The constants c_1 and c_2 are obtained to satisfy the boundary conditions. From these conditions we have $c_2 = g_1(t)$ and

$$c_1 = -\frac{1}{2}\left(\sum_{j=0}^{N} \vartheta_{N,j}^{(2)}\, \varphi(x_j,t) + g_1(t) - g_2(t)\right).$$

By substituting c_1 and c_2 into (3.28) and (3.29), we have

$$w(x_i,t) = \sum_{j=0}^{N} \vartheta_{i,j}^{(2)}\, \varphi(x_j,t) \; - \frac{1}{2}(x_i+1)$$

$$\sum_{j=0}^{N} \vartheta_{N,j}^{(2)}\, \varphi(x_j,t) + \mathbf{Z}_i(t), \quad i = 0(1)N, \tag{3.30}$$

$$\left.\frac{\partial w(x,t)}{\partial x}\right|_{x_i} = \sum_{j=0}^{N} \vartheta_{i,j}^{(1)}\, \varphi(x_j,t) - \frac{1}{2}\left(\sum_{j=0}^{N} \vartheta_{N,j}^{(2)} + g_1(t) - g_2(t)\right), \quad i = 0(1)N, \tag{3.31}$$

which

$$\mathbf{Z_i(t)} = \frac{1}{2}\, x_i(g_2(t) - g_1(t)) + \frac{1}{2}(g_2(t) + g_1(t)), \tag{3.32}$$

Table 9: Max errors for example (4.2) when $N = 4, 8$ and $m = 20$.

t	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.8$
0.05	$7.39E-5$	$3.88E-4$	$1.06E-3$
0.1	$1.38E-4$	$6.41E-4$	$1.71E-3$
0.15	$1.96E-4$	$8.38E-4$	$2.14E-3$
0.2	$2.51E-4$	$1.01E-3$	$2.46E-3$
0.25	$3.03E-4$	$1.15E-3$	$2.70E-3$
0.3	$3.53E-4$	$1.29E-3$	$2.90E-3$
0.35	$4.02E-4$	$1.41E-3$	$3.06E-3$
0.4	$4.49E-4$	$1.52E-3$	$3.20E-3$
0.45	$4.96E-4$	$1.63E-3$	$3.31E-3$
0.5	$5.41E-4$	$1.73E-3$	$3.42E-3$
0.55	$5.86E-4$	$1.83E-3$	$3.52E-3$
0.6	$6.30E-4$	$1.92E-3$	$3.60E-3$
0.65	$6.73E-4$	$2.01E-3$	$3.68E-3$
0.7	$7.16E-4$	$2.10E-3$	$3.76E-3$
0.75	$7.58E-4$	$2.18E-3$	$3.82E-3$
0.8	$7.99E-4$	$2.26E-3$	$3.89E-3$
0.85	$8.40E-4$	$2.33E-3$	$3.95E-3$
0.9	$8.81E-4$	$2.41E-3$	$4.01E-3$
0.95	$9.21E-4$	$2.48E-3$	$4.06E-3$
1	$9.61E-4$	$2.55E-3$	$4.10E-3$

Now, we substitute (3.27), (3.30) and (3.31) into main equation (3.25) to obtain

$$\sum_{j=0}^{N} \vartheta_{i,j}^{(2)}\, {}_0D_t^{\alpha}\varphi(x_j,t) - \frac{1}{2}(x_i+1)$$

$$\sum_{j=0}^{N} \vartheta_{N,j}^{(2)}\, {}_0D_t^{\alpha}\varphi(x_j,t) = \Bigg(\sum_{j=0}^{N} \vartheta_{i,j}^{(2)}\, \varphi(x_j,t)$$

$$-\frac{1}{2}(x_i+1)\sum_{j=0}^{N} \vartheta_{N,j}^{(2)}\, \varphi(x_j,t) + \mathbf{Z}_i(t)\Bigg) f'(x_i)$$

$$+\Bigg[\sum_{j=0}^{N} \vartheta_{i,j}^{(1)}\, \varphi(x_j,t) - \frac{1}{2}\Bigg(\sum_{j=0}^{N} \vartheta_{N,j}^{(2)}\varphi(x_i,t)$$

$$+g_1(t) - g_2(t)\Bigg)\Bigg] f(x_i) + \frac{p(x_i)}{t^{\alpha}\,\Gamma(1-\alpha)}$$

$$+ K_{\alpha}\varphi(x_i,t) - {}_0D_t^{\alpha}\mathbf{Z}_i(t), \qquad i = 0(1)N. \quad (3.33)$$

Now, by using the relationship between Grunwald-Letnikov and Riemann-Liouville we apply the approximation of standard Grunwald formula in the time. Let

$$t_k = k\tau, \qquad k = 0(1)m, \qquad \tau = \frac{T}{m}.$$

Hence, for $t = t_k$ and $k = 0(1)m$ we have

$$ {}_0D_{t_k}^{\alpha}\varphi(x_j,t) = \tau^{-\alpha}\sum_{r=0}^{k} g_{\alpha}^{(r)}\varphi(x_j,t_{k-r}), \quad (3.34)$$

where $g_{\alpha}^{(r)}$ are normalized Grunwald weights functions. If (3.34) substitute (3.33) then for (x_i, t_k) we obtain

$$\tau^{-\alpha}\sum_{r=0}^{k} g_{\alpha}^{(r)}\Bigg(\sum_{j=0}^{N} \vartheta_{i,j}^{(2)}\, \varphi(x_j,t_{k-r}) - \frac{1}{2}(x_i+1)$$

$$\sum_{j=0}^{N} \vartheta_{N,j}^{(2)}\, \varphi(x_j,t_{k-r})\Bigg) + {}_0D_{t_k}^{\alpha}\mathbf{Z}_i(t_k) =$$

$$f'(x_i)\Bigg(\sum_{j=0}^{N} \vartheta_{i,j}^{(2)}\, \varphi(x_j,t_k) - \frac{1}{2}(x_i+1)$$

$$\sum_{j=0}^{N} \vartheta_{N,j}^{(2)}\, \varphi(x_j,t_k)\Bigg) + f(x_i)\Bigg(\sum_{j=0}^{N} \vartheta_{i,j}^{(1)}\, \varphi(x_j,t_k)$$

$$-\frac{1}{2}\sum_{j=0}^{N} \vartheta_{N,j}^{(2)}\varphi(x_i,t_k)\Bigg) + K_{\alpha}\varphi(x_i,t_k)$$

$$+ \frac{p(x_i)}{t_k^{\alpha}\,\Gamma(1-\alpha)} + f'(x_i)\mathbf{Z}_i(t_k), \quad (3.35)$$

for $i = 0(1)N$ and $k = 0(1)m$. Since $g_\alpha^{(0)} = 1$ we obtain

$$\Big[(1-\tau^\alpha f'(x_i))\mathbf{A_i} - \tau^\alpha f(x_i)\mathbf{V_i}\Big]\Phi^k =$$

$$\tau^\alpha\Bigg(K_\alpha \varphi(x_i,t_k) + f'(x_i)\mathbf{Z}_i(t_k) - \frac{1}{2}f(x_i)$$

$$(g_1(t_k) - g_2(t_k)) + \frac{p(x_i)}{t_k^\alpha\,\Gamma(1-\alpha)}$$

$$- {}_0D_{t_k}^\alpha \mathbf{Z}_i(t_k)\Bigg) - \sum_{r=1}^{k} g_\alpha^{(r)}\mathbf{A_i}\Phi^{k-r}, \qquad (3.36)$$

for $i = 0(1)N$ and $k = 0(1)m$. Which to summarize, we define

$$\mathbf{A_i} = [\vartheta_{i,0}^{(2)}, \vartheta_{i,1}^{(2)}, ..., \vartheta_{i,N}^{(2)}]$$

$$-\frac{1}{2}(X_i+1)[\vartheta_{N,0}^{(2)}, \vartheta_{N,1}^{(2)}, ..., \vartheta_{N,N}^{(2)}], \qquad (3.37)$$

$$\mathbf{V_i} = [\vartheta_{i,0}^{(1)}, \vartheta_{i,1}^{(1)}, ..., \vartheta_{i,N}^{(1)}]$$

$$-\frac{1}{2}[\vartheta_{N,0}^{(2)}, \vartheta_{N,1}^{(2)}, ..., \vartheta_{N,N}^{(2)}], \qquad (3.38)$$

$$\Phi^k = [\varphi_{0,k}, \varphi_{1,k}, ..., \varphi_{N,k}]^t, \qquad (3.39)$$

indeed, (3.36) is the following system

$$\mathbb{A}\Phi^k = \mathbb{B}^k. \qquad (3.40)$$

With all unknowns $\varphi(x_i,t_k)$ for $i = 0(1)N$ and $k = 0(1)m$. By solving this system, we can approximate all $\varphi(x_i,t_k)$ from (3.30).

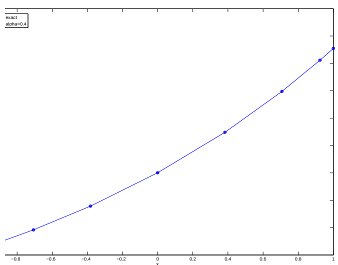

Figure 1: Comparison of numerical solutions of the example (4.1) at some values of t

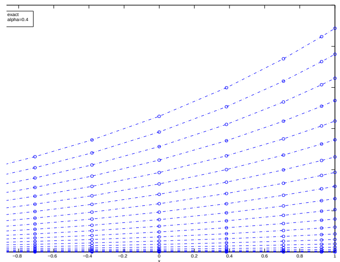

Figure 2: Comparison of numerical solutions of the example (4.1) for some values of t

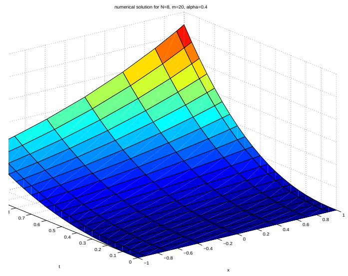

Figure 3: The approximation solution of example (4.1) when $\alpha = 0.4$.

4 Examples

Example 4.1 *Consider the FFPE in [44] by translating* $0 \le x \le 1$ *to* $-1 \le X \le 1$ *as*

$$\frac{\partial w(X,t)}{\partial t} = {}_0D_t^{1-\alpha}\Big[2\frac{\partial}{\partial X}\Big(2d(X)\frac{\partial w}{\partial X}\Big) + f(X,t)\Big], \qquad (4.41)$$

$$(X,t) \in [-1,1]\times[0,1], \qquad 0 \le \alpha \le 1,$$

where $d(X) = e^{(\frac{X+1}{2})}$ *and*

$$f(X,t) = \frac{\Gamma(3+\alpha)}{\Gamma(3)}\,t^2\,e^{(\frac{X+1}{2})} - 2\,e^{(X+1)}\,t^{2+\alpha},$$

with initial condition

$$w(x,0) = 0, \qquad -1 \le X \le 1,$$

and boundary conditions

$$w(-1,t) = t^{2+\alpha}, \quad w(1,t) = e\,t^{2+\alpha}, \quad 0 \le t \le 1,$$

The exact solution of this equation is $w(X,t) = e^{(\frac{X+1}{2})}t^{2+\alpha}$. *The numerical results of this problem are presented in the Tables 1-6 and Figures 1-3.*

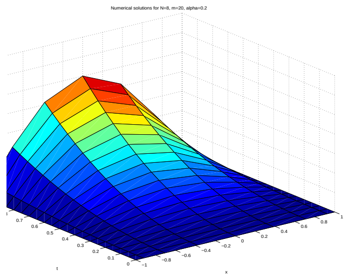

Figure 4: The approximation solution of example (4.2) when $\alpha = 0.2$.

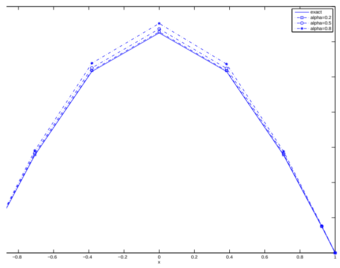

Figure 5: Comparison of numerical solutions of the example (4.2) at $t = 0.2$.

Example 4.2 *Consider the FFPE in [20] by translating* $0 \leq x \leq 1$ *to* $-1 \leq X \leq 1$ *as*

$$\frac{\partial w}{\partial t} = {}_0D_t^{1-\alpha}\left[\left(2\frac{\partial}{\partial X}f(X) + 4K_\alpha \frac{\partial^2 w}{\partial X^2}\right) + g(X,t)\right], \tag{4.42}$$

$$(X,t) \in [-1,1] \times [0,1], \qquad 0 \leq \alpha \leq 1,$$

with initial and boundary conditions

$$w(x,0) = w(-1,t) = w(1,t) = 0,$$

where

$$g(X,t) = \frac{\Gamma(3)}{\Gamma(3-\alpha)}\, t^{2-\alpha} \sin\frac{\pi(X+1)}{2} -$$

$$t^2\, e^{\frac{X^2}{4}}\left[x \sin\frac{\pi(X+1)}{2} + \pi \cos\frac{\pi(X+1)}{2}\right]$$

$$+K_\alpha \pi^2 t^2 \sin\frac{\pi(X+1)}{2},$$

with $K_\alpha = 1$, $f(X) = e^{\frac{X^2}{4}}$. *The exact solution of this equation is* $w(X,t) = t^2 \sin\frac{\pi(X+1)}{2}$. *The numerical results of this problem are presented in the Tables 7-9 and Figures 4-6.*

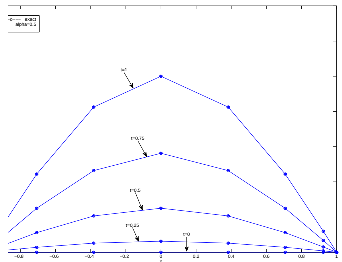

Figure 6: Comparison of numerical solutions of the example (4.2) at some values of t

5 Conclusion

In this paper, fractional Fokker-Planck equation considered as one important fractional PDEs. A new numerical approach for solution of time fractional Fokker-Planck equation presented which is combination of pseudo-spectral successive integration matrix and normalized Grunwald approximations. In the present approach, we used two different discretization of space/time dimensions. This method showed that with less number of points we can approximate the solutions with enough accuracy.

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