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Research Article

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An Explicit Single-step Method for Numerical Solution of Optimal Control Problems

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Abstract

In this research we used forward-backward sweep method(FBSM) in order to solve optimal control problems. In this paper, one hybrid method based on ERK method of order 4 and 5 are proposed for the numerical approximation of the OCP. The convergence of the new method has been proved. This method indicate more accurate numerical results compared with those of ERK method of order 4 and 5 for solving OCP.

Keywords : FBSM; OCP; Stability Analysis; Hybrid methods.

1 Introduction

Optimal control is an effective tool for using physical, economic, engineering, biological and other science models [2, 3, 4, 5]. Analytical solving optimal control problem(OC) is often difficult or not affordable. Hence a lot of numerical methods have been introduced to solve optimal control problems [6]. In general, the numerical methods of solving optimal control problems are divided into three categories [7]: Direct methods, Indirect methods and Dynamic programming(DP). T. R.Goodman and G. N. Lance in 1956 and D. D. Morrison and his colleagues published the first articles in direct methods [8, 9]. One of the reasons for the popularity of numerical methods and their impressive expansion is the availability of many robust and ready-to-use optimization algorithms (NLPsolvers) [10]. Another reason is the software can easily contain different types of inequality constraints. And like indirect methods, there is no need to calculate the co-state equations. Whenever in the optimal trajectory, instead of the initial condition, the final condition is given, direct methods are used. In direct method, an optimal control problem is first discretized. Then it becomes a nonlinear optimization problem or the nonlinear programming problem and then NLP, is solved well, with optimization methods [10]. The main disadvantage of these methods is that they produce only the suboptimal or approximate solutions. The direct methods are divided into three categories: di-

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rect shooting, multiple shooting and collocation. In a direct shooting method, a differential equation system with two-point boundary condition becomes a nonlinear optimization problem. Multiple shooting methods do not require explicit integration and an analytical process. The collocation method does not have a sensitivity to the initialy guess and among these three methods, the collocation is the fastest.Newton first introduced an indirect method in 1686. The necessary conditions were formulated in 1747 by Euler and Lagrange. The same conditions were introduced in 1834 more clearly in the form of the Hamiltonian function. The maximum principle was introduced in 1954 by a Russian mathematician "Pontryagin" [11]. Indirect methods are based on the pontryagin maximum principle(PMP), in which the optimal control problem turns into a two-point boundary value problem. Indirect methods are the most accurate numerical method for solving optimal control problems, flexible and convergent faster [12]. Unlike other numerical methods, they can produce a closed form for the analytical solution of the problem. In the indirect method, analytical processes obtain initial condition equations, and the solution of a differential equation with two-point boundary condition, require more time.The sensitive to co-state variables and a good initial guess for them can not be considered are another disadvantages of the method [12]. Since there are many methods for solving the differential equations, so there is a possibility of further research on indirect methods [3, 13, 14]. Richard Bellman wrote a book, called Dynamic Programming(DP) in 1957 and for the first time introduced (DP). In this method an Optimal Control Problem is solved by algorithms that use Hamilton-Jacoby-Bellman equation. The idea behind these algorithms is Bellman's principle of optimality, which states that each sub-optimal trajectory is self optimal.The searches for the entire state space is the advantage of this method and provides a global optimizer by using rescursive Optimal Controls but the disadvantage is that the dimension of the partial differential equation is larg.

The forward-backward sweep method(FBSM) is indirect numerical method for solving Optimal Control Problems(OP), The first time by Lenhart in the "Optimal Control Applied to Biological Models" Book, was introduced [3]. Convergence of this method was proved by M. Mcasey, L. Moua and W. Han in 2012 [16]. The RungKutta method is used from the 4th-order for numerical solving differential equation. The reason for this naming is that the state equation in this method solved by forward sweep method and the co-state equation solved by backward sweep method. In the FBSM method in recent years, the few articles are written [17, 18, 19]. Instead of using runge-kutta in the FBSM method,in this work, we illustrate one implicit hybrid methods of order 6 and then convert it's into explicit methods using explicit Runge-Kutta of orders 4 and 5 as a predictor of the scheme. the stability and order of trunation error of the methods discussed showing that new methods have wide stability regions by which more accurate results can be obtain compared to the FBSMbased explicit Runge-Kutta methods of orders 4 and 5. we apply numerical methods with wide stability regions as well as good accuracy in order to analyse stiff problems of ODE$_s$ which may be appeared in OCP. The hybrid method is suitable for this purpose.

2 Hybrid methods and order of truncation errors

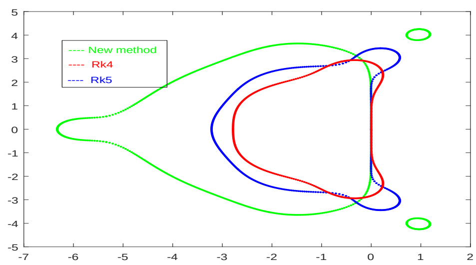


Figure 1: Stability region of New method , Rk4 and Rk5

For the numerical solution of initial value problems (IVP) of the form

$$x' = f(t,x) \ , \quad x \in \mathbb{R}^n \ , x(t_0) = x_0 \ , \quad t_0 \ \leq t \leq t_1, \tag{2.1}$$

where $f : [t_0, t_1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, one can use an explicit or implicit method. Methods based of off-

Table 1: Error of control values in Example 4.1 for FBSM_Rk4 , New6_4.

h	time	FBSM_Rk4	New6_4
$\frac{1}{20}$	0.00	8.6132e-2	1.3865e-3
$\frac{1}{20}$	0.25	2.5778e-2	9.4612e-4
$\frac{1}{20}$	0.50	1.8349e-3	5.6750e-4
$\frac{1}{20}$	0.75	3.6866e-3	2.5284e-4
$\frac{1}{20}$	0.90	2.2211e-3	9.3455e-5
$\frac{1}{20}$	1.00	0.0000000	0.0000000
$\frac{1}{50}$	0.00	3.5290e-2	4.4031e-4
$\frac{1}{50}$	0.25	1.0829e-2	3.1395e-4
$\frac{1}{50}$	0.50	4.2201e-4	1.9102e-4
$\frac{1}{50}$	0.75	1.6950e-3	9.2582e-5
$\frac{1}{50}$	0.90	9.8330e-4	3.4037e-5
$\frac{1}{50}$	1.00	0.0000000	0.0000000
$\frac{1}{100}$	0.00	1.7776e-2	1.5905e-5
$\frac{1}{100}$	0.25	5.0705e-3	1.0676e-4
$\frac{1}{100}$	0.50	1.5505e-4	1.1237e-4
$\frac{1}{100}$	0.75	8.8258e-4	6.6257e-5
$\frac{1}{100}$	0.90	5.0837e-4	2.7671e-5
$\frac{1}{100}$	1.00	0.0000000	0.0000000
$\frac{1}{500}$	0.00	3.5759e-3	1.8647e-7
$\frac{1}{500}$	0.25	1.0096e-3	2.3060e-5
$\frac{1}{500}$	0.50	2.1522e-5	2.3291e-5
$\frac{1}{500}$	0.75	1.8287e-4	1.3492e-5
$\frac{1}{500}$	0.90	1.0446e-4	5.5839e-6
$\frac{1}{500}$	1.00	0.0000000	0.0000000
$\frac{1}{1000}$	0.00	1.7882e-3	6.1507e-8
$\frac{1}{1000}$	0.25	5.0494e-4	1.1518e-5
$\frac{1}{1000}$	0.50	1.0436e-5	1.1616e-5
$\frac{1}{1000}$	0.75	1.1702e-5	6.7209e-6
$\frac{1}{1000}$	0.90	5.2353e-5	2.7788e-6
$\frac{1}{1000}$	1.00	0.0000000	0.0000000

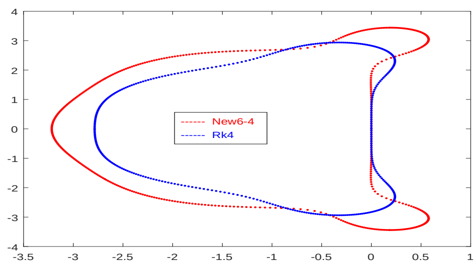


Figure 2: Stability region of Rk4 and New6_4 methods

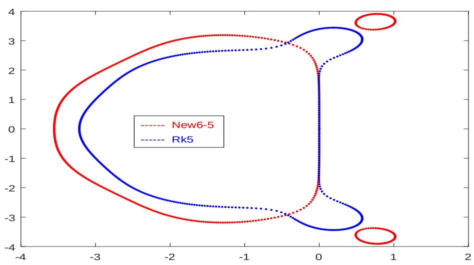


Figure 3: Stability region of Rk5 and New 6_5 methods

step points, such as hybrid BDF, (HBDF), new class of HBDFs and class $2 + 1$ hybrid BDF-like methods have wide stability regions and higher order compared to some Runge-Kutta method and implicit BDF methods [1, 20]. Let us consider the IVP of the form (2.1). Linear k-step methods of the

$$x_{n+1} = \alpha_1 x_n + \alpha_2 x_{n-1} + ... + \alpha_k x_{n-k+1}$$

Table 2: Error of control values in Example 4.1 for FBSM_Rk5 , New6_5.

h	time	FBSM_Rk5	New6_5
$\frac{1}{20}$	0.00	7.3983e-2	4.7207e-4
$\frac{1}{20}$	0.25	1.8308e-2	6.2152e-4
$\frac{1}{20}$	0.50	2.4131e-3	5.1416e-4
$\frac{1}{20}$	0.75	5.6224e-3	2.9090e-4
$\frac{1}{20}$	0.90	2.9802e-3	1.2380e-4
$\frac{1}{20}$	1.00	0.0000000	0.0000000
$\frac{1}{50}$	0.00	3.3399e-2	1.7468e-4
$\frac{1}{50}$	0.25	7.8199e-3	2.3938e-4
$\frac{1}{50}$	0.50	1.2241e-3	1.9668e-4
$\frac{1}{50}$	0.75	2.4652e-3	1.1241e-4
$\frac{1}{50}$	0.90	1.2704e-3	4.5014e-5
$\frac{1}{50}$	1.00	0.0000000	0.0000000
$\frac{1}{100}$	0.00	1.5297e-2	8.5557e-5
$\frac{1}{100}$	0.25	3.6056e-3	1.1775e-4
$\frac{1}{100}$	0.50	6.5866e-4	9.6313e-5
$\frac{1}{100}$	0.75	1.2455e-3	2.2522e-5
$\frac{1}{100}$	0.90	1.3425e-3	5.2522e-5
$\frac{1}{100}$	1.00	0.0000000	0.0000000
$\frac{1}{500}$	0.00	3.0797e-3	1.7427e-5
$\frac{1}{500}$	0.25	7.1793e-4	2.3640e-5
$\frac{1}{500}$	0.50	1.3968e-4	1.9177e-5
$\frac{1}{500}$	0.75	2.5444e-4	1.0348e-5
$\frac{1}{500}$	0.90	1.3213e-4	4.2175e-6
$\frac{1}{500}$	1.00	0.0000000	0.0000000
$\frac{1}{1000}$	0.00	1.5417e-3	8.1780e-6
$\frac{1}{1000}$	0.25	3.5917e-4	1.1407e-5
$\frac{1}{1000}$	0.50	7.0067e-5	9.3035e-6
$\frac{1}{1000}$	0.75	1.2627e-4	4.9284e-6
$\frac{1}{1000}$	0.90	6.6158e-5	2.0463e-6
$\frac{1}{1000}$	1.00	0.0000000	0.0000000

$$+ h\{\beta_0 f_{n+1} + \beta_1 f_n + ... + \beta_k f_{n-k+1}\} \qquad (2.2)$$

has $2k+1$ arbitrary parameter and can be written as

$$\rho(E)x_{n-k+1} - h\sigma(E)f_{n-k+1} = 0$$

where E is the shift operator as $E(x(t)) = x(t+h)$, with the step length h and ρ and σ are first and second characteristic polynomials defined by

$$\rho(\xi) = \xi^k - \alpha_1\xi^{k-1} - \alpha_2\xi^{k-2} - ... - \alpha_k,$$

$$\sigma(\xi) = \beta_0\xi^k + \beta_1\xi^{k-1} + ... + \beta_k \qquad ,$$

To increase the order of k-step methods of the form (2.2), we use a linear combination of the slopes at several points between t_n and t_{n+1} where $t_{n+1} = t_n + h$ and h is the step length on $[t_0, t_1]$. Then, the modified form of (2.2) with m slops is given by

$$x_{n+1} = \sum_{j=1}^{k} \alpha_j x_{n-j+1} + h\sum_{j=0}^{k} \beta_j f_{n-j+1}$$

$$+ h\sum_{j=1}^{m} \gamma_j f_{n+\theta_j} \qquad (2.3)$$

where α_j, β_j, γ_j and θ_j are $2k+2m+1$ arbitrary parameters [20]. Methods of the form [5] with m off-step points are called hybrid methods where $-1 < \theta_j < 3, j = 1, 2, ..., m$. In this work, we set $\alpha_1 = 1, k = 1$ and $m = 4$. Hence, we write (2.3) as

$$x_{n+1} = x_n + h\{\beta_0 f_n + \gamma_1 f_{n+\theta_1} + \gamma_2 f_{n+\theta_2}$$

Table 3: Error of state values in Example 4.1 for FBSM_Rk4 , New6_4.

h	time	FBSM_Rk4	New6_4
$\frac{1}{20}$	0.00	0.0000000	0.0000000
$\frac{1}{20}$	0.25	1.4621e-2	1.2140e-4
$\frac{1}{20}$	0.50	2.1962e-2	1.7791e-4
$\frac{1}{20}$	0.75	2.7528e-2	1.9353e-4
$\frac{1}{20}$	0.90	3.1434e-2	1.8116e-4
$\frac{1}{20}$	1.00	3.4609e-2	1.5980e-4
$\frac{1}{50}$	0.00	0.0000000	0.0000000
$\frac{1}{50}$	0.25	5.8666e-3	4.0885e-5
$\frac{1}{50}$	0.50	9.0092e-3	6.4591e-5
$\frac{1}{50}$	0.75	1.1173e-2	2.2532e-4
$\frac{1}{50}$	0.90	1.2835e-2	8.4307e-5
$\frac{1}{50}$	1.00	1.4129e-2	8.6651e-5
$\frac{1}{100}$	0.00	0.0000000	0.0000000
$\frac{1}{100}$	0.25	3.0457e-3	8.7137e-5
$\frac{1}{100}$	0.50	4.5430e-3	1.3930e-4
$\frac{1}{100}$	0.75	5.6673e-3	1.8525e-4
$\frac{1}{100}$	0.90	6.4624e-3	2.1699e-4
$\frac{1}{100}$	1.00	7.1131e-3	2.4143e-4
$\frac{1}{500}$	0.00	0.0000000	0.0000000
$\frac{1}{500}$	0.25	6.1435e-4	1.7493e-5
$\frac{1}{500}$	0.50	9.1506e-4	2.7834e-5
$\frac{1}{500}$	0.75	1.1404e-3	3.6713e-5
$\frac{1}{500}$	0.90	1.3000e-3	4.2707e-5
$\frac{1}{500}$	1.00	1.4309e-3	4.7251e-5
$\frac{1}{1000}$	0.00	0.0000000	0.0000000
$\frac{1}{1000}$	0.25	3.0736e-4	8.7102e-6
$\frac{1}{1000}$	0.50	4.5766e-4	1.3834e-5
$\frac{1}{1000}$	0.75	5.7023e-4	1.8214e-5
$\frac{1}{1000}$	0.90	6.5001e-4	2.1165e-5
$\frac{1}{1000}$	1.00	7.1541e-4	2.3399e-5

$$+\gamma_3 f_{n+\theta_3} + \gamma_4 f_{n+\theta_4} + \beta_1 f_{n+1}\}, \quad (2.4)$$

where $\alpha_1, \beta_0, \beta_1, \gamma_1$ and θ_1 are arbitrary parameters and $\theta_1 \neq 0$ or 1. Expanding terms y_{n+1}, $f_{n+1}, f_{n-\theta_1+1}$ in Taylor's series about t_n, we can obtain a family of fifth order methods if the equations

$$\alpha_1 = 1$$

$$\beta_1 + \beta_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = 1$$

$$\beta_0 + \theta_1\gamma_1 + \theta_2\gamma_2 + \theta_3\gamma_3$$

$$+\theta_4\gamma_4 = \frac{1}{2}$$

$$\frac{1}{2}\beta_0 + \frac{1}{2}\theta_1^2\gamma_1 + \frac{1}{2}\theta_2^2\gamma_2$$

$$+\frac{1}{2}\theta_3^2\gamma_3 + \frac{1}{2}\theta_4^2\gamma_4 = \frac{1}{6}$$

are satisfied where the principal term of the truncation error is

$$\frac{1}{4!}c_4 h^6 x^{(6)}(t_n) + o(h^7),$$

$$c_4 = 1 - 4\beta_2 - 4\gamma_1\theta_1^3 - 4\gamma_2\theta_2^3$$

$$-4\gamma_3\theta_3^3 - 4\gamma_4\theta_4^3.$$

For more details, we refer the reader to [20]. Considering the following three cases:

$$\beta_1 = 2.243954398806741e-01, \alpha_1 = 1,$$

$$\beta_0 = -3.801777517433751e+00,$$

$$\gamma_1 = -1.831754202513441e+00,$$

$$\gamma_2 = 4.102698254887516e+00,$$

Table 4: Error of state values in Example 4.1 for FBSM_Rk5 , New6_5.

h	time	FBSM_Rk5	New6_5
$\frac{1}{20}$	0.00	0.0000000	0.0000000
$\frac{1}{20}$	0.25	1.4666e-2	2.5309e-4
$\frac{1}{20}$	0.50	2.2030e-2	4.2609e-4
$\frac{1}{20}$	0.75	2.7623e-2	6.1522e-4
$\frac{1}{20}$	0.90	3.1556e-2	7.6831e-4
$\frac{1}{20}$	1.00	3.4759e-2	8.9716e-4
$\frac{1}{50}$	0.00	0.0000000	0.0000000
$\frac{1}{50}$	0.25	5.8739e-3	9.9218e-5
$\frac{1}{50}$	0.50	9.0203e-3	1.6652e-4
$\frac{1}{50}$	0.75	1.1173e-2	2.2532e-4
$\frac{1}{50}$	0.90	1.2855e-2	2.7352e-4
$\frac{1}{50}$	1.00	1.4153e-2	3.5997e-4
$\frac{1}{100}$	0.00	0.0000000	0.0000000
$\frac{1}{100}$	0.25	3.0456e-3	5.0783e-5
$\frac{1}{100}$	0.50	4.5458e-3	8.1714e-5
$\frac{1}{100}$	0.75	5.6711e-3	1.0978e-4
$\frac{1}{100}$	0.90	6.4673e-3	1.2967e-4
$\frac{1}{100}$	1.00	7.1191e-3	1.4521e-4
$\frac{1}{500}$	0.00	0.0000000	0.0000000
$\frac{1}{500}$	0.25	6.1443e-4	1.0199e-5
$\frac{1}{500}$	0.50	9.1518e-4	1.6307e-5
$\frac{1}{500}$	0.75	1.1405e-3	2.1606e-5
$\frac{1}{500}$	0.90	1.3002e-3	2.5193e-5
$\frac{1}{500}$	1.00	1.4311e-3	2.7915e-5
$\frac{1}{1000}$	0.00	0.0000000	0.0000000
$\frac{1}{1000}$	0.25	3.0738e-4	4.9618e-6
$\frac{1}{1000}$	0.50	4.5768e-4	7.8709e-6
$\frac{1}{1000}$	0.75	5.7027e-4	1.0360e-5
$\frac{1}{1000}$	0.90	6.5006e-4	1.2045e-5
$\frac{1}{1000}$	1.00	7.1547e-4	1.3326e-5

$$\gamma_3 = 2.307214186027961e + 00,$$

$$\gamma_4 = -7.761608489596578e - 04,$$

$$\theta_1 = -\frac{75}{288},$$

$$\theta_2 = -\frac{50}{288},$$

$$\theta_3 = \frac{64}{288},$$

$$\theta_4 = 2.39,$$

$$c_4 = -\frac{5}{18}.$$

gives us the following method of order 6 (say New method in this work [20]:

$$x_{n+1} = x_n + h\{\beta_0 f_n + \gamma_1 f_{n+\theta_1} + \gamma_2 f_{n+\theta_2}$$

$$+ \gamma_3 f_{n+\theta_3} + \gamma_4 f_{n+\theta_4} + \beta_1 f_{n+1}\}, \qquad (2.5)$$

where $f_{n+1} = f(t_n, x_{n+1}), f_{n+m} = f(t_n + mh, x_{n+m})$ and $f_n = f(t_n, x_n)$ for $m = v_1, v_2, v_3, v_4$. Note that, x_{n+1}, x_{n+m} and x_n are numerical approximations according to the exact values of the solution $x(t)$ at $t_{n+1} = t_n + h, t_{n+m} = t_n + mh,\ t_n = t_n$ for $m = v_1, v_2, v_3, v_4$. In order to convert method (2.5) into explicit methods at each step, we predict the values of x_{n+1} and x_{n+m} used on the right hand side of the new methods using fourth order explicit Runge-Kutta method as follows :

$$x_{n+1} = x_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \qquad (2.6)$$

$$k_1 = f(t_n, x_n),$$

Table 5: Error of control values in Example 4.2 for FBSM_Rk4 , New6_4.

h	time	FBSM_Rk4	New6_4
$\frac{1}{20}$	0.00	3.4701e-3	1.2011e-4
$\frac{1}{20}$	0.25	4.3191e-3	1.4846e-4
$\frac{1}{20}$	0.50	4.7786e-3	1.6308e-4
$\frac{1}{20}$	0.75	3.9653e-3	1.3426e-4
$\frac{1}{20}$	0.90	2.1497e-3	7.2252e-5
$\frac{1}{20}$	1.00	0.0000000	0.0000000
$\frac{1}{50}$	0.00	1.3670e-3	1.8439e-5
$\frac{1}{50}$	0.25	1.6832e-3	2.2622e-5
$\frac{1}{50}$	0.50	1.8676e-3	2.4973e-5
$\frac{1}{50}$	0.75	1.5734e-3	2.0849e-5
$\frac{1}{50}$	0.90	8.3500e-4	1.0802e-5
$\frac{1}{50}$	1.00	0.0000000	0.0000000
$\frac{1}{100}$	0.00	6.8014e-4	4.5035e-6
$\frac{1}{100}$	0.25	8.4210e-4	5.5421e-6
$\frac{1}{100}$	0.50	9.2682e-4	6.0331e-6
$\frac{1}{100}$	0.75	7.6515e-4	4.8288e-6
$\frac{1}{100}$	0.90	4.1368e-4	2.3730e-6
$\frac{1}{100}$	1.00	0.0000000	0.0000000
$\frac{1}{500}$	0.00	1.3553e-4	1.1636e-7
$\frac{1}{500}$	0.25	1.6766e-4	1.1729e-7
$\frac{1}{500}$	0.50	1.8439e-4	7.0448e-8
$\frac{1}{500}$	0.75	1.5218e-4	8.6533e-8
$\frac{1}{500}$	0.90	8.2419e-5	2.8096e-7
$\frac{1}{500}$	1.00	0.0000000	0.0000000
$\frac{1}{1000}$	0.00	6.7765e-5	1.9364e-8
$\frac{1}{1000}$	0.25	8.3832e-5	5.0544e-8
$\frac{1}{1000}$	0.50	9.2214e-5	1.1402e-7
$\frac{1}{1000}$	0.75	7.6171e-5	2.3860e-7
$\frac{1}{1000}$	0.90	4.1365e-5	3.6307e-7
$\frac{1}{1000}$	1.00	0.0000000	0.0000000

$$k_2 = f(t_n + \frac{1}{2}h, x_n + \frac{1}{2}k_1 h),$$

$$k_3 = f(t_n + \frac{1}{2}h, x_n + \frac{1}{2}k_2 h),$$

$$k_4 = f(t_n + h, x_n + k_3 h).$$

In general, we rewrite method (2.5) using RK4 method as a predictor as follows:

$$\bar{x}_{n+1} = x_n + \tfrac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \quad (2.7)$$

$$\bar{x}_{n+\theta_i} = x_n + \tfrac{\theta_i h}{6}(k_{1i} + 2k_{2i} + 2k_{3i} + k_{4i}), \quad i = 1, 2, 3, 4. \quad (2.8)$$

$$x_{n+1} = x_n + h\{\beta_0 \bar{f}_n + \gamma_1 \bar{f}_{n+\theta_1} + \gamma_2 \bar{f}_{n+\theta_2} + \gamma_3 \bar{f}_{n+\theta_3} + \gamma_4 \bar{f}_{n+\theta_4} + \beta_1 f_{n+1}\}, \quad (2.9)$$

where

$$\begin{aligned} k_{1i} &= f(t_n, x_n), \\ k_{2i} &= f(t_n + \theta_i h, x_n + \theta_i k_1 h), \\ k_{3i} &= f(t_n + \theta_i h, x_n + \theta_i k_2 h), \\ k_{4i} &= f(t_n + \theta_i h, x_n + \theta_i k_3 h), \end{aligned}$$

and $\bar{f}_{n+1} = f(t_n, \bar{x}_{n+1}), \bar{f}_{n+\theta_i} = f(t_n + \theta_i h, x_{n+\theta_i}), f_n = f(t_n, x_n)$.
and say (2.9) is New $6-4$ method in this work. Now, suppose that the order of stage equation (2.6) is 4 . Thus, the difference between exact and numerical solution at $t = t_{n+m} = t_n + mh$,

Table 6: Error of control values in Example 4.2 for FBSM_Rk4 , New6_4.

h	time	FBSM_Rk4	New6_4
$\frac{1}{20}$	0.00	3.4701e-3	1.1952e-4
$\frac{1}{20}$	0.25	4.3191e-3	1.4774e-4
$\frac{1}{20}$	0.50	4.7786e-3	1.6229e-4
$\frac{1}{20}$	0.75	3.9653e-3	1.3361e-4
$\frac{1}{20}$	0.90	2.1497e-3	7.1900e-5
$\frac{1}{20}$	1.00	0.0000000	0.0000000
$\frac{1}{50}$	0.00	1.3670e-3	1.8404e-5
$\frac{1}{50}$	0.25	1.6832e-3	2.2579e-5
$\frac{1}{50}$	0.50	1.8676e-3	2.4925e-5
$\frac{1}{50}$	0.75	1.5734e-3	2.0809e-5
$\frac{1}{50}$	0.90	8.3500e-4	1.0781e-5
$\frac{1}{50}$	1.00	0.0000000	0.0000000
$\frac{1}{100}$	0.00	6.8014e-4	4.4992e-6
$\frac{1}{100}$	0.25	8.4210e-4	5.5368e-6
$\frac{1}{100}$	0.50	9.2682e-4	6.0272e-6
$\frac{1}{100}$	0.75	7.6515e-4	4.8240e-6
$\frac{1}{100}$	0.90	4.1368e-4	2.3704e-6
$\frac{1}{100}$	1.00	0.0000000	0.0000000
$\frac{1}{500}$	0.00	1.3553e-4	1.1633e-7
$\frac{1}{500}$	0.25	1.6766e-4	1.1724e-7
$\frac{1}{500}$	0.50	1.8439e-4	7.0402e-8
$\frac{1}{500}$	0.75	1.5218e-4	8.6571e-8
$\frac{1}{500}$	0.90	8.2419e-5	2.8098e-7
$\frac{1}{500}$	1.00	0.0000000	0.0000000
$\frac{1}{1000}$	0.00	6.7765e-5	1.1368e-8
$\frac{1}{1000}$	0.25	8.3832e-5	5.0549e-8
$\frac{1}{1000}$	0.50	9.2214e-5	1.1403e-7
$\frac{1}{1000}$	0.75	7.6171e-5	2.3861e-7
$\frac{1}{1000}$	0.90	4.1365e-5	3.6307e-7
$\frac{1}{1000}$	1.00	0.0000000	0.0000000

$m = \theta_1, \theta_2, \theta_3, \theta_4$ and 1 is

$$y(t_{n+m}) - y_{n+m} = C_m h^p y^{(p)}(t_n) + O(h^{p+1}) \tag{2.10}$$

where C_m is the error constant of the method (2.7) with corresponding m which can take only one of the values $\theta_1, \theta_2, \theta_3, \theta_4$, together with the value 1 related to method (2.5) . The difference operator associated to method (2.7), of order 4 can be written as

$$y(t_{n+1}) - y_{n+1} = Ch^p y^{(p)}(t_n) + O(h^{p+1}) \tag{2.11}$$

where C is the error constant of the method (2.7). Therefore, we have the following theorem:

Theorem 2.1 *Given that formula* (2.7) *is of order p then p is equal to* 5.

Proof. Suppose that m can only take one of the values $\theta_1, \theta_2, \theta_3, \theta_4$ and y_n is exact. From (2.7) and (2.9) one can write

$$\begin{aligned} y(t_{n+1}) - y_{n+1} &= h\gamma_1[f(t_{n+\theta_1}, y(t_{n+\theta_1})) \\ &\quad - f(t_{n+\theta_1}, \overline{y}_{n+\theta_1})] \qquad (2.12) \\ &+ h\gamma_2[f(t_{n+\theta_2}, y(t_{n+\theta_2})) - f(t_{n+\theta_2}, \overline{y}_{n+\theta_2})] \\ &+ h\gamma_2[f(t_{n+\theta_2}, y(t_{n+\theta_2})) - f(t_{n+\theta_2}, \overline{y}_{n+\theta_2})] \\ &+ h\gamma_3[f(t_{n+\theta_3}, y(t_{n+\theta_3})) - f(t_{n+\theta_3}, \overline{y}_{n+\theta_3})] \\ &+ h\gamma_4[f(t_{n+\theta_4}, y(t_{n+\theta_4})) - f(t_{n+\theta_4}, \overline{y}_{n+\theta_4})] \\ &+ h\beta_1[f(t_{n+1}, y(t_{n+1})) - f(t_{n+1}, \overline{y}_{n+1})] \\ &+ Ch^6 y^{(6)}(t_n) + O(h^7) \end{aligned}$$

Considering properties of the $IVPs$ of the form

Table 7: Error of state values in Example 4.2 for FBSM_Rk4 , New6_4.

h	time	FBSM_Rk4	New6_4
$\frac{1}{20}$	0.00	0.0000000	0.0000000
$\frac{1}{20}$	0.25	7.6093e-4	9.2272e-4
$\frac{1}{20}$	0.50	1.3473e-3	1.1392e-3
$\frac{1}{20}$	0.75	1.6619e-3	1.0879e-3
$\frac{1}{20}$	0.90	1.5965e-3	1.0283e-3
$\frac{1}{20}$	1.00	1.3722e-3	9.9712e-4
$\frac{1}{50}$	0.00	0.0000000	0.0000000
$\frac{1}{50}$	0.25	2.9166e-4	1.5290e-4
$\frac{1}{50}$	0.50	5.3599e-4	1.9288e-4
$\frac{1}{50}$	0.75	6.6223e-4	1.8488e-4
$\frac{1}{50}$	0.90	6.4345e-4	1.7403e-4
$\frac{1}{50}$	1.00	5.6052e-4	1.6858e-4
$\frac{1}{100}$	0.00	0.0000000	0.0000000
$\frac{1}{100}$	0.25	1.5096e-4	4.5027e-5
$\frac{1}{100}$	0.50	2.6759e-4	5.5677e-5
$\frac{1}{100}$	0.75	3.3208e-4	5.3225e-5
$\frac{1}{100}$	0.90	3.2262e-4	5.0264e-5
$\frac{1}{100}$	1.00	2.8225e-4	4.8636e-5
$\frac{1}{500}$	0.00	0.0000000	0.0000000
$\frac{1}{500}$	0.25	3.0175e-5	3.9036e-6
$\frac{1}{500}$	0.50	3.3511e-5	4.8713e-6
$\frac{1}{500}$	0.75	6.6518e-5	4.7175e-6
$\frac{1}{500}$	0.90	6.4805e-5	4.4925e-6
$\frac{1}{500}$	1.00	5.6936e-5	4.3669e-6
$\frac{1}{1000}$	0.00	0.0000000	0.0000000
$\frac{1}{1000}$	0.25	1.5101e-5	1.6531e-6
$\frac{1}{1000}$	0.50	2.6789e-5	2.0845e-6
$\frac{1}{1000}$	0.75	3.3327e-5	2.0560e-6
$\frac{1}{1000}$	0.90	3.2505e-5	1.9897e-6
$\frac{1}{1000}$	1.00	2.8603e-5	1.9591e-6

(2.1), for some values such as η_m and η_1 belong to intervals $(\overline{y}_{n+m}, y(t_{n+m}))$ and $(\overline{y}_{n+1}, y(t_{n+1}))$ respectively, we can write

$$f(t_{n+\theta_1}, y(t_{n+\theta_1})) - f(t_{n+\theta_1}, \overline{y}_{n+\theta_1})$$
$$= \frac{\partial f}{\partial y}(t_{n+\theta_1}, \eta_{n+\theta_1})(y(t_{n+\theta_1} - \overline{y}_{n+\theta_1}),$$
$$f(t_{n+\theta_2}, y(t_{n+\theta_2})) - f(t_{n+\theta_2}, \overline{y}_{n+\theta_2})$$
$$= \frac{\partial f}{\partial y}(t_{n+\theta_2}, \eta_{n+\theta_2})(y(t_{n+\theta_2} - \overline{y}_{n+\theta_2}),$$
$$f(t_{n+\theta_3}, y(t_{n+\theta_3})) - f(t_{n+\theta_3}, \overline{y}_{n+\theta_3})$$
$$= \frac{\partial f}{\partial y}(t_{n+\theta_3}, \eta_{n+\theta_3})(y(t_{n+\theta_3 - \overline{y}_{n+\theta_3}}),$$
$$f(t_{n+\theta_4}, y(t_{n+\theta_4})) - f(t_{n+\theta_4}, \overline{y}_{n+\theta_4})$$
$$= \frac{\partial f}{\partial y}(t_{n+\theta_4}, \eta_{n+\theta_4})(y(t_{n+\theta_4} - \overline{y}_{n+\theta_4}),$$
$$f(t_{n+1}, y(t_{n+1})) - f(t_{n+1}, \overline{y}_{n+1})$$
$$= \frac{\partial f}{\partial y}(t_{n+1}, \eta_{n+1})(y(t_{n+1}) - \overline{y}_{n+1}).$$

Therefore, by using (2.11), we have

$$y(t_{n+1}) - y_{n+1}$$
$$= h\gamma_1 \left[\frac{\partial f}{\partial y}(t_{n+\theta_1}, \eta_{n+\theta_1})(y(t_{n+\theta_1}) - \overline{y}_{n+\theta_1})\right]$$
$$+ h\gamma_2 \left[\frac{\partial f}{\partial y}(t_{n+\theta_2}, \eta_{n+\theta_2})(y(t_{n+\theta_2}) - \overline{y}_{n+\theta_2})\right]$$
$$+ h\gamma_3 \left[\frac{\partial f}{\partial y}(t_{n+v_3}, \eta_{n+\theta_3})(y(t_{n+\theta_3}) - \overline{y}_{n+\theta_3})\right]$$
$$+ h\gamma_4 \left[\frac{\partial f}{\partial y}(t_{n+\theta_4}, \eta_{n+\theta_4})(y(t_{n+\theta_4}) - \overline{y}_{n+\theta_4})\right]$$
$$+ h\beta_1 \left[\frac{\partial f}{\partial y}(t_{n+1}, \eta_{n+1})(y(t_{n+1}) - \overline{y}_{n+1})\right]$$
$$+ Ch^6 y^{(6)}(t_n) + O(h^7).$$

Table 8: Error of state values in Example 4.2 for FBSM_Rk5 New6_5.

h	time	FBSM_Rk5	New6_5
$\frac{1}{20}$	0.00	0.0000000	0.0000000
$\frac{1}{20}$	0.25	6.1668e-4	2.3405e-5
$\frac{1}{20}$	0.50	1.0191e-3	3.7002e-5
$\frac{1}{20}$	0.75	1.0609e-3	4.3374e-5
$\frac{1}{20}$	0.90	7.6032e-4	4.4561e-5
$\frac{1}{20}$	1.00	3.3646e-4	4.4467e-5
$\frac{1}{50}$	0.00	0.0000000	0.0000000
$\frac{1}{50}$	0.25	2.2361e-4	4.0296e-5
$\frac{1}{50}$	0.50	3.9866e-4	4.1735e-5
$\frac{1}{50}$	0.75	4.1786e-4	2.5403e-5
$\frac{1}{50}$	0.90	2.9720e-4	1.0122e-5
$\frac{1}{50}$	1.00	1.3288e-4	3.9547e-7
$\frac{1}{100}$	0.00	0.0000000	0.0000000
$\frac{1}{100}$	0.25	1.2012e-4	7.7332e-6
$\frac{1}{100}$	0.50	1.9791e-4	5.0291e-6
$\frac{1}{100}$	0.75	2.0556e-4	3.4083e-6
$\frac{1}{100}$	0.90	1.4756e-4	1.0529e-5
$\frac{1}{100}$	1.00	6.6259e-5	1.6129e-5
$\frac{1}{500}$	0.00	0.0000000	0.0000000
$\frac{1}{500}$	0.25	2.3926e-5	3.9745e-7
$\frac{1}{500}$	0.50	3.9414e-5	1.3850e-6
$\frac{1}{500}$	0.75	4.0965e-5	3.0023e-6
$\frac{1}{500}$	0.90	2.9488e-5	4.3775e-6
$\frac{1}{500}$	1.00	1.3391e-5	5.5201e-6
$\frac{1}{1000}$	0.00	0.0000000	0.0000000
$\frac{1}{1000}$	0.25	1.1972e-5	3.0292e-7
$\frac{1}{1000}$	0.50	1.9730e-5	8.0510e-7
$\frac{1}{1000}$	0.75	2.05335e-5	1.5816e-6
$\frac{1}{1000}$	0.90	1.4827e-5	2.2426e-6
$\frac{1}{1000}$	1.00	6.8099e-6	2.7954e-6

Applying equation (2.7) to this gives us

$$y(t_{n+1}) - y_{n+1} =$$

$$h\gamma_1\left[\frac{\partial f}{\partial y}(t_{n+\theta_1},\eta_{n+\theta_1})C_1h^p y^{(p)}(t_n)\right]$$

$$+h\gamma_1\left[O(h^{p+1})\right]$$

$$+h\gamma_2\left[\frac{\partial f}{\partial y}(t_{n+\theta_2},\eta_{n+\theta_2})C_2h^p y^{(p)}(t_n)\right]$$

$$+h\gamma_2\left[+O(h^{p+1})\right]$$

$$+h\gamma_3\left[\frac{\partial f}{\partial y}(t_{n+\theta_3},\eta_{n+\theta_3})C_3h^p y^{(p)}(t_n)\right]$$

$$+h\gamma_3\left[O(h^{p+1})\right]$$

$$+h\gamma_4\left[\frac{\partial f}{\partial y}(t_{n+\theta_4},\eta_{n+\theta_4})C_4h^p y^{(p)}(t_n)\right]$$

$$+h\gamma_4\left[O(h^{p+1})\right]$$

$$+h\beta_1\left[\frac{\partial f}{\partial y}(t_{n+1},\eta_{n+1})C_5h^p y^{(p)}(t_n)\right]$$

$$+h\beta_1\left[O(h^{p+1})\right]$$

$$+Ch^6y^{(6)}(t_n) + O(h^7)$$

$$= h^4\left[\frac{\partial f}{\partial y}(t_{n+\theta_1},\eta_{n+\theta_1})c_1\gamma_1 h^{p-4+1}y^{(p)}(t_n)\right]$$

$$+h^4\left[\frac{\partial f}{\partial y}(t_{n+\theta_2},\eta_{n+\theta_2})c_2\gamma_2 h^{p-4+1}y^{(p)}(t_n)\right]$$

$$h^4\left[\frac{\partial f}{\partial y}(t_{n+\theta_3},\eta_{n+\theta_3})c_3\gamma_3 h^{p-4+1}y^{(p)}(t_n)\right]$$

$$+h^4\left[\frac{\partial f}{\partial y}(t_{n+\theta_4},\eta_{n+\theta_4})c_4\gamma_4 h^{p-4+1}y^{(p)}(t_n)\right]$$

$$+h^4\left[\frac{\partial f}{\partial y}(t_{n+1},\eta_{n+1})c_5 h^{p-4+1}y^{(p)}(t_n)\right].$$

$$+Cy^{(p)}(t_n)\} + O(h^{p+1})$$

Table 9: Error of control values in Example 4.3 for FBSM_Rk4 , New6_4.

h	time	FBSM_Rk4	New6_4
$\frac{1}{20}$	0.00	1.5934e-2	2.6650e-4
$\frac{1}{20}$	0.25	1.3300e-2	2.2249e-4
$\frac{1}{20}$	0.50	9.9187e-3	1.6594e-4
$\frac{1}{20}$	0.75	5.5760e-3	9.3306e-5
$\frac{1}{20}$	0.90	2.3988e-3	4.0145e-5
$\frac{1}{20}$	1.00	0.0000000	0.0000000
$\frac{1}{50}$	0.00	6.3424e-3	4.2176e-5
$\frac{1}{50}$	0.25	5.5251e-3	3.6776e-5
$\frac{1}{50}$	0.50	3.9478e-3	2.6311e-5
$\frac{1}{50}$	0.75	2.2971e-3	1.5324e-5
$\frac{1}{50}$	0.90	9.5479e-4	6.3736e-6
$\frac{1}{50}$	1.00	0.0000000	0.0000000
$\frac{1}{100}$	0.00	3.1660e-3	1.0386e-5
$\frac{1}{100}$	0.25	2.6426e-3	8.7098e-6
$\frac{1}{100}$	0.50	1.9706e-3	6.5234e-6
$\frac{1}{100}$	0.75	1.2980e-3	4.3110e-6
$\frac{1}{100}$	0.90	4.7660e-4	1.5877e-6
$\frac{1}{100}$	1.00	0.0000000	0.0000000
$\frac{1}{500}$	0.00	6.3250e-4	2.4620e-7
$\frac{1}{500}$	0.25	5.2791e-4	2.4589e-7
$\frac{1}{500}$	0.50	3.9365e-4	2.1164e-7
$\frac{1}{500}$	0.75	2.2128e-4	1.3380e-7
$\frac{1}{500}$	0.90	9.5196e-5	6.1166e-7
$\frac{1}{500}$	1.00	0.0000000	0.0000000
$\frac{1}{1000}$	0.00	3.1628e-4	7.0038e-8
$\frac{1}{1000}$	0.25	2.6396e-4	1.8071e-8
$\frac{1}{1000}$	0.50	1.9681e-4	1.4796e-8
$\frac{1}{1000}$	0.75	1.1063e-4	2.3142e-8
$\frac{1}{1000}$	0.90	4.7591e-5	1.3557e-
$\frac{1}{1000}$	1.00	0.0000000	0.0000000

Thus, it can be concluded that the method (2.6) is of order p and so the proof is completed.

By following the same way as presented above, it can be proved that the methods (2.5) using RK5 method as a predictor (Runge-kutta of order 5) of the form

$$\bar{x}_{n+1} = x_n + \tfrac{h}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6), \tag{2.13}$$

$$\bar{x}_{n+\theta_i} = x_n + \tfrac{\theta_i h}{90}(7k_{1i} + 32k_{3i} + 12k_{4i} + 32k_{5i} + 7k_{6i}) \tag{2.14}$$

$$, i = 1, 2, 3, 4.$$

$$x_{n+1} = x_n + h\{\beta_0 \bar{f}_n + \gamma_1 \bar{f}_{n+\theta_1} + \gamma_2 \bar{f}_{n+\theta_2} + \gamma_3 \bar{f}_{n+\theta_3} + \gamma_4 \bar{f}_{n+\theta_4} + \beta_1 f_{n+1}\}, \tag{2.15}$$

where

$$k_{1i} = f(t_n, x_n), k_{2i} = f(t_n + \theta_i h/2, x_n + \theta_i k_1 h/2),$$

$$k_{3i} = f(t_n + \theta_i h/4, x_n + \theta_i/16(3k_{i1} + k_{2i}),$$

$$k_{4i} = f(t_n + \theta_i h/2, x_n + \theta_i k_{3i} h/2),$$

$$k_{5i} = f(t_n + 3\theta_i h/4, x_n + \theta_i/16(-3k_{2i} + 6k_{3i} + 9k_{4i})),$$

$$k_{6i} = f(t_n + \theta_i h, x_n + \theta_i/7(k_{1i} + 4k_{2i} + 6k_{3i} - 12k_{4i} + 8k_{5i})),$$

and $\bar{f}_{n+1} = f(t_n, \bar{x}_{n+1}), \bar{f}_{n+\theta_i} = f(t_n + \theta_i h, x_{n+\theta_i}), f_n = f(t_n, x_n)$.
and say (2.15) is New $6-5$ method in this work.

Table 10: Error of control values in Example 4.3 for FBSM_Rk5 New6_5.

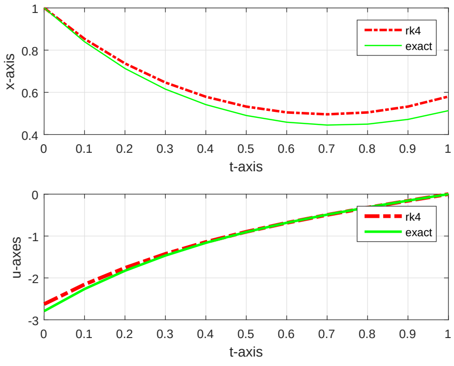
h	time	FBSM_Rk5	New6_5
$\frac{1}{20}$	0.00	4.1885e-3	7.0173e-3
$\frac{1}{20}$	0.25	3.4961e-3	5.8614e-5
$\frac{1}{20}$	0.50	2.6071e-3	4.3738e-5
$\frac{1}{20}$	0.75	1.4656e-3	2.4603e-5
$\frac{1}{20}$	0.90	6.3053e-4	1.0588e-5
$\frac{1}{20}$	1.00	0.0000000	0.0000000
$\frac{1}{50}$	0.00	1.6816e-3	1.1069e-5
$\frac{1}{50}$	0.25	1.4649e-3	9.6769e-6
$\frac{1}{50}$	0.50	1.0467e-3	6.9486e-6
$\frac{1}{50}$	0.75	6.0902e-4	4.0579e-6
$\frac{1}{50}$	0.90	2.5313e-4	1.6905e-6
$\frac{1}{50}$	1.00	0.0000000	0.0000000
$\frac{1}{100}$	0.00	8.4195e-4	2.6348e-6
$\frac{1}{100}$	0.25	7.0247e-4	2.2397e-6
$\frac{1}{100}$	0.50	5.2402e-4	1.6985e-6
$\frac{1}{100}$	0.75	2.9457e-4	9.6968e-7
$\frac{1}{100}$	0.90	1.2672e-4	4.2077e-7
$\frac{1}{100}$	1.00	0.0000000	0.0000000
$\frac{1}{500}$	0.00	1.6869e-4	6.3034e-8
$\frac{1}{500}$	0.25	1.4077e-4	1.2225e-8
$\frac{1}{500}$	0.50	1.0494e-4	1.9156e-8
$\frac{1}{500}$	0.75	5.8985e-5	2.5593e-8
$\frac{1}{500}$	0.90	2.5372e-5	1.4612e-8
$\frac{1}{500}$	1.00	0.0000000	0.0000000
$\frac{1}{1000}$	0.00	8.4447e-5	1.4732e-7
$\frac{1}{1000}$	0.25	7.0448e-5	8.2581e-8
$\frac{1}{1000}$	0.50	5.2506e-5	3.3309e-8
$\frac{1}{1000}$	0.75	2.9503e-5	3.9019e-9
$\frac{1}{1000}$	0.90	1.2689e-5	1.9229e-9
$\frac{1}{1000}$	1.00	0.0000000	0.0000000

Table 11: The optimal states of methods in Example 4.4 $(x1, x2)$.

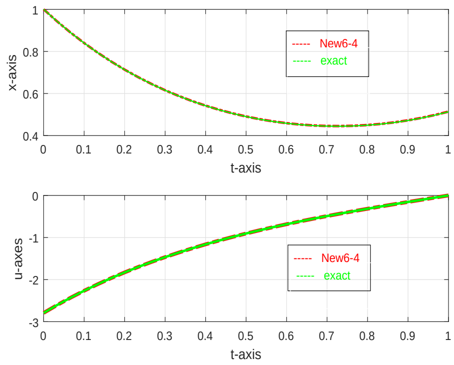
method	x1	x2
Rk4	5.7229e-2	1.7403e-4
New6_4	5.7229e-2	1.7390e-4
Rk5	5.7373e-2	1.8272e-4
New6_5	5.7227e-2	1.7446e-4

Table 12: The optimal states of methods in Example 4.4 $(x3, x4)$.

method	x3	x4
Rk4	2.0982e-4	0
New6_4	2.0973e-4	0
Rk5	2.1996e-4	0
New6_5	2.1026e-4	0

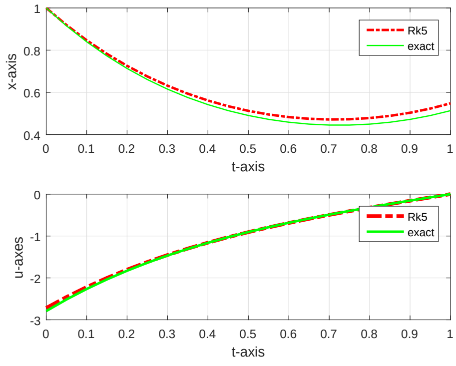


(a)

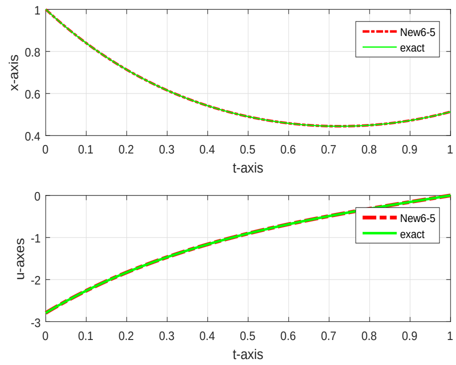


(b)

Figure 4: (a) The optimal state and control of Example 4.1, where $h = \frac{1}{20}$ ($FBSM\_rk4$). (b) The optimal state and control of Example 4.1 (new6_4 method), where $h = \frac{1}{20}$.



(a)



(b)

Figure 5: (a) The optimal state and control of Example 4.1, where $h = \frac{1}{20}$ ($FBSM\_rk5$). (b) The optimal state and control of Example 4.1 (new6_5 method), where $h = \frac{1}{20}$.

3 Stability analysis of the new methods

Now we want to examine the stability analysis of new method. We consider Dahlquist test problem $x' = \lambda x$ to investigate the stability region of the method presented in this study. Using the Dahlquist test problem to the methods (2.5) inserting $p = 4$, the following equations can be obtained:

$$\bar{x}_{n+1} = \left(1 + \bar{h} + \frac{(\bar{h})^2}{2!} + \frac{(\bar{h})^3}{3!} + \frac{(\bar{h})^4}{4!}\right) x_n, \tag{3.16}$$

$$\bar{x}_{n+m} = \left(1 + \frac{m\bar{h}}{1!} + \frac{(m\bar{h})^2}{2!} + \frac{(m\bar{h})^3}{3!}\right) x_n + \left(\frac{(m\bar{h})^4}{4!}\right) x_n, m = \theta_1, \theta_2, \theta_3, \theta_4), \tag{3.17}$$

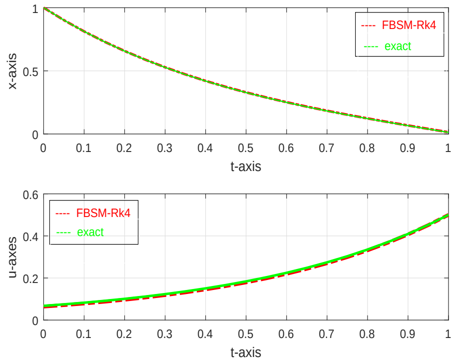
$$x_{n+1} = x_n + h\{\beta_0 f_n + \gamma_1 f_{n+\theta_1} + \gamma_2 f_{n+\theta_2} + \gamma_3 f_{n+\theta_3} + \gamma_4 f_{n+\theta_4} + \beta_1 f_{n+1}\}. \tag{3.18}$$

where $\bar{h} = h\lambda$. By substituting (3.16) and (3.17) into (3.18), the following equation is obtained:

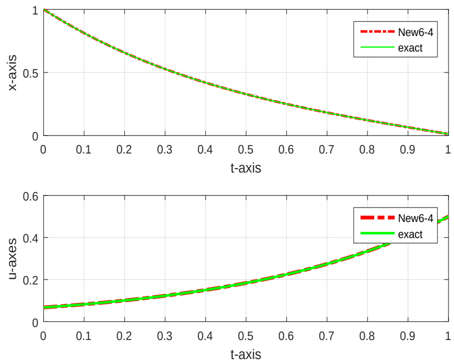
$$x_{n+1} = x_n + \bar{h}\left\{\beta_0 x_n + \beta_1 x_n \left(1 + \bar{h} + \frac{(\bar{h})^2}{2!}\right)\right\} + \bar{h}\beta_1 x_n \left(\frac{(\bar{h})^3}{3!} + \frac{(\bar{h})^4}{4!}\right) + \bar{h}\left\{x_n \sum_{i=1}^{4} \gamma_i \left(1 + (\theta_i \bar{h}) + \frac{(\theta_i \bar{h})^2}{2!} + \frac{(\theta_i \bar{h})^3}{3!}\right)\right\} + \bar{h}\left\{x_n \sum_{i=1}^{4} \gamma_i \left(\frac{(\theta_i \bar{h})^4}{4!}\right)\right\}. \tag{3.19}$$

By inserting $x_n = r^n$ into (3.19) and dividing by r^n we can obtain:

$$r^{n+1} = r^n \left\{1 + a_1 \bar{h} + a_2 \bar{h}^2 + a_3 \bar{h}^3\right\}$$

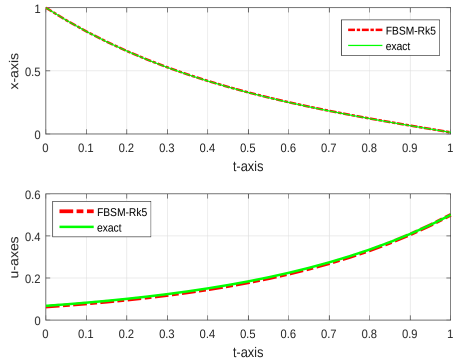


(a)

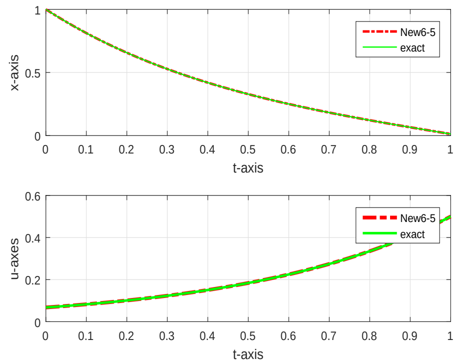


(b)

Figure 6: (a) The optimal state and control of Example 4.2, where $h = \frac{1}{20}$ $(FBSM\_rk4)$. (b) The optimal state and control of Example 4.2 $(New6\_4method)$, where $h = \frac{1}{20}$.



(a)



(b)

Figure 7: (a) The optimal state and control of Example 4.2, where $h = \frac{1}{20}$ $(FBSM\_rk5)$. (b) The optimal state and control of Example 4.2 (New6_5 method), where $h = \frac{1}{20}$.

$$+r^n\left\{+a_4\bar{h}^4 + a_5\bar{h}^5\right\}$$

$$\Rightarrow r = 1 + a_1\bar{h} + a_2\bar{h}^2 + a_3\bar{h}^3 + a_4\bar{h}^4 + a_5\bar{h}^5$$

$$a_1 = \beta_0 + \beta_1 + \sum_{i=1}^{4}\gamma_i$$

$$a_2 = \beta_1 + \sum_{i=1}^{4}(\theta_i\gamma_i)$$

$$a_3 = \frac{1}{2}(\beta_1 + \sum_{i=1}^{4}(\theta_i^2\gamma_i)$$

$$a_4 = \frac{1}{6}(\beta_1 + \sum_{i=1}^{4}(\theta_i^3\gamma_i)$$

$$a_4 = \frac{1}{24}(\beta_1 + \sum_{i=1}^{4}(\theta_i^4\gamma_i).$$

which is the stability polynomial of the method (2.9).
By following the same way for $p = 5$ we can be obtained:

$$\bar{x}_{n+1} = \left(1 + \bar{h} + \frac{(\bar{h})^2}{2!} + \frac{(\bar{h})^3}{3!} + \frac{(\bar{h})^4}{4!} + \frac{(\bar{h})^5}{5!}\right)x_n, \tag{3.20}$$

$$\bar{x}_{n+m} = \left(1 + \frac{m\bar{h}}{1!} + \frac{(m\bar{h})^2}{2!} + \frac{(m\bar{h})^3}{3!}\right)x_n$$

$$+\left(\frac{(m\bar{h})^4}{4!} + \frac{(m\bar{h})^5}{5!}\right)x_n, m = \theta_1, \theta_2, \theta_3, \theta_4, \tag{3.21}$$

$$x_{n+1} = x_n + h\{\beta_0 f_n + \gamma_1 f_{n+\theta_1} + \gamma_2 f_{n+\theta_2}$$

$$+\gamma_3 f_{n+\theta_3} + \gamma_4 f_{n+\theta_4} + \beta_1 f_{n+1}\}. \tag{3.22}$$

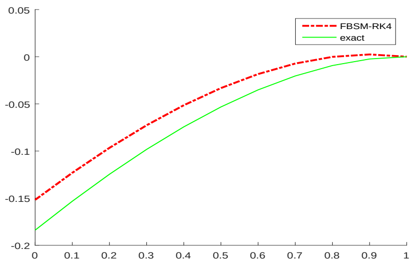


Figure 8: The optimal curves of the problem in Example 4.3 (($FBSM\_rk4$), $h = \frac{1}{20}$)

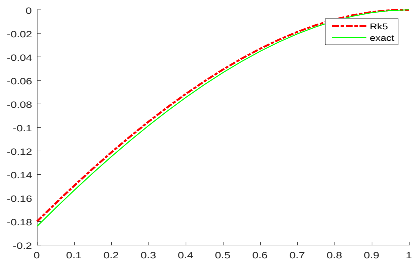


Figure 10: The optimal curves of the problem in Example 4.3 (($FBSM\_rk5$), $h = \frac{1}{20}$)

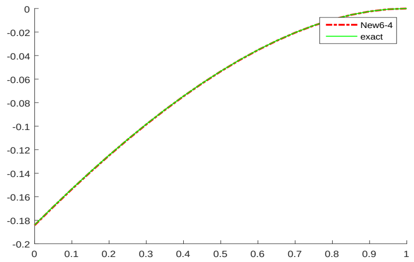


Figure 9: The optimal curves of the problem in Example 4.3 (New6_4 method, $h = \frac{1}{20}$)

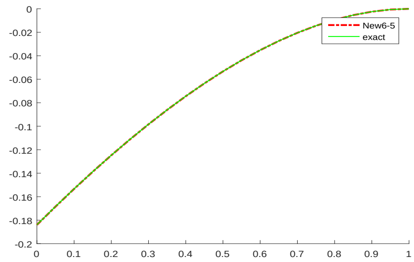


Figure 11: The optimal curves of the problem in Example 4.3 (New6_5 method, $h = \frac{1}{20}$)

where $\bar{h} = h\lambda$. By substituting (3.20) and (3.21) into (3.22), the following equation is obtained:

$$x_{n+1} = x_n + \bar{h}\left\{\beta_0 x_n + \beta_1 x_n\left(1 + \bar{h} + \frac{(\bar{h})^2}{2!}\right)\right\}$$
$$+\bar{h}\beta_1 x_n\left\{\left(\frac{(\bar{h})^3}{3!} + \frac{(\bar{h})^4}{4!} + \frac{(\bar{h})^5}{5!}\right)\right\}$$
$$+\bar{h}\left\{x_n\sum_{i=1}^{4}\gamma_i\left(1 + (\theta_i\bar{h}) + \frac{(\theta_i\bar{h})^2}{2!} + \frac{(\theta_i\bar{h})^3}{3!}\right)\right\}$$
$$+\bar{h}\left\{x_n\sum_{i=1}^{4}\gamma_i\left(\frac{(\theta_i\bar{h})^4}{4!} + \frac{(\theta_i\bar{h})^5}{5!}\right)\right\}.$$

By inserting $x_n = r^n$ into (3.19) and dividing by r^n we can obtain:

$$r^{n+1} = r^n\left\{1 + a_1\bar{h} + a_2\bar{h}^2 + a_3\bar{h}^3\right\}$$
$$+r^n\left\{+a_4\bar{h}^4 + a_5\bar{h}^5 + a_6\bar{h}^6\right\}$$
$$\Rightarrow r = 1 + a_1\bar{h} + a_2\bar{h}^2 + a_3\bar{h}^3 + a_4\bar{h}^4 + a_5\bar{h}^5 + a_6\bar{h}^6$$

$$a_1 = \beta_0 + \beta_1 + \sum_{i=1}^{4}\gamma_i$$

$$a_2 = \beta_1 + \sum_{i=1}^{4}(\theta_i\gamma_i)$$

$$a_3 = \frac{1}{2}(\beta_1 + \sum_{i=1}^{4}(\theta_i^2\gamma_i)$$

$$a_4 = \frac{1}{6}(\beta_1 + \sum_{i=1}^{4}(\theta_i^3\gamma_i)$$

$$a_5 = \frac{1}{24}(\beta_1 + \sum_{i=1}^{4}(\theta_i^4\gamma_i)$$

$$a_6 = \frac{1}{120}(\beta_1 + \sum_{i=1}^{4}(\theta_i^5\gamma_i).$$

which is the stability polynomial of the method. (2.15)
We show the stability of the New methods,in Figures 1-3 and compared ERK method of order 4 and 5. It can be seen that the stability region of the new methods is larger, and this proves the efficiency of the new method.

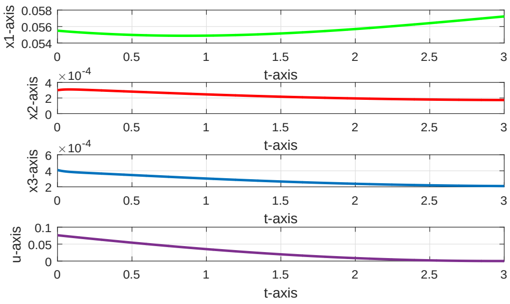


Figure 12: The optimal curves of the problem in Example 4.4 ($FBSM\_Rk4$)

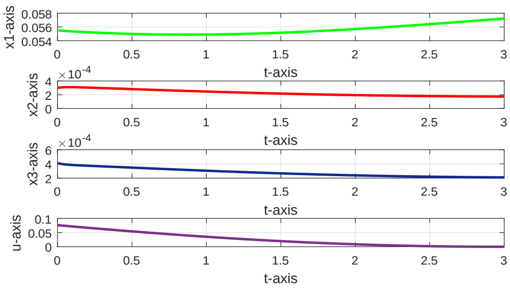


Figure 13: The optimal curves of the problem in Example 4.4 ($new6\_4$)

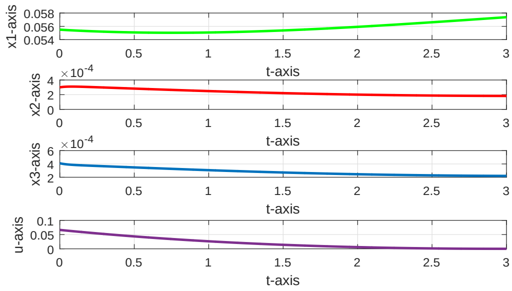


Figure 14: The optimal curves of the problem in Example 4.4 ($FBSM\_Rk5$)

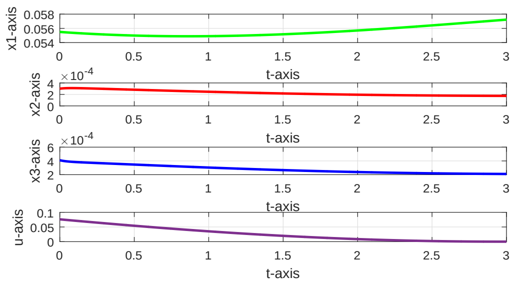


Figure 15: The optimal curves of the problem in Example 4.4 (new6_5)

4 Numerical results using FBSM and new methods

Example 4.1 *Consider the following optimal control problem:*

$$\min_u \int_0^1 3x(t)^2 + u(t)^2 dt$$
$$st. \quad x'(t) = x(t) + u(t), \qquad x(0) = 1.$$

Analytical solutions which are as follows[8]: $u^*(t) = \frac{3e^{-4}}{3e^{-4}+1}e^{2t} - \frac{3}{3e^{-4}+1}e^{-2t}$, $x^*(t) = \frac{3e^{-4}}{3e^{-4}+1}e^{2t} + \frac{1}{3e^{-4}+1}e^{-2t}$. The state variable at the end point is $x = 0.51314537669$. And variable control endpoint is equal to zero. Matlab implementation of the three methods of Example 4.1 was determined as follows. The results are shown in Figures 4, 5 and in Tables 1-4. These results show that the new methods are more accurate than the Runge-Kutta of rank 4.

Figure 4 shows that the $New6\_4$ method is more accurate than the FBSM mothod based on Runge-Kutta of rank 4. ($FBSM\_Rk4$) Because of the large stability area of the $New6\_4$ shows better performance than the ($FBSM\_Rk4$) method for small h. Figure 5 shows that the $New6\_5$ method is more accurate than the FBSM mothod based on Runge-Kutta of rank 5. ($FBSM\_Rk5$) Because of the large stability area of the $New6\_5$ shows better performance than the ($FBSM\_Rk5$) method for small h.

Tables 1, 2 results shows all over time of t in $[0, 1]$, the new methods the better approximation for the control variable values. it also shows that the new methods for each h has better performance and this is due to the high stability of the new methods.

The results of Tabels 3, 4 shows that the new methods for different h is more accurate than the $FBSM\_Rk4$ and $FBSM\_Rk5$ methods for state variable values.

Example 4.2 *Consider the following optimal control problem with a payoff term:*

$$\min_u x(T) + \int_0^T u(t)^2 dt$$
$$\text{st} \quad x'(t) = \alpha x(t) - u(t), \; x(0) = x_0 > 0$$

The analytical solution of this problem is as follows [8]:

$$H = u^2 + \lambda(\alpha x - u) \;, \quad \frac{\partial H}{\partial u} = 2u - \lambda = 0$$
$$at \;\; u^* \quad \Rightarrow u^* = \tfrac{\lambda}{2} \;,$$
$$\lambda' = -\frac{\partial H}{\partial x} = -\partial\lambda \Rightarrow \lambda = ce^{-\alpha t}, \lambda(T) = 1$$
$$\Rightarrow \lambda(t) = e^{\alpha(T-t)} \Rightarrow u^*(t) = \frac{e^{\alpha(T-t)}}{2},$$
$$x' = \alpha x - u = \alpha x - u = \alpha x - \frac{e^{\alpha(T-t)}}{2},$$
$$x(0) = x_0. \Rightarrow x^*(t) = x_0 e^{\alpha t} + e^{\alpha T}\frac{e^{-\alpha t} - e^{\alpha t}}{4\alpha}.$$

The numerical solution of this problem related to x(t), and u(t) are obtained and their results have been plotted in the following figures with $\alpha = -2$.
Figure 6 shows that the $New6\_4$ method is more accurate than the FBSM mothod based on Runge-Kutta of rank 4 .($FBSM\_Rk4$) Because of the large stability area of the $New6\_4$ shows better performance than the ($FBSM\_Rk4$) method for small h. Figure 7 shows that the $New6\_5$ method is more accurate than the FBSM mothod based on Runge-Kutta of rank 5 .($FBSM\_Rk5$) Because of the large stability area of the $New6\_5$ shows better performance than the ($FBSM\_Rk5$) method for small h.
Tables 5, 6 results shows all over time of t in $[0, 1]$, the new methods the better approximation for the control variable values. it also shows that the new methods for each h has better performance and this is due to the high stability of the new methods.
The results of Tabels 7, 8 shows that the new methods for different h is more accurate than the $FBSM\_Rk4$ and $FBSM\_Rk5$ methods for state variable values.

Example 4.3 *Consider the following optimal control problem for a fixed T:*

$$\begin{array}{l} \min_u \int_0^T (\int_0^t x(\eta)d\eta + u(t)^2)dt \\ st. \quad x'(t) = -x(t) + u(t), \quad x(0) = a. \end{array}$$

solution which is as follows (see [19]):
To convert the problem into standard form, we add another state to make the system two dimensional $x_1(t) := x(t)$ and $x_2(t) := (\int_0^t x(\eta)d\eta)$, and redefine $x(t) := [x_1(t), x_2(t)]^T$. Thus our new problem is

$$\min_u \int_0^T (x_2(t) + u(t)^2)dt$$
$$st, x_1'(t) = -x(t) + u(t),$$
$$x_2'(t) = x_1(t),$$
$$x(0) = [a, 0]^T$$

Analytical solution is as follows:

$$H = (x_2 + u^2) + \lambda_1(-x_1 + u) + \lambda_2 x_1$$
$$\frac{\partial H}{\partial u} = 2u + \lambda_1 = 0$$
$$at \quad u^*$$
$$\Rightarrow u^* = -\frac{1}{2}\lambda_1,$$
$$\lambda_1(T) = \lambda_2(T) = 0 \Rightarrow \lambda_1{}' = -\frac{\partial H}{\partial x_1}$$
$$= \lambda_1 - \lambda_2, \lambda_2' = -\frac{\partial H}{\partial x_2} = -1$$

$\Rightarrow \lambda_1(t) = -(t-T) - 1 + e(t-T) \;, u^*(t) = -\frac{1}{2}\lambda_1(t) = \frac{1}{2}(1 + t - T - e(t-T))$,
Matlab implementation of the three methods of Example 4.2 was determined as follows. The results are shown in Figures 4, 5 and in Tables 5.
Figure 8, 9 shows that the $New6\_4$ method is more accurate than the FBSM mothod based on Runge-Kutta of rank 4 .($FBSM\_Rk4$) Because of the large stability area of the $New6\_4$ shows better performance than the ($FBSM\_Rk4$) method for small h. Figure 10, 11 shows that the $New6\_5$ method is more accurate than the FBSM mothod based on Runge-Kutta of rank 5

.($FBSM\_Rk5$) Because of the large stability area of the $New6\_5$ shows better performance than the ($FBSM\_Rk5$) method for small h. Numerical results of table 5 show that the h increases both new method and FBSM-Rk4 more accurate, but the precision of the new method is three more dijits than the FBSM-Rk4 method.
Tables 9, 10 results shows all over time of t in $[0, 1]$, the new methods the better approximation for the control variable values. it also shows that the new methods for each h has better performance and this is due to the high stability of the new methods.

Example 4.4 *Consider the following optimal control problem [5, 21],*

$$\min_u \int_0^3 (Ax_3 + u^2)^2 dt$$
$$st. \quad \begin{aligned} x_1{}' &= b - b(px_2 + qx_2) - bx_1 - \beta x_1 x_3 - ux_1,, \\ x_2{}' &= bpx_2 + \beta x_1 x_3 - (e+b)x_2, \\ x_3{}' &= ex_2 - (g+b)x_3, \\ x_4{}' &= b - bx_4. \end{aligned}$$

With initial conditions $x1(1) = 0.0555$, $x2(1) = 0.0003$, $x3(1) = 0.00041$, $x4(1) = 1$ and the parameters $b = 0.012$, $p = 0.65$, $q = 0.65$, $\beta = 527.59$, $e = 36.5$, $g = 30.417$ and $A = 100$.

It is not easy to solve this, analytically and it is necessary to use numerical method.
The results obtained in the Table 11 and figures 11-14 shows that the new methods compete with Runge-Kutta methods in stiff and several variable problems.

5 Conclusion

In this work, one hybrid methods of orders 6 was presented and then, Runge - Kutta methods of order 4 and 5 were used as predictor scheme to gain whole method of the same orders. In section 2, order of truncation error was investigated for the explicit hybrid based on Rune - Kutta method. Then, stability analysis of the method was discussed which shows that the stability region of the new method is wider compared to explicit Runge - Kutta method of order 4 and 5. Finally, four examples of optimal control problems are solved using Matlab, FBSM scheme and presented method. Numerical results to solve the examples presented by Tables 1- 12 and therefore one can conclude that hybrid methods, have a good performance in getting small end errors in solving optimal control problems numerically. Example 4 also shows that this method has good implementation to solve OCP stiff problems.

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