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Research Article

The Generalized Returns to Scale for Multiplicative Models in Data Envelopment Analysis

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Abstract

Conventional Data Envelopment Analysis (DEA) models define returns to scale (RTS) based on some local information about the proportional variation in outputs with respect to the proportional variation in inputs. Generalized RTS has been introduced to compute the rate of variation in outputs with respect to the variation in inputs up to the most productive scale size (MPSS) pattern. In this paper, we address the generalized RTS in the multiplicative models and we propose an algorithm to calculate the rate of variations in different intervals. We also demonstrate that the non-discretionary factors can be easily taken into account in the algorithm.

Keywords : Data envelopment analysis; Generalized returns to scale; Most productive scale size; Multiplicative models.

1 Introduction

Data envelopment analysis has been recognized as an efficient technique in evaluation of decision making units (DMUs) with multiple inputs and multiple outputs. Since its introduction by Charnes et al. [11], many DEA models have been proposed in literature (Banker et al. [6], Cooper et al. [14]). Returns to scale (RTS) is one of the important economic notion that has been investigated within the framework of DEA (Banker et al. [7]. It provides useful information about the optimal size of DMUs and whether the expansion or contraction of the units is beneficial.

Boussemart et al. [10] deriving a notion of alphaup-returns to scale, proposed a specification of strictly increasing and decreasing returns to scale in multi-output technologies. Allahyar and Rostami- Malkhalifeh [1] determine the type of the right and left RTS for each efficient DMU particularly. Yang et al. [30] analyzed the directional RTS of national biological institutes in China. Ding et al. [15] provided a radial measurements of efficiency and a procedure that is unaffected by multiple optima for estimating returns to scale for the production process possessing multi-components.

Sahoo et al. [26] attempting to resolve the shortcoming of [2], proposed a general non-radial DEA

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model to determine both the most productive scale size and the returns to scale characterizations of production units in the presence of negative data. Taleb et al. [28] discussed the RTS for an output oriented integer-valued DEA model. Mirbolouki and Allahyar [18] proposed a parameter free procedure for detecting the right and left RTS classification and to determine the value of the right and left RTS of efficient DMUs.

RTS studies mainly focus on the qualitative aspects, including whether RTS are "increasing," "decreasing," or "constant." The quantitive aspect of RTS is related to elasticity. It is the rate of the proportional variation of outputs with respect to the proportional variation of inputs in a local sense; i.e., in a sufficiently small interval of variations (Cooper et al. [14], Banker et al. [7] and Hadjicostas and Soteriou [17]). There is a literature which is directed to elasticities in DEA. Examples are the treatment of scale elasticities in Banker et al. [7], Podinovski [22] and Banker et al. [8]. Podinovski et al. [21] expressed the idea of elasticity measures as directional derivatives of the optimal value in linear programs and used it for the case of scale elasticity in VRS technology. Atici and Podinovski [3] extended the approach proposed by [22], analyzing and calculating a class of mixed partial elasticity measures in variable returns-to-scale (VRS) production technologies, to the constant returns-to-scale (CRS). Zelenyuk [34] noticed the scale elasticity measurement based on directional distance function for multi-output–multi-input technologies, and used directional distance function via the DEA estimator to get an estimation of the scale elasticity. Noticing the directional RTS as a generalization of traditional RTS by considering the non-proportional changes of inputs and outputs, Yang & Liu [29] proposed the definitions of directional scale elasticity and directional RTS in the DEA framework and estimates the directional RTS using DEA models. There are another researches on scale elasticities and RTS characterization in within the DEA context (e.g., [16], [23], [24], and [20]).

There are problems in using the standard DEA models, to obtain scale elasticity estimates which is related to the piecewise linear character of the frontiers for these models. There is yet another class of models referred to as "multiplicative models," which were introduced by this name into the DEA literature in Charnes et al. [13]– see also Banker et al. [5] – and extended in Charnes et al. [12]. Although not used very much in applications in DEA, these multiplicative models can provide advantages for extending the range of potential uses for DEA. For instance, they are not confined to efficiency frontiers that are concave. They can be formulated to allow the efficiency frontiers to be concave in some regions and non-concave elsewhere. See Banker and Maindiratta [8]. They can also be used to obtain "exact" estimates of elasticities.

The elasticity measure shows the rate of variation only in a neighborhood of inputs, and it may be insufficient for a decision being made by the manager. Zarepisheh and Soleimani-damaneh [32] introduced the generalized RTS to overcome this issue. They introduced an algorithm which is capable of determining the rate of variation until reaching the most productive scale size (MPSS) unit which is the best position from the manager's point of view. The rate of variation tells manager how beneficial is the expansion or contraction of the units in different intervals until reaching their optimal size. Having these information available will help the manager to make a better decision. For example, the manager may decide to expand/contact a unit up to the certain level when expansion/contraction is beneficial enough rather than up to the MPSS pattern. To the best of our knowledge, so far no study has been conducted to develop the concept of generalized RTS for multiplicative models. One of our goal in this paper is expanding the global elasticity notion to the *multiplicative models* where the production possibility set (PPS) is non-convex. The ordinary convexity postulate is replaced by geometric convexity in multiplicative models that implies the piece-wise linear frontiers usually employed in DEA are replaced by a frontier that is piece-wise Cobb-Douglas(= log linear) (Banker and Maindiratta [8]). If in an empirical application there are a priori reasons to believe that marginal products are increasing in some regions, then the log-linear model is the appropriate DEA model for the analysis. Banker and Maindiratta [8] introduced a model to determine MPSS pattern and after that, Banker et al. [7], presented a two-stage method to identify the RTS situation. Recently, Zarepisheh et al. [33] proved that both RTS situation and MPSS pattern can be identified by solving a single optimization model. However, as stated before, the RTS is just a local notion which can be misleading, and in section 2 of this paper we want to provide a global information about the rate of variation in outputs with respect to the variation in inputs in multiplicative models.

Another aim of this paper is demonstrating that the generalized RTS algorithm can be modified in order to take care of the *non-discretionary factors*. Standard data envelopment analysis implicitly assumes that all inputs and outputs are discretionary, i.e., can be controlled by the management of each DMU and varied at its discretion. However, there may exist exogenously fixed or non-discretionary (ND) inputs or outputs that are beyond the control of a DMUs management, which also need to be considered (Ruggiero [25], Syrjanen [27] and Muniz et al. [19]). Instances from the DEA literature include snowfall or weather forecast in evaluating the efficiency of maintenance units, soil characteristics and topography in different farms, number of competitors in the branches of a restaurant chain, etc.

So, our contribution is twofold: first, Given the importance of multivariate models (linear logarithmic, Cub Douglas) in the theory of production functions, and the ability to properly estimate scale elasticity, we combine [32] and [33] in order to attain generalized RTS and MPSS pattern by solving a single optimization model; second, given that in many applied studies of DEA, we may encounter non-discretionary data, the development of a proposed method for this type of data will also be considered.

The structure of the article is as follows. In section 2 the multiplicative models are reviewed and then an approach to determine RTS will be introduced. Moreover, we will show that the global rate of variation can be calculated in the existence of non-discretionary factors. Section 3 includes an example to clarify the approach. Section 4 concludes the article.

2 Preliminaries

Consider n DMUs where each DMU_j $(j = 1,\ldots,n)$, produces s outputs y_{rj} $(r = 1,\ldots,s)$, using m inputs x_{ij} $(i = 1,\ldots,m)$. Define $x_j = (x_{1j}, x_{2j},\ldots,x_{mj})^T$ and $y_j = (y_{1j}, y_{2j},\ldots,y_{sj})^T$ as input and output vectors of DMU_j, respectively. Also $X = [x_1, x_2,\ldots,x_n]$ and $Y = [y_1, y_2,\ldots,y_n]$ are $m \times n$ and $s \times n$ matrices of inputs and outputs, respectively. The production possibility set T is represented as

$$T = \{(x, y) \in \mathrm{R}_+^{m+s} \mid y \textit{ can be produced by } x\}$$

Banker, Charnes and Cooper [6] defined the following PPS based on some postulates. This set is denoted by T_v, regarding the prevalence of variable returns to scale assumption of the production technology.

$$T_v = \{(x, y) \in \mathrm{R}_+^{m+s} \mid X\lambda \le x,\ Y\lambda \ge y,$$
$$e\lambda = 1,\ \lambda \ge 0\},$$

where e is a vector with all components equal to one. Banker and Maindiratta [8] replaced the (ordinary) convexity postulate of BCC by geometric convexity, and introduced the following PPS.

$$T_m = \{(x, y) \in \mathrm{R}_+^{m+s} | \prod_{j\in J} x_{ij}^{\lambda_j} \le x_i,$$
$$i = 1, ..., m,\ \prod_{j\in J} y_{rj}^{\lambda_j} \ge y_r, r = 1, ..., s,$$
$$\sum_{j\in J} \lambda_j = 1,\ \lambda_j \ge 0,\ j \in J\}, \tag{2.1}$$

where $J = \{1, 2, ..., n\}$. PPS has a key role in DEA, and most of the concepts, such as RTS, are defined based on that. Increasing returns to scale (IRS) prevail at (x_o, y_o) if there exists a neighborhood around x_o(in T) such that in this neighborhood a proportional increase in all inputs leads to a greater proportional increase in all outputs. In this case, a (possible) reduction in all inputs leads to a greater reduction in all outputs (see Zarepisheh et al. [33]). Decreasing returns to scale (DRS) prevail at (x_o, y_o) if there exists a neighborhood around x_o(in T) such that in this neighborhood a proportional decrease in all inputs leads to a smaller proportional decrease in all outputs. It can be proved that an increase in all inputs leads to a smaller increase in all outputs for DRS units (see Zarepisheh et al. [33]). Constant returns to scale (CRS) prevail at (x_o, y_o) if neither IRS nor DRS prevail at this point.

3 Generalized RTS for Multiplicative Models

The aforementioned definition of RTS can be expressed from the mathematical viewpoint by the aid of function $\alpha(\beta) = \max\{\alpha \mid (\beta x_o, \alpha y_o) \in T\}$. If $\beta > 1$, then $\alpha(\beta)$ defines the maximum proportional increase in outputs when the inputs are increased proportionally by factor β. In this case, if $\alpha(\beta) > \beta$ for all β in the right neighborhood of 1, then proportional increase in inputs is beneficial and so IRS prevail at DMU$_o$. For $\beta < 1$, $\alpha(\beta)$ is the minimum proportional decrease in outputs when the inputs are decreased proportionally by factor β. In this case, if $\alpha(\beta) > \beta$ for all β in the left neighborhood of 1, then proportional decrease in inputs is beneficial and hence DRS prevail at DMU$_o$. For IRS (DRS) unit, the so-called elasticity measure which is the rate of proportional increase (decrease) in outputs to the proportional increase (decrease) in inputs, defined by $E = \underset{\beta\to 1}{Lim}\frac{\alpha(\beta)-1}{\beta-1}$, reveals how beneficial is the expansion (contraction) of DMU$_o$ in the local sense. For a global point of view, we refer to $\rho = \frac{\alpha(\beta)-1}{\beta-1}$ as the rate of benefit. In traditional DEA models, the RTS situation is determined based on the local information. Then, the DMU is expanded/contracted until it reaches its optimal size (MPSS pattern). Zarepisheh et al. [31] introduced an algorithm to calculate the *global rate of variation* in outputs to the variation in inputs in T_v. In fact, these information show that how beneficial is the expansion/contraction of the unit in different intervals until reaching its optimal size. So, the manager may decide to expand/contract the unit until a certain level, when it is beneficial enough, rather than until reaching the MPSS pattern. Extending the same concept for geometrical production possibility set, consider a set of DMUs: {DMU$_A$,...,DMU$_E$} each DMU utilizing one input to produce one output, as depicted in Fig. 1 (which data can seen in Table 1).

If we examine these DMUs with the method introduced by Zarepisheh et al. [33], then IRS prevail at A and B, CRS prevail at C and D, and DRS prevail at E. As an example, we consider the relation between the increase in outputs with respect to the increase in inputs for DMU$_A$ in a non-local viewpoint. DMUs A and B are efficient, and the geometrical convex of them (curve AB) is also efficient. For each (x,y) on AB we have,

$$\begin{cases} x = x_A^{\lambda}.x_B^{1-\lambda} \Rightarrow x = 1_A^{\lambda}.2^{1-\lambda} = 2^{1-\lambda} \\ y = y_A^{\lambda}.y_B^{1-\lambda} \Rightarrow y = 1_A^{\lambda}.5^{1-\lambda} = 5^{1-\lambda} \end{cases}$$

$\Rightarrow (x, y) = (2^{1-\lambda}, 5^{1-\lambda}), \quad for\, \lambda \in [0, 1]$.
Rewriting $1-\lambda$ as λ, the parametric equation of AB curve is $(2^{\lambda}, 5^{\lambda})$. According the definition of $\alpha(\beta)$, for $\beta \in [1, 2]$, $(\beta x_A, \alpha(\beta) y_A) = (\beta, \alpha(\beta))$ is on curve AB. So, there is a $\lambda \in [0, 1]$ in which $\beta = 2^{\lambda}$and $\alpha(\beta) = 5^{\lambda}$. From $\beta = 2^{\lambda}$we have $\lambda = Ln\,\beta/Ln\,2$. Substituting λ in $\alpha(\beta) = 5^{\lambda}$, we have,

$\alpha(\beta) = 5^{Ln\beta/Ln2} \Rightarrow Ln(\alpha(\beta)) = Ln\beta/Ln2Ln5 = (Ln5/Ln2)Ln\beta = Ln\beta^{Ln5/Ln2} \Rightarrow \alpha(\beta) = \beta^{Ln5/Ln2}$

Since (Ln 5/Ln 2)>1 and $\beta \in [1, 2]$, we have $\alpha(\beta) > \beta$, i.e., the proportional increase in inputs from DMU$_A$ to DMU$_B$ is beneficial with the rate of benefit $\rho = \frac{\beta^{\frac{\ln 5}{\ln 2}}-1}{\beta-1}$. Let us assume that we increased the inputs and we reached DMU$_B$. At this position, $\alpha(\beta) = \beta^{\frac{\ln 2}{\ln 1.5}}$ for $\beta \in [1, 1.5]$, and so, the rate of benefit is $\rho = \frac{\beta^{\frac{\ln 2}{\ln 1.5}}-1}{\beta-1}$ which is less that the rate of benefit in the previous step. That means, the proportional increase in inputs from DMU$_B$ to DMU$_C$ is still beneficial, but not as beneficial as the increase from DMU$_A$ to DMU$_B$. At DMU$_C$, $\alpha(\beta) = 1$ for $\beta \in [1, 2]$, meaning, the increase in inputs is not beneficial anymore and DMU$_C$ and DMU$_D$ are CRS. DMU$_C$ and DMU$_D$ are considered as the MPSS patterns for DMU$_A$.

Finding the rate of benefit in different intervals in the multiple-input, multiple-output case is not a simple task. In the sequel, we deal with this issue. Hereafter, we aim at introducing an algorithm which is capable of calculating the global RTS for multiplicative models. We also concur with Banker et al. [4] who claim that RTS has an unambiguous meaning only if a DMU is on the efficient frontier, and define RTS only for points on the efficient frontier.

Table 1: Data for 5 DMUs

	A	B	C	D	E
Input	1	2	3	6	7
Output	1	5	10	20	22

Table 2: Optimal simplex table of Problem (4.5)

	z	λ_1	λ_2	λ_3	λ_4	λ_5	α	R.H.S
z	1	0	0	0.24	1.16	1.42	0	0
λ_1	0	1	0	-0.58	-1.58	-1.80	0	1
λ_2	0	0	1	1.58	2.58	2.8	0	0
α	0	0	0	0.24	1.16	1.42	1	0

Table 3: Table 3 after updating R.H.S

	z	λ_1	λ_2	λ_3	λ_4	λ_5	α	R.H.S
z	1	0	0	0.24	1.16	1.42	0	1.16
λ_1	0	1	0	-0.58	-1.58	-1.80	0	0
λ_2	0	0	1	1.58	2.58	2.8	0	1
α	0	0	0	0.24	1.16	1.42	1	1.16

Table 4: Table 3 after dual simplex pivoting

	z	λ_1	λ_2	λ_3	λ_4	λ_5	α	R.H.S
z	1	0.42	0	0	0.49	0.66	0	1.7
λ_3	0	-1.7	0	1	2.7	3.08	0	0
λ_2	0	2.7	1	0	-1.7	-2.08	0	1
α	0	0.42	0	0	0.49	0.66	1	1.16

Table 5: Table 4 after updating R.H.S. and performing dual simplex pivoting

	z	λ_1	λ_2	λ_3	λ_4	λ_5	α	R.H.S
z	1	1.2	0.28	0	0	0.05	0	2.3
λ_3	0	2.58	1.58	1	0	-0.22	0	1
λ_4	0	-1.58	-0.58	0	1	1.22	0	0
α	0	1.2	0.28	0	0	0.05	1	2.3

In order to calculate $\alpha(\beta)$ for (x_o, y_o) in T_m, we should solve the following model with β as a parameter.

$$\alpha(\beta) = \max \alpha \tag{3.2}$$

$$s.t.$$

$$\prod_{j \in J} x_{ij}^{\lambda_j} \leq \beta x_{io},\ i = 1, ..., m,$$

$$\prod_{j \in J} y_{rj}^{\lambda_j} \geq \alpha y_{ro},\quad r = 1, ..., s,$$

$$\sum_{j \in J} \lambda_j = 1, \qquad \lambda_j \geq 0 \qquad \forall j \in J.$$

where $J = \{1, 2, ..., n\}$. This non-linear optimization model can be easily converted to the following equivalent linear problem by exploiting the ln function.

$$\bar{\alpha}(\bar{\beta}) = \max \bar{\alpha} \tag{3.3}$$

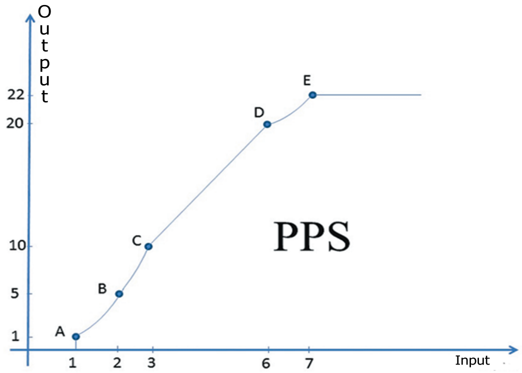

Figure 1: The Production Possibility Set for Multiplicative Model

$$s.t.$$

$$\sum_{j\in J}\lambda_j \bar{x}_{ij} \le \bar{\beta} + \bar{x}_{io},\ i = 1, ..., m,$$

$$\sum_{j\in J}\lambda_j \bar{y}_{rj} \ge \bar{\alpha} + \bar{y}_{ro},\quad r = 1, ..., s,$$

$$\sum_{j\in J}\lambda_j = 1,\qquad \lambda_j \ge 0 \qquad \forall j \in J.$$

where "-" denotes the *ln* function and $J = \{1, 2, ..., n\}$. We start to compute $\bar{\alpha}(\bar{\beta})$ for $\bar{\beta} \ge 0$. First, we solve Problem (3.3) for $\bar{\beta} = 0$. Since (x_o, y_o) is an efficient unit, the optimal objective value is equal to zero. Then, we do parametric analysis by perturbing the right hand side vector in the direction of $\begin{pmatrix} 1_{m\times 1} \\ 0_{(s+1)\times 1} \end{pmatrix}$ for non-negative parameters (see Bazaraa et al. [9] for details about parametric analysis). It gives us the optimal value of $\bar{\alpha}(\bar{\beta})$ as a linear function of parameter $\bar{\beta}$ on different intervals. Here, we just need the first interval and let us denote the slope of the linear function and the length of the first interval by m_1^+ and $\bar{\beta}_1^+$ respectively. Since $\bar{\alpha}(0) = 0$, we have $\bar{\alpha}(\bar{\beta}) = m_1^+\bar{\beta}$ for $\bar{\beta} \in [0, \bar{\beta}_1^+]$. By taking into consideration that $\bar{\beta} = \ln(\beta)$ and $\bar{\alpha} = \ln(\alpha)$, we have $\alpha(\beta) = \beta^{m_1^+}$ for $\beta \in [1, \exp(\bar{\beta}_1^+)] = [1, \beta_1^+]$. If $m_1^+ = 1$, then as it can be seen later as a conclusion of Theorem 3.1, CRS prevail (x_o, y_o). If $m_1^+ > 1$, then the proportional increase of the inputs from x_o to $\beta_1^+ x_o$ is beneficial and m_1^+ shows how beneficial it is. By increasing the inputs to $\beta_1^+ x_o$, we reach the point $(\beta_1^+ x_o, \alpha(\beta_1^+)y_o)$ in PPS. To investigate the rate of benefit for further increase in inputs, we just need to replace (x_o, y_o) by $(\beta_1^+ x_o, \alpha(\beta_1^+)y_o)$ and repeat the aforementioned process. In this case, we do not need to solve Problem (3.3) from the scratch. Instead, we update the right hand side of the previous optimal simplex table accordingly and perform the dual simplex algorithm to get the optimal simplex table corresponding to $(\beta_1^+ x_o, \alpha(\beta_1^+)y_o)$ (see numerical example in Section 3). If the case $m_1^+ < 1$ occurs, that means the proportional increase in inputs is not favorable. In this case, we should examine the proportional decrease in inputs.
To investigate the proportional decrease in inputs (when $m_1^+ < 1$), we should compute $\bar{\alpha}(\bar{\beta})$ for $\bar{\beta} < 0$. To this end, after solving Problem (3.3) for $\bar{\beta} = 0$, we perform the parametric analysis in the direction of $\begin{pmatrix} 1_{m\times 1} \\ 0_{(s+1)\times 1} \end{pmatrix}$ for negative parameters. Let suppose that $\bar{\alpha}(\bar{\beta}) = m_1^-\bar{\beta}$ for $\bar{\beta} \in [\bar{\beta}_1^-, 0]$. It means that $\alpha(\beta) = \beta^{m_1^-}$ for $\beta \in [\exp(\bar{\beta}_1^-), 1] = [\beta_1^-, 1]$. If $m_1^- \ge 1$, then the proportional decrease in the inputs is not beneficial and since $m_1^+ < 1$ the proportional increase in the inputs is not beneficial as well and hence CRS prevail at (x_o, y_o). If $m_1^- < 1$, then the proportional decrease in the inputs is beneficial and DRS prevail at (x_o, y_o). Like the previous case, to calculate the rate of benefit up to the MPSS, we just need to replace (x_o, y_o) by $(\beta_1^- x_o, \alpha(\beta_1^-)y_o)$ and repeat the process.

Theorem 3.1. *If $(x_o, y_o) \in T_m$ is an efficient unit and the proportional decrease in the inputs is possible for that, then $m_1^+ \le m_1^-$.*

Proof. By definition of m_1^+ and m_1^-, we have $(\beta_1 x_o, \beta_1^{m_1^+} y_0) \in T_m$ and $(\beta_2 x_o, \beta_2^{m_1^-} y_0) \in T_m$, where $\beta_1 \in [1, \beta_1^+]$ and $\beta_2 \in [\beta_1^-, 1]$. Due to the geometrical convexity property of T_m, we have $(\beta_1^{\lambda}\beta_2^{1-\lambda}x_o, \beta_1^{\lambda m_1^+}\beta_2^{(1-\lambda)m_1^-} y_0) \in T_m$ for each $\lambda \in [0, 1]$. We consider $\tilde{\lambda} = \frac{\ln(\beta_2)}{\ln(\beta_2)-\ln(\beta_1)}$ for which $\beta_1^{\tilde{\lambda}}\beta_2^{1-\tilde{\lambda}} = 1$. Since (x_o, y_o) is an efficient unit, $\beta_1^{\tilde{\lambda} m_1^+}\beta_2^{(1-\tilde{\lambda})m_1^-} \le 1$ which means $m_1^+ \le m_1^-$. □

According to the above theorem and what was discussed before, if $m_1^+ = 1$, then $m_1^- \ge 1$ and so CRS prevail at (x_o, y_o).

3.1 Generalized RTS in the Existence of Non-Discretionary Factors

Up to now, we implicitly assumed that all inputs and outputs are discretionary, i.e., can be controlled by the management of each DMU and varied at its discretion. Thus, we calculated the proportional variation in all outputs with respect to the proportional variation in all inputs. However, there may exist non-discretionary inputs or outputs that are beyond the control of a DMU's management. In this case we should redefine the generalized RTS as the proportional variation in discretionary outputs with respect to the proportional variation in discretionary inputs. In this section, we aim to show that the proposed algorithm can be easily modified to take care of non-discretionary factors.

Suppose that the input and output variables may each be partitioned into subsets of discretionary (D) and non-discretionary (N) variables. Thus,

$$I = \{1, 2, ..., m\} = I_D \bigcup I_N \text{ with } I_N \bigcap I_D = \emptyset$$

and

$$O = \{1, 2, ..., s\} = O_D \bigcup O_N \text{ with } O_N \bigcap O_D = \emptyset$$

where I_D, O_D and I_N, O_N refer to discretionary (D) and non-discretionary (N) input, I, and output, O, variables, respectively. Then, we modify Problem (3.2) as follows:

$$\begin{aligned} \alpha(\beta) &= \max \alpha \\ & s.t. \\ & \prod_{j\in J} x_{ij}^{\lambda_j} \leq \beta x_{io}, \ i \in I_D \\ & \prod_{j\in J} x_{ij}^{\lambda_j} \leq x_{io}, \quad i \in I_N \\ & \prod_{j\in J} y_{rj}^{\lambda_j} \geq \alpha y_{ro}, \quad r \in O_D \\ & \prod_{j\in J} y_{rj}^{\lambda_j} \geq y_{ro}, \quad r \in O_N \\ & \sum_{j\in J} \lambda_j = 1, \quad \lambda_j \geq 0 \quad \forall j \in J, \end{aligned} \tag{3.4}$$

Problem (3.3) should also be modified accordingly. After solving the modified version of Problem (3.3), we only need to modify the direction in which we perturb the right hand side vector. In this case, we only perturb the right hand side components corresponding to the discretionary inputs. The other parts of the algorithm would remain the same.

4 Numerical Example

To verify the proposed algorithm, we apply it to example of the preceeding section.

Example 4.1. *We consider DMU_A as a unit under assessment. Problem (3.3) corresponding to this unit is:*

$$\begin{aligned} & \max \quad \alpha \\ & \lambda_2 \ln 2 + \lambda_3 \ln 3 + \lambda_4 \ln 6 + \lambda_5 \ln 7 \leq 0 \\ & \lambda_2 \ln 5 + \lambda_3 \ln 10 + \lambda_4 \ln 20 + \lambda_5 \ln 22 \geq \alpha \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1 \\ & (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \geq 0 \end{aligned} \tag{4.5}$$

Table 2 illustrates the optimal simplex tableau of Problem (4.5).

To do the parametric analysis in the direction of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, we should compute the maximum value of $\bar{\beta}_1^+$ for which $B^{-1}(b + \bar{\beta}_1^+ b') \geq 0$, where

$$B^{-1} = \begin{pmatrix} -1.44 & 0 & 1 \\ 1.44 & 0 & 0 \\ 2.32 & -1 & 0 \end{pmatrix}, \ B^{-1}b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$b' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

It can be easily shown that $\bar{\beta}_1^+ = 0.693$. The slope of the optimal objective function is equal to $m_1^+ = (0\ 0\ 1)B^{-1}b' = 2.321$. Therefore,

$$\begin{aligned} \alpha(\beta) = \beta^{m_1^+} = \beta^{2.321}, \ \ \forall \beta \in [1, \exp(\bar{\beta}_1^+)] \\ = [1, 1.999]. \end{aligned}$$

It means that if we proportionally increase the inputs by $\beta \in [1, 1.999]$, then the outputs increase proportionally by $\alpha(\beta) = \beta^{2.321}$. Since $1.999 \approx 2$ and $2.321 \approx \frac{\ln 5}{\ln 2}$, these results coincide with what we obtained in previous section without using this algorithm.

Now, we update the R.H.S. column by replacing it with $B^{-1}(b+\bar{\beta}_1^{+}b')$. Table 3 depicts the next table.

According to dual simplex algorithm, λ_1 leaves the basis and λ_3 enters the basis. Table 4 represents the table after dual simplex pivoting.

We should again do the parametric analysis with the data in Table 4. In this case, we have $\bar{\beta}_2^{+} = 0.405$ and $m_2^{+} = 1.709$. Thus,

$$\alpha(\beta) = \beta^{1.709}, \quad \forall \beta \in [1, 1.499].$$

Since $1.499 \approx 1.5$ and $0.405 \approx \frac{\ln 2}{\ln 1.5}$, the algorithm's result is consistent with what we obtained before in Example 4.1.

The next step is updating the R.H.S. and implementing the dual simplex algorithm. Dual simplex makes λ_2 leave the basis and λ_4 enter the basis. Table 5 is the Table 5.

Regarding Table 4, $\bar{\beta}_3^{+} = 0.693$ and $m_3^{+} = 1$.

Hence, we reach the MPSS pattern at $(\exp(\bar{\beta}_1^{+})\exp(\bar{\beta}_2^{+})x_0, \alpha(\exp(\bar{\beta}_1^{+}))\alpha(\exp(\bar{\beta}_2^{+}))y_o)$ $= (3, 10)$, and algorithm terminates.

5 Conclusion

RTS has been introduced to provide useful information about the optimal size of units and whether the expansion or contraction of each unit is beneficial. However, the traditional DEA models provide these information based on some local investigation about the proportional variation in outputs with respect to the variation in inputs. These local information may not be enough for manager to make a good decision because they do not tell how beneficial is the expansion or the contraction of each unit. The generalized RTS has been introduced to overcome this drawback, and in this paper we proposed an algorithm to provide global information about the rate of benefit in multiplicative models. We also explained how the algorithm can be modified in order to include the non-discretionary inputs and outputs.

There are several ways to develop the present study. In the proposed models, only ordinary data and non-dicretionary inputs were considered. Development of the model to include undesirable outputs, which play a significant role in environmental data envelopment analysis studies, can be considered as a pathway for research. Studying the Generalized return to scale for units with network structures, developing this concept for GDEA, paying attention to fuzzy, interval, integer and negative data can also be a roadmap for future studies.

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