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Research Article

Dynamical Control of Computations Using the Iterative Methods to Solve Absolute Value Equation

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Abstract

In this work, the absolute value equation (AVE) $Ax - |x| = b$ is solved by the Gauss-Seidel and Jacobi iterative methods based on the stochastic arithmetic, where A is an arbitrary square matrix whose singular values exceed one. An algorithm is proposed to find the optimal number of iterations in the given iterative scheme and obtain the optimal solution with its accuracy. To this aim, the CESTAC [1§]method and the CADNA [2¶]library are applied which allows us to estimate the round-off error effect on any computed result. The classical criterion to terminate the iterative procedure is replaced by a criterion independent of the given accuracy (ϵ) such that the best solution is evaluated numerically. Numerical examples are solved to validate the results and show the efficiency and importance of using the stochastic arithmetic in place of the floating-point arithmetic. Moreover, this method is applied to solve two-point boundary value problem.

Keywords : Iterative Method; Absolute Value Equation; Stochastic Arithmetic; CESTAC Method; CADNA library.

1 Introduction

We consider the absolute value equation (AVE):

$$Ax - |x| = b. \tag{1.1}$$

where $A \in R^{n\times n}$, $b \in R^n$ and $|x|$ denotes the vector with absolute values of each component of x. A slightly more general form of AVE was introduced in [25] and investigated in a more general context in Mangasarian [21]. In [15] Cottle et al. presented the general NP-hard linear complementarity problem (LCP) that subsumes many mathematical programming problems that can be formulated as an absolute value equation such as AVE. Eq. (1.1) was investigated in detail theoretically in [20] and a bilinear program was prescribed there for the special case when the singular values of A exceed one. Mangasarian [22] generalized LCP, and the standard LCP if 1 is not an eigenvalue of A. Based on the LCP reformulation, sufficient conditions for the existence and nonexistence of solutions are given.

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Rohn provided a theorem of the alternatives for a more general form of AVE, $Ax+B|x|= b$, and enlightened the relation between the AVE and the interval matrix [26]. Prokopyev proved that AVE can be equivalently reformulated as a standard LCP without any assumption [24] on A and B, and discussed unique solvability of AVE. In the recent work of Mangasarian, a semismooth Newton's method is proposed for solving AVE, which largely shortens the computation time than the succession of linear programs (SLP) method [23]. It shows that the semismooth Newton's iterates are well defined and bounded when the singular values of matrix A exceed one. A generalized Newton's method, which has global and finite convergence, was proposed for the AVE by Zhang et al. The method utilizes both the semismooth and the smoothing Newton's steps, in which the semismooth Newton's step guaranteed the finite convergence and the smoothing Newton's step contributed to the global convergence [36].
For solving Eq. (1.1) by an iterative method, we usually use the common strategy to stop the iterations. For a given tolerance $\epsilon > 0$, we can use the$\| x_n - x_{n-1} \| < \epsilon$ as stopping criterion where, $\{x_n\}$ is the sequence of approximate vectors obtained from an iterative process such that

$$\lim_{n\to\infty} x_n = x. \tag{1.2}$$

If this termination criterion is used then the number of significant digits that are common to corresponding entries of x_n and x cannot be specified. One of the problems in iteration methods is to choose the value ϵ. When ϵ is chosen too large, then the iterative process is stopped too soon, and consequently the approximate solution has a poor accuracy. On the contrary, when ϵ is chosen too small, it is possible, due to the numerical instabilities, that many useless iterations are performed without improving the accuracy of the solution[28]. In this paper, in order to obtain the optimal iteration, optimal solution and eliminate the useless iterations, the results are computed by using the stochastic arithmetic and the CESTAC method. In this method, the usual floating-point arithmetic is replaced by a random arithmetic. Consequently, each result appears as a random variable. This approach leads to concepts, stochastic numbers and stochastic arithmetic.
In recent years, the CESTAC method used to validate many problems in mathematics and physic such as interpolation polynomials [2], numerical integration [1, 4], linear algebra [27] and other topics [6, 7, 8, 9, 28]. This paper is organized as follows. In section 2, a brief description of stochastic round-off analysis, the CESTAC method and the CADNA library is described . In section 3, a theorem is proved in order to show that the significant digits common between two sequential results are almost equal to the significant digits common between $x^{(k)}$ and the exact solution x^*. Also, an algorithm is presented in order to solve the sample examples of AVE via the CADNA library. In section 4, three sample examples are given which are computed by using the proposed algorithm based on the CESTAC method.

2 Preliminaries

Let x be a real number, then x in the computer is represented as follows:

$$X = x - \epsilon_1 2^{E_1-\mathcal{P}}\alpha_1,$$

where, ϵ_1 is the sign of x and $2^{-\mathcal{P}}\alpha_1$ is the lost part of the mantissa due to round-off error and E_1 is the binary exponent of the result. In the floating-point arithmetic and double precision case, $\mathcal{P} = 53$ and usually $-1 \leqslant \alpha_1 \leqslant 1$. In the CESTAC method, α_1 is considered as random variable uniformly distributed on $[-1, 1]$. The CESTAC method is applied in order to implement an algorithm, any result X is a random variable with mean μ and variance σ^2. Hence, a stochastic arithmetic should be used. When the last mantissa bit is perturbed N times, the samples $X_1, X_2, \ldots, X_N$ are produced which are used to estimate μ and σ^2. Then, the mean of these samples is considered as estimation of the result and the variance of them is used to determine the accuracy of the result. The main idea of the CESTAC method is to evaluate the number of significant digits of the result X effectively. In the CESTAC method to find samples for the obtained random variables, we perturb the last mantissa bit of the value X [6]. The algorithm of CESTAC method is as follows:

1. Find N sample for X as

   $$X_1, X_2, \ldots, X_N,$$

   by means of the perturbation of the last bit of the mantissa.

2. Compute

   $$X_{ave} = \frac{\sum_{i=1}^{N} X_i}{N},$$

3. Compute

   $$S_X^2 = \frac{1}{N-1} \sum_{i=1}^{N} (X_i - X_{ave})^2,$$

4. Compute

   $$C_{X_{ave},X} = log_{10} \frac{\sqrt{N}|X_{ave}|}{\tau_\beta S_X},$$

   as the common significant digits between the exact value X and the approximate value X_{ave}, (If $N = 3$ and $\beta = 0.05$ then $\tau_\beta = 4.303$.).

5. if

   $$C_{X_{ave},X} \leqslant 0 \quad or X_{ave} = 0,$$

   then write $X = @.0$.

Definition 2.1 *A computed result X by using the CESTAC method is an "informatical zero", denoted by @.0, if and only if, $X_{ave} = 0$ or $C_{X_{ave},X} \leqslant 0$.*

CADNA is a library for programs written in FORTRAN77, FORTRAN90, or in C++ which allows the computation using stochastic arithmetic by automatically implementing the CESTAC method. CADNA is able to estimate the accuracy of the computed results, and to detect numerical instabilities occurring during the run. During the run, as soon as a numerical anomaly (for example, appearance of the informatical zero in a computation or a criterion) occurs, a message is written in a special file called Cadna-stability-f90.lst. The user must consult this file after the program is performed. For more details about the stochastic arithmetic, the CESTAC method and the CADNA library, we refer the reader to [7, 8, 9, 10, 11, 12, 13, 14, 18, 28, 29, 30, 31, 32, 33].

3 Main idea

In this section, in order to solve an AVE, an algorithm is presented based on the CESTAC method to validate the results and is implemented by the CADNA library. At first, we recall some theorems, which are necessary to confirm the performance of the proposed algorithm.
For Eq. (1.1), we use the following decomposition:

$$A = D + L + U, \tag{3.3}$$

in which D is the diagonal of matrix A, L its strict lower part and U its strict upper part. It is always assumed that the diagonal entries of matrix A are all non-zero. In each iteration of the iterative method for solving Eq. (1.1) , a lower triangular absolute value system $(D + L)x - |x| = b$, should be solved where b is a fixed vector.
By rewriting Eq. (1.1) we get:

$$\begin{aligned} (D + L)x - |x| &= -Ux + b, \\ Dx - |x| &= -(L + U)x + b. \end{aligned} \tag{3.4}$$

Let $E = \text{diag}(\text{sign}(x))$, we can define the iterative methods for solving Eq. (1.1) as follows:

$$\begin{aligned} &Dx^{(k+1)} - |x^{(k)}| = -(L + U)x^{(k)} + b, \\ &k = 0, 1, 2, ..., \end{aligned} \tag{3.5}$$

$$\begin{aligned} &Dx^{(k+1)} = -(L + U - E)x^{(k)} + b, \\ &k = 0, 1, 2, ..., \end{aligned} \tag{3.6}$$

$$\begin{aligned} &(D + L)x^{(k+1)} - |x^{(k)}| = -Ux^{(k)} + b, \\ &k = 0, 1, 2, ..., \end{aligned} \tag{3.7}$$

$$\begin{aligned} &(D + L)x^{(k+1)} = (-U + E)x^{(k)} + b, \\ &k = 0, 1, 2, ... \end{aligned} \tag{3.8}$$

where $x^{(0)}$ is a given initial guess.
The Jacobi and Gauss-Seidel methods are shown by Eqs. (3.6) and (3.6) respectively. In general, iterative techniques for solving Eq. (1.1) by Eqs. (3.6) and (3.6) involve a process that converts the system of the form $x = Px + c$ for some fixed matrix P and vector c, where in Jacobi method $P = -D^{-1}(L + U - E)$, $c = D^{-1}b$ and in Gauss- Seidel method $P = (L + D)^{-1}(-U + E)$, $c = (L + D)^{-1}b$.
After the initial vector $x^{(0)}$ is selected, the sequence of approximate solution vectors is generated by computing:

$$x^{(k+1)} = Px^{(k)} + c, k = 1, 2, 3, \ldots. \tag{3.9}$$

Table 1: The results for Jacobi Method in Example 4.1.

k	x	$\| x^{k+1} - x^k \|$
1	$x_1 = 1.000000000000000E + 000$ $x_2 = -0.818181818181818E + 000$ $x_3 = 0.666666666666666E + 000$ $x_4 = -0.153846153846153E + 001$	0.153846153846153E+001
2	$x_1 = 0.10484848484848E + 001$ $x_2 = -0.876668785759694E + 000$ $x_3 = 0.986790986790986E + 000$ $x_4 = -0.105648197955890E + 001$	0.481979558902635E+000
$\vdots$	$\vdots$	$\vdots$
41	$x_1 = 0.100000000000000E + 001$ $x_2 = -0.100000000000000E + 001$ $x_3 = 0.999999999999999E + 000$ $x_4 = -0.100000000000000E + 001$	0.3E-014
42	$x_1 = 0.100000000000000E + 001$ $x_2 = -0.100000000000000E + 001$ $x_3 = 0.100000000000000E + 001$ $x_4 = -0.100000000000000E + 001$	@.0

Table 2: The results for Gauss-Seidel Method in Example 4.1.

k	x	$\| x^{k+1} - x^k \|$
1	$x_1 = 0.100000000000000E + 001$ $x_2 = -0.909090909090908E + 000$ $x_3 = 0.818181818181818E + 000$ $x_4 = -0.112587412587412E + 001$	0.112587412587412E+001
2	$x_1 = 0.102727272727272E + 001$ $x_2 = -0.949713922441194E + 000$ $x_3 = 0.986956982411527E + 000$ $x_4 = -0.101949239571617E + 001$	0.16877516422970E+000
$\vdots$	$\vdots$	$\vdots$
14	$x_1 = 0.100000000000000E + 001$ $x_2 = -0.100000000000000E + 001$ $x_3 = 0.999999999999999E + 000$ $x_4 = -0.100000000000000E + 001$	0.21E-013
15	$x_1 = 0.100000000000000E + 001$ $x_2 = -0.999999999999999E + 000$ $x_3 = 0.100000000000000E + 001$ $x_4 = -0.999999999999999E + 000$	@.0

Table 3: The results for Gauss-Seidel Method in Example 4.2.

n	2	4	8	10	16	32	50
optimal iteration-Jacobi method	18	37	54	56	63	70	97
optimal iteration-Gauss-Seidel method	27	26	28	28	30	30	29

Therfore, the following corollaries can be deduced [5]:

Corollary 3.1 *For any $x^{(0)} \in \mathbb{R}^n$, the sequence $\{x^{(k)}\}_{l=0}^{\infty}$ defined by $x^{(k+1)} = Px^{(k)} + c$, for each $k \geq 1$, converges to the unique solution of $x = Px + c$ if and only if $\rho(P) < 1$.*
where $\rho(P)$ is the spectral radius of matrix P.

Corollary 3.2 *If $\| P \|_2 < 1$ and c is a given vector, then the sequence$\{x^{(k)}\}_{k=0}^{\infty}$ defined by $x^{(k+1)} = Px^{(k)} + c$ converges, for any $x^{(0)} \in \mathbb{R}^n$, to a vector $x \in \mathbb{R}^n$, with $x = Px + c$, and the following error bounds hold:*

$$\begin{aligned} &(i) \ \| x - x^{(k)} \|_2 \leq \| P \|_2^k . \| x^{(0)} - x \|_2, \\ &(ii) \ \| x - x^{(k)} \|_2 \\ &\leq \frac{\| P \|_2^k}{1 - \| P \|_2} \| x^{(1)} - x^{(0)} \|_2 . \end{aligned} \tag{3.10}$$

Corollary 3.3 *Suppose that Eq. (1.1) is solvable. Let $(A - I)$ be an strictly row diagonally dominant matrix having positive diagonal entries. If*

$$\| (D + L)^{-1} U \|_2 < 1 - \| (D + L)^{-1} \|_2, \tag{3.11}$$

then Eq. (1.1) has a unique solution x^* and the sequence $x^{(k)}$ obtained from (3.6) and (3.6) converges to the solution x^*.
To show accuracy of result, we must compare the common significant digits of each corresponding components of approximate solution and exact solution. At first, the following definition is given which is applied for the proof of the next theorem[27].

Definition 3.1 *For two distinct real vectors $x, y \in \mathbb{R}^n$, the number of common significant digits defined as:*

$$C_{x,y} = \log_{10} \frac{||x + y||_2}{2\sqrt{n}||x - y||_2}. \tag{3.12}$$

If $x = y$ then $C_{x,y} = +\infty$.

Now, similar to the discussion in [27], the following theorem can be proved for computing of the common significant digits of each corresponding components of the computed solution and the exact solution for the mentioned iterative methods.

Theorem 3.1 *Let $x^{(k+1)} = Px^{(k)} + c$, be convergent iterative method obtained from Eqs. (3.6) and (3.8) to the exact solution x^* of Eq. (1.1), and $c \neq 0$, Then for sufficiently large value of k, we have:*

$$\begin{aligned} &log_{10}(1 - \| P \|_2) \leqslant C_{x^{(k)},x^*} - C_{x^{(k)},x^{(k+1)}} \\ &\leqslant log_{10}(1 + \| P \|_2). \end{aligned} \tag{3.13}$$

Proof: According to Eq. (3.12)

$$\begin{aligned} &C_{x^{(k)},x^*} - C_{x^{(k)},x^{(k+1)}} = \\ &\log \frac{\|x^{(k)} + x^*\|_2}{2\sqrt{n}\|x^{(k)} - x^*\|_2} \\ &- \log \frac{\|x^{(k)} + x^{(k+1)}\|_2}{2\sqrt{n}\|x^{(k)} - x^{(k+1)}\|_2} = \\ &\log \frac{\|x^{(k)} + x^*\|_2}{\|x^{(k)} + x^{(k+1)}\|_2} + \\ &\log \frac{\|x^{(k)} - x^{(k+1)}\|_2}{\|x^{(k)} - x^*\|_2}. \end{aligned} \tag{3.14}$$

By increasing k, $x^{(k)} \simeq x^*$ and $x^{(k+1)} \simeq x^*$, then first term of Eq. (3.14) is almost equal to zero. Let $x^{(k)} - x^* = T^{(k)}$. Then one can see

$$x^{(k)} - x^* = P^{(k)}(x_0 - x^*) = P^{(k)}T^{(0)}. \tag{3.15}$$

Furthermore, for the second term of Eq. (3.14), we can write

$$\begin{aligned} &\frac{\|x^{(k)} - x^{(k+1)}\|_2}{\|x^{(k)} - x^*\|_2} = \\ &\frac{\|x^{(k)} - x^* + x^* - x^{(k+1)}\|_2}{\|x^{(k)} - x^*\|_2} \\ &\leq \frac{\|x^{(k)} - x^*\|_2 + \|x^{(k+1)} - x^*\|_2}{\|x^{(k)} - x^*\|_2} \end{aligned} \tag{3.16}$$

$$\begin{aligned} &= 1 + \frac{\|x^{(k+1)} - x^*\|_2}{\|x^{(k)} - x^*\|_2} = \\ &1 + \frac{\|P^{(k+1)}T^{(0)}\|_2}{\|P^{(k)}T^{(0)}\|_2} \leq 1 + \|P\|_2. \end{aligned}$$

According to Eq. (3.15)

$$x^{(k+1)} - x^{(k)} = P^k(I - P)T^{(0)}.$$

Therefore,

$$\frac{\|x^{(k)}-x^{(k+1)}\|_2}{\|x^{(k)}-x^*\|_2}=\frac{\|P^k(I-P)T^{(0)}\|_2}{\|P^kT^{(0)}\|_2} \geq \frac{\|P^kT^{(0)}\|_2-\|P\|_2\|P^kT^{(0)}\|_2}{\|P^kT^{(0)}\|_2}= 1-\|P\|_2. \tag{3.17}$$

According to (3.16) and (3.17)

$$1-\|P\|_2 \leq \frac{\|x^{(k)}-x^{(k+1)}\|_2}{\|x^{(k)}-x^*\|_2} \leq 1+\|P\|_2.$$

Finally

$$\log(1-\|P\|_2) \leq \log \frac{\|x^{(k)}-x^{(k+1)}\|_2}{\|x^{(k)}-x^*\|_2} \leq \log(1+\|P\|_2), \tag{3.18}$$

which completes the proof.
If $\| P \|_2 \ll 1$, then Eq.(3.13) shows that the common significant digits of each corresponding components of the solution in two successive iteration and the common significant digits of the computed solution and the exact solution are almost equal. It means that:

$$C_{x^{(k)},x^*} \simeq C_{x^{(k)},x^{(k+1)}}.$$

In the sequel, an algorithm is proposed to apply for implementation the CESTAC method by the CADNA library for solving an AVE.

Algorithm 1
1. type ($double\_st$) The list of the real variables.
2. call cadna-init(-1)
3. $k=0$
4. $cin >> x_0$
5. do
6. {
7. $x^{(k+1)}=Px^{(k)}+c, \quad k=0,1,2,\ldots$
8. $cout << "x=", strp(x^{k+1})$
9. $k=k+1$
10. }
11. while $((x^{(k+1)}-x^{(k)})!=@.0$)
12. cadna-end().

The function "Strp" in the output instruction shows only the significant digits of the value. The successive values $x^{(k)}$ are computed and at each iteration. When $x^{(k+1)}-x^{(k)}=@.0$, Then $x^{(k+1)}$ and $x^{(k)}$ are equal stochastically. The computations of the sequence $x^{(k)}$ are stopped when for an index like k_{opt}, the number of common significant digits in the difference between $x^{(k_{opt})}$ and $x^{(k+1)}$ become zero. In this case, one can say, before k_{opt}th iteration, $x^{(k+1)}-x^{(k)}$ has exact significant digits. But, the computations after k_{opt} iteration are useless. In other words, the number of iteration in k_{opt} has been optimized. Also, according to Theorem (3.1), the significant digits of the last approximation $x^{(k)}$ are in common with the value of the exact solution x^*. Therefore, $x^{(k)}$ is an approximation of x^*.

4 Numerical Examples

In this section, three examples of AVE are solved by applying the CADNA library based on Algorithm 1. The programs have been provided by C++ code in double precision on Linux operating system.

Example 4.1 *[35] Consider AVE problem where the data (A, b) are:*

$$A=\begin{bmatrix}10&1&2&0\\1&11&3&1\\0&2&12&1\\1&7&0&13\end{bmatrix}, \qquad b=\begin{bmatrix}10\\-9\\8\\-20\end{bmatrix}$$

Since all singular values of A exceed 1, $SVD(A) = (17.4349, 12.2629, 9.6389, 7.5984)$, this problem is uniquely solvable. By $x^{(0)} = [0,0,0,0]^T$, and applying two mentioned iterative methods, Algorithm 1 converges to the unique solution $x=(1,1,1,-1)^T$. According to Tables 1 and 2, the Jacobi method at optimal iteration 42 and the Gauss-Seidel method at optimal iteration 15 converge to the exact solution. The optimal iteration and optimal solutions of this AVE are listed in Tables 1 and 2.

Example 4.2 *[35] Consider AVE problem where the matrix A given by:*

$a_{ii}=4n, a_{i+1,i}=a_{i,i+1}=n$, else $a_{ij}=0.5, i=1,2,\ldots n$. Let $b=(A-I)e$ where I is the identity matrix order n and e is $n\times 1$ vector whose elements are all equal to unity such that $x=(1,1,...,1)^T$ is the unique solution. The optimal iterations of this AVE in order to verify n by two

iterative methods are listed in Table 3. As we observe in the table, when n increases, the optimal number of iterations for the Gauss-Seidel method is less than the Jacobi method. Therefore the Gauss-Seidel method is faster than the Jacobi method.

Example 4.3 *[35] In this example, we are going to apply the proposed algorithm by applying the iterative methods to find the numerical solution of two-point boundary value problem of the form:*

$$\begin{cases} \frac{d^2u}{dt^2} - |u| = f(t), f(t) \in C[a,b]. \\ u(a) = u_a, u(b) = u_b, a \leqslant t \leqslant b. \end{cases} \tag{4.19}$$

We consider a uniform mesh as:

$$\begin{cases} t_i = a + ih, i = 0, 1, 2, \ldots, n, n+1, \\ h = \frac{b-a}{n+1} \end{cases}$$

where n is number of insert point, and h is the step-size of mesh, and $t_0 = a$ and $t_n = b$ are the boundary-points of the mesh. Assume that, (4.19) has unique solution. That is there exists a function $u(t) \in C^2[a,b]$ satisfied (4.19). Hence, substituting the mesh-points t_i into (4.19), we obtain:

$$\begin{cases} u''(t_i) - |u(t_i)| = f(t_i), i = 0, 1, 2, \ldots, n. \\ u(t_0) = u_a, u(t_{n+1}) = u_b. \end{cases} \tag{4.20}$$

By denoting $u(t_i) = u_i$ and using finite differences method to discrete Eq. (4.20), it can be found that:

$$\begin{cases} \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} - |u_i| = f(t_i), \\ i = 0, 1, 2, \ldots, n. \\ u_0 = u_a, u_{n+1} = u_b. \end{cases} \tag{4.21}$$

Problem (4.21) shows a system of linear algebraic equations as the following absolute value equation:

$$Ax - |x| = b,$$

where

$$A = \frac{1}{h^2} \begin{bmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & 2 & \end{bmatrix}_{n \times n},$$

$$b = \begin{bmatrix} f(t_1) - \frac{u_0}{h^2} \\ f(t_2) \\ \vdots \\ f(t_{n-1}) \\ f(t_n) - \frac{u_{n+1}}{h^2} \end{bmatrix}_{n \times 1}, x = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix}_{n \times 1}.$$

Consider the two-point boundary value problems as:

$$\begin{cases} u''(t) - |u(t)| = (1 - t^2), 0 \leqslant t \leqslant 1. \\ u(0) = -1, u(1) = 0. \end{cases} \tag{4.22}$$

By $n = 10$ the matrix A and vector b are as follows:

$$A = 121 \begin{bmatrix} -2 & 1 & & & & & & & \\ 1 & -2 & 1 & & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & & 1 & -2 & 1 & & \\ & & & & & 1 & 2 & & \end{bmatrix},$$

$$b = \begin{bmatrix} 121.9917 \\ 0.9669 \\ 0.9256 \\ \vdots \\ 0.3306 \\ 0.1736 \end{bmatrix}_{10 \times 1},$$

with $x^{(0)} = [0, 0, \ldots, 0, 0]^T$, the Jacobi iterative method after 794 iterations and the Gauss-Seidel method after 420 iterations, converge to the unique solution. In this case, the optimal solution listed as:

$$u_{opt} = \begin{cases} u_1 = -0.974466234231605E + 000 \\ u_2 = -0.932682867046500E + 000 \\ u_3 = -0.875200119565475E + 000 \\ u_4 = -0.802834562544975E + 000 \\ u_5 = -0.716662367858472E + 000 \\ u_6 = -0.618010414514031E + 000 \\ u_7 = -0.508445322712095E + 000 \\ u_8 = -0.389760506571099E + 000 \\ u_9 = -0.263961352523191E + 000 \\ u_{10} = -0.133248645872639E + 000 \end{cases}.$$

5 Conclusion

In this essay, we proposed an algorithm in order to approximate the solution of an AVE in the stochastic arithmetic. We have seen that by using the CESTAC method based on the stochastic arithmetic, and using the iterative methods to approximate the solution of AVE, we are able to validate the results step by step. We obtained the solution with the least number of iterations on the stochastic arithmetic. Also, the useless iterations are put away. By using the optimal termination

criterion which uses the computational zero, the iterative process is stopped correctly and computational time is saved, because many useless operations and iterations are not performed. Consequently, we observe that by simple calculation it is possible to determine the accuracy of the computed solution and, the CADNA library based on the stochastic arithmetic can play an important role to rely the numerical solution of AVE.

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