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Research Article

On the k-ary Moment Map

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Abstract

The moment map is a mathematical expression of the concept of the conservation associated with the symmetries of a Hamiltonian system. By considering topological properties of the moment map we are lead to the definition of the abstract moment map. This map is defined from G-manifold M to dual Lie algebra of G. We will interested study maps from G-manifold M to spaces that are more general than dual Lie algebra of G. These maps help us to reduce the dimension of a manifold much more.

Keywords : Lie algebra; Hamiltonian system; k-ary moment map; moment map; reduction.

1 Introduction

THe mathematical formulation of the fundamental laws of motion, whose foundation was laid out by Galileo and Newton, is known as classical mechanics. The classical or Newtonian mechanics mainly refers to the implications of the motion laws which highly dependents on the Euclidean geometry. In contrast, Hamiltonian mechanics is a new viewpoint that allows us to describe the motion equation, regardless of the Euclidean geometry of the system. The generalization of the Hamiltonian mechanics, which allows taking manifolds as our phase space, is possible by using symplectic geometry. In other words, the phase space of a Hamiltonian system is a symplectic space. This even dimensional geometry is defined by a closed and non-degenerate 2-differential form whose its first stages has been formed more than two centuries ago by an introduction of lagrangian generalized coordinates.

In this field, symmetries on the phase space of the Hamiltonian systems are important. The symmetry of the Hamiltonian system is describable by the action of a Lie group on the phase space which leads to conservation laws on the system. The connection between conservation laws and symmetries was formalized by Noether. The conserved quantities, have been used by the founders of classical mechanics to eliminate degrees of freedom in the particular systems under investigation. These procedures called reduction theory.

The moment map is a mathematical expression of the concept of the conservation associated with the symmetries of a Hamiltonian system and is a fundamental tool for the study of symplectic reduction of symplectic manifolds. In fact,

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given the symplectic action of a Lie group on a symplectic manifold that has a moment map, one quotient a level set of the moment map by the proper action of a isotropy group to form a new symplectic manifold. The moment map has been interoduced by Kostant [7] and Souriau [13]. Kostant introduced the moment map to generalize a theorem. Souriau introduced the moment map in his article. He also studied its physical properties moment map and finally got its formal definition and its name.

By considering topological properties of the moment map we are lead to the definition of the abstract moment map. This definition introduced by Viktor Ginzburg, Victor Guillemin, and Yael Karshon [6]. Abstract moment map is a generalization of moment map on symplectic manifold when there is not symplectic structure, but the relation between the map and the G-action is kept. This map is an equivariant mapping from G-manifold M to $\mathfrak{g}^*$. In this paper we will interested study maps from G-manifold M to spaces that are more general than $\mathfrak{g}^*$ such that abstract moment maps are among examples of this maps. These maps help us to reduce the dimension of a manifold much more.

2 Preliminaries

Let us now recall some definitions and proposition. In what follows, we will suppose that every our manifolds, actions and maps to be smooth. All over the text, M will be a smooth manifold and G a Lie group acting on M. The Lie algebra of G is $\mathfrak{g}$. We will denote the quotient space by M/G. We will denote the fixed point set of the action by M^G where is submanifolds of M. Hereafter we outline some preliminary definitions and proposition of the Lie group actions that are useful in the subsequent discussion [12, 14].

Definition 2.1 *Let us be given a action ψ of G on manifold M. For all $X \in \mathfrak{g}$, we define one-parameter group Ψ of diffeomorphisms on M as $\Psi(t,p) = \psi_{exp(tX)}(p)$. The one-parameter group Ψ is a complete flow on M. The infinitesimal generator corresponding to this action is denoted by X^M:*

$$X^M(q) = \frac{d}{dt}(\psi_{exp(tX)}(q))\Big|_{t=0}.$$

Definition 2.2 *For any $g \in G$,a Lie group G acts on itself by inner automorphisms*

$$C_g := L_g \circ R_{g^{-1}} : G \to G \ : \ x \mapsto gxg^{-1},$$

where R_g right translation by g. We called differentiable map C_g at identity element adjoint representation and denoted by Ad_g, i.e. $Ad_g := (C_{g*})_e : \mathfrak{g} \longrightarrow \mathfrak{g}$. Now define the following map

$$\begin{aligned} Ad : G &\longrightarrow \mathrm{Aut}(\mathfrak{g}) \\ g &\longmapsto Ad_g \end{aligned}.$$

The map Ad is called adjoint action Lie group on Lie algebra.

Proposition 2.1 *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any $X \in \mathfrak{g}$ and $g \in G$:*

$$exp(Ad_g(X)) = g\ exp(X)\ g^{-1}.$$

Proposition 2.2 *Let a lie group action ψ of G on M. For every $g \in G$ and $X \in \mathfrak{g}$:*

$$(Ad_g(X))^M = \psi^*_{g^{-1}} X^M.$$

Proof. Let $q \in M$, according to Definition 2.1 and Proposition 2.1, we have

$$\begin{aligned}
(Ad_g(X))^M &= \frac{d}{dt}\psi_{exp(t\ Ad_g(X))}(q)\Big|_{t=0} \\
&= \frac{d}{dt}\psi_{(g\ exp(tX)\ g^{-1})}(q)\Big|_{t=0} \\
&= \frac{d}{dt}\psi_g \circ \psi_{exp(tX)}(\psi_{g^{-1}}(q))\Big|_{t=0} \\
&= (\psi_{g*})_{\psi_{g^{-1}}(q)} \frac{d}{dt}\psi_{exp(tX)}(\psi_{g^{-1}}(q))\Big|_{t=0} \\
&= (\psi_{g*})_{\psi_{g^{-1}}(q)} X^M(\psi_{g^{-1}}(q)) \\
&= \psi_{g*} X^M = \psi^*_{g^{-1}} X^M.
\end{aligned}$$

Definition 2.3 *Assuming that $\mathcal{T}^k(\mathfrak{g}^*)$ is the vector space of all k-covariant tensor on $\mathfrak{g}$. The coadjoint action of G on $\mathcal{T}^k(\mathfrak{g}^*)$, for $k \in \mathbb{N}$, is defined as follows*

$$\begin{aligned} Ad^{\#} : G &\longrightarrow \mathrm{Aut}(\mathcal{T}^k(\mathfrak{g}^*)) \\ g &\longmapsto \left\{ \begin{aligned} Ad^{\#}_g : \mathcal{T}^k(\mathfrak{g}^*) &\longrightarrow \mathcal{T}^k(\mathfrak{g}^*) \\ \xi &\longmapsto Ad^{\#}_g(\xi) \end{aligned} \right. \end{aligned},$$

where for any

$$X = (X_1, ..., X_k) \in \underbrace{\mathfrak{g} \times ... \times \mathfrak{g}}_{k \text{ times}},$$

$Ad_g^{\#}(\xi)_{(X)}$ *defined by*

$$\begin{aligned} Ad_g^{\#}(\xi)_{(X)} &:= (C^*_{g^{-1}})_e(\xi)_{(X)} \\ &= \xi((C_{g^{-1}_*})_e X) \\ &= \xi(Ad_{g^{-1}}(X)). \end{aligned}$$

The map $Ad^{\#}$ is called coadjoint action Lie group on dual of Lie algebra.

3 k-ary moment map

The moment map is map of G-manifold M to $\mathfrak{g}^*$. In this section, we define maps from G-manifold M to spaces that are more general than $\mathfrak{g}^*$.

Definition 3.1 ***(k-ary moment map)*** *Let a action $\psi : G \times M \longrightarrow M$. A k-ary moment map on M is a map $\mu : M \to \mathcal{T}^k(\mathfrak{g}^*)$ with the following two properties*

1. *μ is G-equivariant, i.e. for any $g \in G$ the following diagram commutes.*

$$3pcM[d]_{\psi_g}[r]^{\mu}\mathcal{T}^k(\mathfrak{g}^*)[d]^{Ad_g^{\#}}$$
$$M[r]^{\mu}\mathcal{T}^k(\mathfrak{g}^*)$$

2. *For any closed subgroup H of G, the map $\mu^H : M \longrightarrow T^k(\mathfrak{h}^*)$ defined by*

$$\mu^H := i^* \circ \mu : M \longrightarrow \mathcal{T}^k(\mathfrak{h}^*),$$

is locally constant on M^H.

Remark 3.1 *For $k = 1$, Definition 3.1 is equivalent to definition Yael Karshon of abstract moment map in [6].*

Lemma 3.1 *If for any Lie algebra element*

$$X = (X_1, ..., X_k) \in \underbrace{\mathfrak{g} \times ... \times \mathfrak{g}}_{k \text{ times}},$$

the function μ^X where for each $q \in M$ defined by $\mu^X(q) = \mu(q)(X_1, ..., X_k)$, is locally constant on the set of zeros of the corresponding vector field $(X_1^M, ..., X_k^M)$, then the second requirement in Definition 3.1 to hold.

Proof. For any Lie algebra element

$$X = (X_1, ..., X_k) \in \underbrace{\mathfrak{g} \times ... \times \mathfrak{g}}_{k \text{ times}},$$

let closed one-parameter subgroups H_i as $\{exp(tX_i)|\ t \in \mathbb{R}\}$ in G, corresponding with left invariant vector field $X_1, ..., X_k$ respectively. If ψ is action of G on M, then

$$\begin{aligned} &(X_1^M(q), ..., X_k^M(q)) = \\ &\left(\frac{d}{dt}(\psi_{exp(tX_1)}(q))\Big|_{t=0}, ..., \frac{d}{dt}(\psi_{exp(tX_k)}(q))\Big|_{t=0} \right) \\ &= \left(\frac{d}{dt}\psi|_{H_1}(q)\Big|_{t=0}, ..., \frac{d}{dt}\psi|_{H_k}(q)\Big|_{t=0} \right). \end{aligned}$$

So the zeros of the corresponding vector field $(X_1^M, ..., X_k^M)$ equals to fixed points $\psi|_{H_1}(q), ..., \psi|_{H_k}(q)$. therefore enough to take $H = H_1 \cap ... \cap H_k$.

Example 3.1 *Let (M, ω) be a Hamiltonian G-manifold and let $\mu : M \longrightarrow \mathfrak{g}^*$ be a moment map. Recall, that this means is G-equivariant with respect to the given action ψ of G on M and the coadjoint action $Ad^{\#}$ of G on $\mathfrak{g}^*$, and for any $X \in \mathfrak{g}$ satisfies Hamilton's equation $\iota_{X^M}\omega = d\mu^X$. Then μ is a 1-ary moment map. Indeed, the function μ^X is locally constant on the set of zeros of the corresponding vector field X^M, because $d\mu^X = \iota_{X^M}\omega = 0$.*

Definition 3.2 *Suppose ψ is action of G on manifold M. For each $g \in G$, the map $\psi_g : M \longrightarrow M$ is diffeomorphism of M to M. The Riemannian metric $< \cdot, \cdot >$ on G-manifold M is invariant, whenever for any $g \in G$, $q \in M$ and $X, Y \in TM$:*

$$< X, Y >_q = < \psi_{g_*}(X), \psi_{g_*}(Y) >_{\psi_g(q)} .$$

Example 3.2 *(See [4]). If X^M is a vector field on M that generates a circle action, then the function*

$$\mu(q) = < X^M(q), X^M(q) >,$$

where $< \cdot, \cdot >$ is any invariant Riemannian metric, is a 1-ary moment map.

Theorem 3.1 *Let a action ψ of G on manifold M. For any invariant Riemannian metric $< \cdot, \cdot >$ on M, the following map is a 2-ary moment map.*

$$\begin{aligned} \mu : & M \longrightarrow \mathcal{T}^2(\mathfrak{g}^2) \\ & q \longmapsto \begin{cases} \mu(q) : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R} \\ \mu(q)(X,Y) =< X^M, Y^M >_q \end{cases} . \end{aligned}$$

Proof. First, we show that the map μ is G-equivariant. Suppose $(X,Y) \in \mathfrak{g} \times \mathfrak{g}$, $g \in G$ and $q \in M$. We have

$$\begin{aligned} & Ad_g^{\#} \circ \mu(q)(X,Y) = \\ & \mu(q)\Big(Ad_{g^{-1}}(X), Ad_{g^{-1}}(Y)\Big) = \\ & < \Big(Ad_{g^{-1}}(X)\Big)^M, \Big(Ad_{g^{-1}}(Y)\Big)^M >_q . \end{aligned} \tag{3.1}$$

According to the Proposition 2.2, for each $Z \in \mathfrak{g}$, $(Ad_{g^{-1}}(Z))^M = \psi_g^* Z^M$. Therefore (3.1) becomes

$$\begin{aligned} & < \psi_g^* X^M, \psi_g^* Y^M >_q = \\ & < \psi_{g^{-1}}{}_* X^M, \psi_{g^{-1}}{}_* X^M >_q \\ & =< \psi_{g^{-1}}{}_* X^M, \psi_{g^{-1}}{}_* Y^M >_{\psi_{g^{-1}}(\psi_g(q))} . \end{aligned} \tag{3.2}$$

That because $< \cdot, \cdot >$ is invariant Riemannian metric on M, then (3.2) becomes

$$\begin{aligned} < X^M, Y^M >_{\psi_g(q)} &= \mu(\psi_g(q))(X,Y) \\ &= \mu \circ \psi_g(q)(X,Y). \end{aligned}$$

For the second requirement in Definition 2-ary moment map, we use Lemma 4.1. Let (X^M, Y^M) a vector field corresponding $(X,Y) \in \mathfrak{g} \times \mathfrak{g}$. If $q \in M$ satisfying $(X^M(q), Y^M(q)) = (0,0)$, then the map

$$\begin{aligned} \mu^{(X,Y)}(q) &=< X^M, Y^M >_q \\ &=< X^M(q), Y^M(q) >= 0, \end{aligned}$$

is constant. Therefore the μ is a 2-ary moment map.

Example 3.3 *Let $\psi : SO(3) \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ induced by $\psi(A,q) = Aq$. Euclidean inner product $< \cdot, \cdot >$ on space $\mathbb{R}^3$ is invariant Riemannian metric under the angular action ψ. Notice that for each $X \in \mathfrak{so}(3)$, there is linear isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$.*

$$X = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \longleftrightarrow \hat{X} \in \mathbb{R}^3.$$

Using this isomorphism, the multiplication $X \in \mathfrak{so}(3)$ in $q \in \mathbb{R}^3$ defined as $Xq = \hat{X} \times q$, where $\times$ is the cross product. We find that the infinitesimal generator corresponding to $X \in \mathfrak{so}(3)$ is

$$\begin{aligned} X^{\mathbb{R}^3}(q) &= \frac{d}{dt}(\psi_{exp(tX)}(q))\Big|_{t=0} \\ &= \frac{d}{dt} exp(tX)q\Big|_{t=0} = Xq = \hat{X} \times q. \end{aligned}$$

Now the map μ is defined as follows is a 2-ary moment map

$$\begin{aligned} \mu : & \mathbb{R}^3 \longrightarrow \mathcal{T}^2(\mathfrak{so}(3)^*) \\ & q \longmapsto \begin{cases} \mu(q) : \mathfrak{so}(3) \times \mathfrak{so}(3) \longrightarrow \mathbb{R} \\ \mu(q)(X,Y) =< \hat{X} \times q, \hat{Y} \times q > \end{cases} . \end{aligned}$$

4 Properties of the k-ary moment map

In this section, we study several propositions and theorems of functions and examine their general properties.

Proposition 4.1 *Suppose M_1 and M_2 are G-manifolds with actions φ and ψ respectively. If $f : M_1 \to M_2$ is G-equivariant map and $\mu : M_2 \longrightarrow \mathcal{T}^k(\mathfrak{g}^*)$ is a k-ary moment map on M_2, then $\mu \circ f$ is a k-ary moment map on M_1.*

Proof. First, we show that for any $g \in G$ the following diagram is commutative.

$$\begin{array}{ll} 3pcM_1[d]_{\varphi_g}[r]^f M_2[d]_{\psi_g}[r]^\mu & \mathcal{T}^k(\mathfrak{g}^*)[d]^{Ad_g^{\#}} \\ M_1[r]^f M_2[r]^\mu & \mathcal{T}^k(\mathfrak{g}^*) \end{array}$$

Since μ is a k-ary moment map and f is G-equivariant map, then $\mu \circ \psi_g = Ad_g^{\#} \circ \mu$ and $f \circ \varphi_g = \psi_g \circ f$. Therefore,

$$\begin{aligned} (\mu \circ f) \circ \varphi_g &= \mu(f \circ \varphi_g) = \mu(\psi_g \circ f) \\ &= (\mu \circ \psi_g) \circ f = (Ad_g^{\#} \circ \mu) \circ f \\ &= Ad_g^{\#} \circ (\mu \circ f). \end{aligned}$$

Now, we show that for any closed subgroup H of G, the $(\mu \circ f)^H : M_1 \to T^k(\mathfrak{h}^*)$ is locally constant on the submanifold M_1^H. Assuming that $q \in M_1^H$, then for any $g \in G$, $\varphi_g(q) = q$ and $f(\varphi_g(q)) = f(q)$. Since f is G-equivariant, then for any $g \in G$:

$$\psi_g(f(q)) = f(\varphi_g(q)) = f(q),$$

therefore $f(q) \in M_2^H$. Consequently the map $(\mu \circ f)^H(q) = \mu^H(f(q))$ is constant.

Example 4.1 *Let a lie group action ψ of G on M. For $q \in M$, consider the isotropy group $G_q = \{g \in G \mid \psi(g,q) = q\}$. Define the action $\varphi : G \times G/G_q \longrightarrow G/G_q$, induced by $\varphi(g, hG_q) = ghG_q$. Then the map*

$$\begin{cases} f : G/G_q \longrightarrow M \\ f(hG_q) = \psi(h,q) \end{cases}$$

is G-equivariant, because for $q \in M$:

$$\begin{aligned} f \circ \varphi_g(hG_q) &= f(ghG_q) = \psi(gh,q) \\ &= \psi_g(\psi(h,q)) = \psi_g \circ f(hG_q). \end{aligned}$$

Now, if $\mu : M \longrightarrow \mathcal{T}^k(\mathfrak{g}^*)$ is a k-ary moment map on M, then according to Proposition 4.1, $\mu \circ f$ is a k-ary moment map on G/G_q.

Theorem 4.1 *Assuming that M_1, M_2 are G-manifolds with actions φ and ψ. Define the action $\varphi \times \psi$ of G on a product manifold $M_1 \times M_2$ as follows*

$$\begin{cases} \varphi \times \psi : G \times (M_1 \times M_2) \longrightarrow (M_1 \times M_2) \\ (\varphi \times \psi)(g,(q_1,q_2)) = (\varphi(g,q_1), \psi(g,q_2)) \end{cases}.$$

If for $i = 1,2$, μ_i is k-ary moment maps on M_i, then G-manifold $M_1 \times M_2$ has k-ary moment map $\mu_1 \oplus \mu_2$ where defined by $\mu_1 \oplus \mu_2(q_1,q_2) = \mu_1(q_1) + \mu_2(q_2)$.

Proof. We prove $\mu_1 \oplus \mu_2$ satisfies two properties k-ary moment map. First, we show that for any $g \in G$ the following diagram is commutative.

$$3pcM_1 \times M_2[d]_{(\varphi\times\psi)_g}[r]^{\mu_1\oplus\mu_2}\mathcal{T}^k(\mathfrak{g}^*)[d]^{Ad_g^\#} \\ M_1 \times M_2[r]^{\mu_1\oplus\mu_2}\mathcal{T}^k(\mathfrak{g}^*)$$

For any $g \in G$, $(q_1,q_2) \in M_1 \times M_2$ and

$$X = (X_1, ..., X_k) \in \underbrace{\mathfrak{g} \times ... \times \mathfrak{g}}_{k \text{ times}},$$

we have

$$\begin{aligned} Ad_g^\# \circ & (\mu_1 \oplus \mu_2)^X(q_1,q_2) \\ &= Ad_g^\#(\mu_1^X(q_1) + \mu_2^X(q_2)) \\ &= Ad_g^\# \circ \mu_1^X(q_1) + Ad_g^\# \circ \mu_2^X(q_2). \end{aligned} \tag{4.3}$$

Since for $i = 1,2$, μ_i is k-ary moment maps, then $Ad_g^\# \circ \mu_i^X(q_i) = \mu_i^X \circ \varphi_g(q_i)$. Therefore (4.3) becomes

$$\begin{aligned} \mu_1^X \circ \varphi_g(q_1) &+ \mu_2^X \circ \psi_g(q_2) \\ &= (\mu_1^X \oplus \mu_2^X)(\varphi_g(q_1), \psi_g(q_2)) \\ &= (\mu_1 \oplus \mu_2)^X \circ (\varphi \times \psi)_g(q_1,q_2). \end{aligned}$$

In the following, we show that for any closed Lie subgroup H of G, the map

$$(\mu_1 \oplus \mu_2)^H : (M_1 \times M_2) \longrightarrow \mathcal{T}^k(\mathfrak{h}^*)$$

is locally constant on $(M_1 \times M_2)^H$. Assuming that $(q_1,q_2) \in (M_1 \times M_2)^H$. Since for any $g \in H$

$$\begin{aligned} (\varphi \times \psi)_g(q_1,q_2) &= (\varphi_g(q_1), \psi_g(q_2)) = (q_1,q_2) \\ &\Rightarrow \quad \varphi_g(q_i) = q_i\ ;\ (i = 1,2), \end{aligned}$$

then $q_i \in M_i^H$. Since μ_i^H is locally constant on M_i^H, thus the

$$(\mu_1 \oplus \mu_2)^H(q_1,q_2) = \mu_1^H(q_1) + \mu_2^H(q_2)$$

is constant.

Theorem 4.2 *Let G_1 and G_2 be finite dimension Lie groups. Consider actions $\varphi : G_1 \times M_1 \to M_1$ and $\psi : G_2 \times M_2 \to M_2$. Define the map F as follows*

$$F : ((g_1,g_2),(q_1,q_2)) \longmapsto ((g_1,q_1),(g_2,q_2))$$

Using the map F and actions φ and ψ, define the action Ψ of Lie group $G_1 \times G_2$ on a manifold $M_1 \times M_2$ as $\Psi := (\varphi \times \psi) \circ F$. If for $i = 1,2$, μ_i is k-ary moment maps on M_i, then $G_1 \times G_2$-manifold $M_1 \times M_2$ has k-ary moment map.

Proof. Since G_1, G_2 are finite dimension Lie groups, then Lie algebra of $G_1 \times G_2$ is $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ and there is injective linear transformation

$$T : \mathcal{T}^k(\mathfrak{g}_1^*) \oplus \mathcal{T}^k(\mathfrak{g}_2^*) \longrightarrow \mathcal{T}^k((\mathfrak{g}_1 \oplus \mathfrak{g}_2)^*).$$

Assuming that $\mu_1 : M_1 \longrightarrow \mathcal{T}^k(\mathfrak{g}_1^*)$ and $\mu_2 : M_2 \longrightarrow \mathcal{T}^k(\mathfrak{g}_2^*)$ are k-ary moment maps, thus we prove map μ defined by

$$\mu := M_1 \times M_2 \xrightarrow{\mu_1 \times \mu_2} \mathcal{T}^k(\mathfrak{g}_1^*) \oplus \mathcal{T}^k(\mathfrak{g}_2^*) \\ \xrightarrow{T} \mathcal{T}^k((\mathfrak{g}_1 \oplus \mathfrak{g}_2)^*),$$

is k-ary moment map. It is sufficient to prove that μ satisfies two properties of k-ary moment map.
1) For any $(g_1, g_2) \in G_1 \times G_2$ the following diagram is commutative.

$$3pcM_1 \times M_2[d]_{\Psi_{(g_1,g_2)}}[r]^{\mu}\mathcal{T}^k((\mathfrak{g}_1 \oplus \mathfrak{g}_2)^*)[d]^{Ad^{\#}_{g_1} \times Ad^{\#}_{g_2}} \\ M_1 \times M_2[r]^{\mu}\mathcal{T}^k((\mathfrak{g}_1 \oplus \mathfrak{g}_2)^*)$$

For any $X \in \underbrace{(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \times ... \times (\mathfrak{g}_1 \oplus \mathfrak{g}_2)}_{k \text{ times}}$ and $q = (q_1, q_2) \in M_1 \times M_2$, we have

$$\begin{aligned} &\mu \circ \Psi_{(g_1,g_2)}(q)(X) \\ &= \mu \circ ((\varphi \times \psi) \circ F(q_1, q_2))(X) \\ &= \mu(\varphi_{g_1}(q_1), \psi_{g_2}(q_2))(X) \\ &= T \circ (\mu_1 \times \mu_2)(\varphi_{g_1}(q_1), \psi_{g_2}(q_2))(X) \\ &= T(\mu_1 \circ \varphi_{g_1}(q_1), \mu_2 \circ \psi_{g_2}(q_2))(X). \end{aligned} \tag{4.4}$$

Since for $i = 1, 2$, μ_i is k-ary moment maps, then $Ad^{\#}_{g_i} \circ \mu_i = \mu_i \circ \varphi_{g_i}$. Therefore (4.4) becomes

$$\begin{aligned} &T(Ad^{\#}_{g_1} \circ \mu_1(q_1), Ad^{\#}_{g_2} \circ \mu_2(q_2))(X) \\ &= T(\mu_1(q_1), \mu_2(q_2))((Ad_{g_1^{-1}} \times Ad_{g_2^{-1}})(X)) \\ &= T(\mu_1 \times \mu_2)(q)((Ad_{g_1^{-1}} \times Ad_{g_2^{-1}})(X)) \\ &= (Ad^{\#}_{g_1} \times Ad^{\#}_{g_2}) \circ (T \circ (\mu_1 \times \mu_2))(q)(X) \\ &= (Ad^{\#}_{g_1} \times Ad^{\#}_{g_2}) \circ \mu(q)(X). \end{aligned}$$

2) For any closed Lie subgroup $H \times K$ of Lie group $G_1 \times G_2$, the map

$$\mu^{H \times K} : (M_1 \times M_2) \longrightarrow \mathcal{T}^k((\mathfrak{h} \oplus \mathfrak{k})^*)$$

is locally constant on the $(M_1 \times M_2)^{H \times K}$. Suppose that $(q_1, q_2) \in (M_1 \times M_2)^{H \times K}$. Since for any $(g_1, g_2) \in H \times K$, we see

$$\begin{aligned} \Psi_{(g_1,g_2)}(q_1, q_2) &= (\varphi \times \psi) \circ F((g_1, g_2), (q_1, q_2)) \\ &= (\varphi_g(q_1), \psi_g(q_2)) = (q_1, q_2), \end{aligned}$$

then $q_1 \in M_1^H$, $q_2 \in M_2^K$. Since μ_1^H, μ_2^K are locally constant on the M_1^H and M_2^K respectively and T is a injective linear transformation, thus the

$$\begin{aligned} \mu^{H \times K}(q_1, q_2) &= T \circ (\mu_1 \times \mu_2)^{H \times K}(q_1, q_2) \\ &= T \circ (\mu_1^H \times \mu_2^K)(q_1, q_2) \\ &= T(\mu_1^H(q_1), \mu_2^K(q_2)), \end{aligned}$$

is constant.

Definition 4.1 *(See [1]). Assuming that G is a Lie group and $\mathfrak{g}$ its Lie algebra. Let $\Lambda^k(\mathfrak{g}^*) \subseteq \mathcal{T}^k(\mathfrak{g}^*)$ set of all alternating k-linear maps $c : \underbrace{\mathfrak{g} \times ... \times \mathfrak{g}}_{k \text{ times}} \longrightarrow \mathbb{R}$. Define a linear operator $\partial_G : \Lambda^k(\mathfrak{g}^*) \to \Lambda^{k+1}(\mathfrak{g}^*)$ by*

$$\partial_G(c)(X_0, X_1, ..., X_k) = \\ \sum_{i<j} (-1)^{i+j} c([X_i, X_j], X_0, .., \hat{X}_i, .., \hat{X}_j, .., X_k).$$

∂_G is a differential operator in $\bigoplus_{k=0}^{\infty} \Lambda^k(\mathfrak{g}^*)$ and satisfying $\partial_G^2 = 0$.

Proposition 4.2 *Assuming that M is G-manifold. If $\mu : M \to \Lambda^k(\mathfrak{g}^*)$ is k-ary moment map, then M has $k+1$-ary moment map $\bar{\mu} : M \to \Lambda^{k+1}(\mathfrak{g}^*)$ so that*

$$3pcM[dr]_{\bar{\mu} =: \partial_G \circ \mu}[r]^{\mu}\Lambda^k(\mathfrak{g}^*)[d]^{\partial_G} \\ \Lambda^{k+1}(\mathfrak{g}^*)$$

Proof. $\bar{\mu}$ is G-equivariant with respect to the given action ψ on M and the coadjoint action. Since ∂_G is a differential operator and $Ad^{\#}_g = (C^*_{g^{-1}})_e$, for any $g \in G$, $X = (X_1, ..., X_k) \in \underbrace{\mathfrak{g} \times ... \times \mathfrak{g}}_{k \text{ times}}$ and $q \in M$, we have

$$\begin{aligned} Ad^{\#}_g \circ \bar{\mu}(q)(X) &= Ad^{\#}_g \circ (\partial_G \circ \mu(q))(X) \\ &= (Ad^{\#}_g \circ \partial_G) \circ \mu(q)(X) \\ &= \partial_G \circ (Ad^{\#}_g \circ \mu(q)(X)). \end{aligned} \tag{4.5}$$

Since μ is G-equivariant then (4.5) becomes

$$\begin{aligned} \partial_G \circ (\mu \circ \psi_g(q)(X)) &= (\partial_G \circ \mu) \circ \psi_g(q)(X) \\ &= \bar{\mu} \circ \psi_g(q)(X). \end{aligned}$$

For second properties of k-ary moment map, it is sufficient to prove that for any closed Lie subgroup H and $q \in M^H$, the $\bar{\mu}^H(q)$ is constant.

Since $q \in M^H$ then $\mu^H(q)$ is constant map in $\Lambda^k(\mathfrak{g}^*)$. Therefore $\partial_G(\mu^H(q)) = 0$ and consequently

$$\bar{\mu}^H(q) = (\partial_G \circ \mu)^H(q) = \partial_G \circ \mu^H(q) = 0,$$

thus $\bar{\mu}^H(q)$ is constant map.

5 Reduction and applications

The reduction theory is an old subject, going back to the early roots of mechanics. Methods of this theory represent a synthesis of various techniques of elimination of variables from classical mechanics that are based on the existence of conserved quantities. Early examples are the reduction to the center of the mass frame in the n-body problem. The modern form of symplectic reduction theory begins with the works of Meyer [10], Marsden and Weinstein [8]. In this abstract formulation, we have a symplectic manifold M with a moment map μ for a symplectic G-action. If for regular value ξ of $\mathfrak{g}^*$, the G_ξ acts freely and properly on the level manifold $\mu^{-1}(\xi)$. Thus the symplectic reduced manifold defined to be the quotient manifold $M_\xi := \mu^{-1}(\xi)/G_\xi$. Therefore if G of dimension r, acting on symplectic manifold M, then the dimension of the symplectic reduced manifold may be reduced by $2r$.

The symplectic reduction was first generalized to Pre-symplectic reduction for the manifolds that have closed but not non-degenerate two-forms [2]. The Yael Karshon et al. [4] generalized this procedure to an abstract moment map, while keeping track of various additional structures. In this section, we generalize the concept of reduction for any manifold that is equipped with k-ary moment map and reduced the dimension of a manifold by much more than $2r$. We need short mathematical backgrounds on free and proper actions and some facts about equivariant cohomology.

Definition 5.1 *(See [9]). The action ψ of a G on a manifold M is called a free if, for all $g \in G$ and $q \in M$, $\psi(g, q) = q$ implies $g = e$. Also ψ called proper when the map*

$$\Psi : G \times M \longrightarrow M \times M \;:\; (g, q) \mapsto (\psi(g, q), q)$$

is a proper (i.e., inverse images of compact sets are compact).

Theorem 5.1 *(See [11]). If a Lie group G acts freely and properly on M, then there is a unique structure of a manifold on M/G and the orbit map $\pi : M \longrightarrow M/G$ is a principal G-bundle.*

Definition 5.2 *(See [5, 3]).* ***(Equivariant cohomology)*** *Let G be a compact Lie group acting on a manifold M. There exists a topological space E with a free G action whose E is contractible. The equivariant cohomology of G-manifold M for all $k \in \mathbb{N}$ is defined to be*

$$H^k_G(M) := H^k_{dR}(\frac{E \times M}{G}),$$

where the cartesian product of M with E is equipped with the diagonal action of G which is free. This definition is well defined and independent of the choice of the space E because $(E \times M)/G$ is unique up to homotopy equivalence.

Corollary 5.1 *If G acts freely on M, $H^k_G(M) = H^k_{dR}(M/G)$ for all $k \in \mathbb{N}$, since $(E \times M)/G$ is a fiber bundle over M/G with a contractible fiber E.*

Theorem 5.2 ***(Reduction for k-ary moment maps)*** *Assuming that M is G-manifold with k-ary moment map μ. If $\xi \in \mathcal{T}^k(\mathfrak{g}^*)$ is a regular value of μ and that the isotropy group G_ξ under the $Ad^\#$ action on $\mathcal{T}^k(\mathfrak{g}^*)$, acts freely and properly on $\mu^{-1}(\xi)$, than $M_\xi := \mu^{-1}(\xi)/G_\xi$ is a smooth manifold and*

$$\dim M_\xi = \dim M - (\dim G)^k - \dim G_\xi.$$

Also, the orbit map $\pi : \mu^{-1}(\xi) \longrightarrow M_\xi$ is a principal G-bundle and for all $k \in \mathbb{N}$

$$H^k_G(\mu^{-1}(\xi)) = H^k_{dR}(M_\xi).$$

Therefore for any equivariant cohomology class $[\omega]$ on M, there exists a de Rham cohomology class $[\omega_\xi]$ on M_ξ so that $\pi^*\omega_\xi = i^*_\xi\omega$, where $i_\xi : \mu^{-1}(\xi) \hookrightarrow M$ is the inclusion map.

Proof. It is clear from Theorem 5.1 and Corollary 5.1.

Corollary 5.2

1. *Considering all the conditions of Theorem 5.2, the condition necessary for the existence a manifold M_ξ is that dimensions M, G and G_ξ satisfies*

$$\sqrt[k]{\dim M - \dim G_\xi} > \dim G.$$

2. *In particular, if ξ is a regular value of $\mathcal{T}^k(\mathfrak{g}^*)$ and fixed by the coadjoint action on $\mathcal{T}^k(\mathfrak{g}^*)$, than $G_\xi = G$ and*

$$\dim M_\xi = \dim M - (\dim G)^k - \dim G.$$

Example 5.1 *(**N particles in** $\mathbb{R}^3$) To illustrate these identities, Consider a physical system of N particles moving in $\mathbb{R}^3$. the configuration of this system is described by a $q = (q_1, ..., q_N) \in \mathbb{R}^{3N}$, that*

$$q_i = ({q_i}^1, {q_i}^2, {q_i}^3) \in \mathbb{R}^3$$

describing the position of the ith particle. Let $SO(3)$ act on the configuration space $\mathbb{R}^{3n}$ by rotations. As in Example 3.3, the free and proper angular action $SO(3)$ on $\mathbb{R}^{3N}$ has 2-ary moment map μ defined by

$$\begin{aligned} \mu : &\mathbb{R}^{3N} \longrightarrow \mathcal{T}^2(\mathfrak{so}(3)^*) \\ &q \longmapsto \begin{cases} \mu(q) : \mathfrak{so}(3) \times \mathfrak{so}(3) \longrightarrow \mathbb{R} \\ \mu(q)(X, Y) = < \hat{X} \times q, \hat{Y} \times q > \end{cases} . \end{aligned}$$

Let regular value $0 \in \mathcal{T}^2(\mathfrak{so}(3)^*)$. Since the origin is fixed by the coadjoint action then $SO(3)_0 = SO(3)$. Let $SO(3)$ acts freely and properly on $\mathbb{R}^{3N}$ then $\mathbb{R}^{3N}_0 := \mu^{-1}(0)/SO(3)$ is a smooth manifold and

$$\dim \mathbb{R}^{3N}_0 = 3N - 3^2 - 3.$$

In this example, we could reduce the dimension of the configuration of N particles in $\mathbb{R}^3$ by using the concept k-ary moment maps. The equivariant cohomology of this reduced space equals the equivariant cohomology of the configuration space.

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