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Research Article

Coincidence and Common Fixed Point Results for α-(ψ, φ)-Contractive Mappings in Metric Spaces

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Abstract

Recently Samet et al. introduced the notion of α-ψ-contractive type mappings and established some fixed point theorems in complete metric spaces. In this paper, we introduce α-(ψ, φ)-contractive mappings and stablish coincidence and common fixed point theorems for two mapping in complete metric spaces. We present some examples to illustrate our results. As application, we establish an existence and uniqueness theorem for a solution of some integral equations.

Keywords : Fixed point; Coincidence point; Common fixed point; α-(ψ, φ)-contraction; Altering distance function; Integral equations.

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1 Introduction

Fixed point theory has wide and endless applications in many fields of engineering and science. Its core, the Banach contraction principle(see [3]), has attracted many researchers have generalize it in different aspects. Recently Samet et al. in [20] introduced the following concepts.

Let (X, d) be a metric space, T a self-map on X, $\alpha : X \times X \to [0, +\infty)$ be a function and $\psi : [0, +\infty) \to [0, +\infty)$ a nondecreasing function with $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t \geq 0$, where ψ^n is the nth iterate of ψ. Then T is called an α-ψ-contraction mapping whenever $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$. Also, T is called α-admissible whenever $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ [20].

In [20] the following theorem is proved.

Theorem 1.1 *([20], Theorems 2.1, 2.2 and 2.3) Let (X, d) a complete metric space and T be an α-admissible α-ψ-contractive mapping on X. Suppose that one of the following assertions holds:*

i) T is continuous,

ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.

If there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then there exists a $u \in X$ such that $Tu = u$.

Further if, for any $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, then T has

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a unique fixed point in X. More details about α-ψ-contraction can be found in [8, 9, 11, 13, 18, 20].

Khan et al. in [12] initiated the use of a control function in metric fixed point theory, which they called an altering distance function

Definition 1.1 *([12]) A function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function, if the following properties are satisfied:*

i) ψ is continuous and non-decreasing,

ii) $\psi(t) = 0$ if and only if $t = 0$.

Altering distance has been used in metric fixed point theory in a number of recent papers. (See, for examle [2, 6, 7, 14, 15, 17, 19].) Alber and Guerre-Delabriere [1] introduced the concept of weak contractions in Hilbert spaces. This concept was extended to metric spaces in [16].

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be weakly contractive (see [16]) if, for all $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)),$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is an altering distance function.

Choudhury et al. [4] introduced the concept of a generalized weakly contractive condition as follows.

Let (X, d) be a metric space, T a self-mapping of X. We shall call T a generalized weakly contractive mapping (see [4]) if, for all $x, y \in X$,

$$\begin{aligned}\psi(d(Tx, Ty)) &\leq \psi(m(x, y)) \\ &\quad -\varphi(\max\{d(x, y), d(y, Ty)\}),\end{aligned}$$

where

$$\begin{aligned}m(x, y) := \max\{&d(x, y), d(x, Tx), d(y, Ty), \\ &\frac{1}{2}[d(x, Ty) + d(y, Tx)]\},\end{aligned}$$

ψ is an altering distance function and $\varphi : [0, +\infty) \to [0, +\infty)$ is a continuous function with $\varphi(t) = 0$ if and only if $t = 0$.

We introduce the concept of an α-(ψ, φ)-contractive type mapping as follows.

Definition 1.2 *Let f and T be two self maps on a metric space (X, d). Then f and T are said to satisfy an α-(ψ, φ)-contractive condition if there exists a function $\alpha : X \times X \to [0, +\infty)$ such that, for all $x, y \in X$, we have*

$$\begin{aligned}\alpha(Tx, Ty)\psi(d(fx, fy)) \leq \psi(M(x, y)) \\ -\varphi(d(Tx, Ty)),\end{aligned} \tag{1.1}$$

where

$$\begin{aligned}M(x, y) = \max\{&d(Tx, Ty), d(Tx, fx), \\ &d(Ty, fy), \frac{1}{2}[d(Tx, fy) + d(fx, Ty)]\},\end{aligned} \tag{1.2}$$

$\psi : [0, +\infty) \to [0, +\infty)$ is an altering distance function, and $\varphi : [0, +\infty) \to [0, +\infty)$ is a continuous function with $\varphi(t) = 0$ if and only if $t = 0$.

If T is the identity map on X, then f is an α-(ψ, φ)-contractive mapping.

In this paper we establish some coincidence and common fixed point results for two self-mappings of a complete metric spaces which satisfying an α-(ψ, φ)-contractive condition. In addition, we study the existence and uniqueness of solutions for a class of integral equations.

We begin with the following definitions.

Definition 1.3 *([10]) Let f and g be self mappings of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. If $w = x$, then x is called a common fixed point of f and g.*

Definition 1.4 *([10]) Let (X, d) be a metric space and $f, g : X \to X$. The pair (f, g) is said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.*

Definition 1.5 *([10]) Let f and g be self mappings of a set X. The pair (f, g) is said to be weakly compatible if they commute at each coincidence point; (i.e. $fgx = gfx$, $x \in X$ whenever $fx = gx$).*

It is known that compatible mappings are weakly compatible mappings, but the converse is not true [5].

2 Coincidence and common fixed point results

We begin our study with the following result.

Theorem 2.1 *Let (X,d) a complete metric space and $\alpha : X \times X \to [0,+\infty)$ be a function such that, for all $x,y,z \in X$, we have*

$$\alpha(x,y) \geq 1 \& \alpha(y,z) \geq 1 \Rightarrow \alpha(x,z) \geq 1. \quad (2.3)$$

Let $f,T : X \to X$ satisfy α-(ψ,φ)-contractive condition (1.1).

We assume the following hypotheses:

i) $fX \subseteq TX$,

ii) f and T are continuous,

iii) the pair (f,T) is compatible,

iv) for all $x \in X$, we have

$$\alpha(fx,fy) \geq 1, \quad \forall\, y \in T^{-1}(fx). \quad (2.4)$$

Then f and T have a coincidence point $u \in X$; that is, $fu = Tu$.

Proof. Let x_0 be an arbitrary point in X. Since $fX \subseteq TX$, there exists an $x_1 \in X$ such that $Tx_1 = fx_0$. Again from $fX \subseteq TX$, we can choose $x_2 \in X$ such that $Tx_2 = fx_1$. Continuing this process we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$y_n = fx_n = Tx_{n+1}, \ \forall n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \quad (2.5)$$

Without loss of generality we may assume that $y_n \neq y_{n+1}$ for each n. For, if $y_n = y_{n+1}$ for some n then, from (2.5), $Tx_{n+1} = y_n = y_{n+1} = fx_{n+1}$, and T and f have a coincidence point.

Equation (2.5) implies that $x_{n+1} \in T^{-1}(fx_n)$. Using (2.4),

$$\alpha(y_n, y_{n+1}) = \alpha(fx_n, fx_{n+1}) \geq 1, \ \forall n \in \mathbb{N}_0. \quad (2.6)$$

Using (2.6) and induction, it follows that

$$\alpha(y_m, y_n) \geq 1, \ \forall m,n \in \mathbb{N}_0 \text{with } m < n. \quad (2.7)$$

Set $x = x_n$, $y = x_{n+1}$ in (1.1) and use (2.6) to obtain

$$\begin{aligned} &\psi(d(y_n, y_{n+1})) \quad (2.8)\\ &\leq \alpha(y_{n-1}, y_n)\psi(d(y_n, y_{n+1}))\\ &= \alpha(Tx_n, Tx_{n+1})\psi(d(fx_n, fx_{n+1}))\\ &\leq \psi(M(x_n, x_{n+1})) - \varphi(d(Tx_n, Tx_{n+1}))\\ &< \psi(M(x_n, x_{n+1})), \end{aligned}$$

since $d(Tx_n, Tx_{n+1}) = d(y_{n-1}, y_n) \neq 0$. Using property (ii) of ψ,

$$\begin{aligned} d(y_n, y_{n+1}) &< M(x_n, x_{n+1})\\ &= \max\{d(Tx_n, Tx_{n+1}),\\ &\qquad d(Tx_n, fx_n),\\ &\qquad d(Tx_{n+1}, fx_{n+1}),\\ &\qquad \frac{1}{2}[d(Tx_n, fx_{n+1})\\ &\qquad + d(fx_n, Tx_{n+1})]\}\\ &= \max\{d(y_{n-1}, y_n),\\ &\qquad\qquad d(y_n, y_{n+1})\}. \end{aligned}$$

since $d(fx_n, Tx_{n+1}) = 0$ and

$$\begin{aligned} \frac{d(Tx_n, fx_{n+1})}{2} &= \frac{d(y_{n-1}, y_{n+1})}{2}\\ &\leq \frac{1}{2}[d(y_{n-1}, y_n)\\ &\qquad + d(y_n, y_{n+1})]\\ &\leq \max\{d(y_{n-1}, y_n),\\ &\qquad d(y_n, y_{n+1})\}. \end{aligned}$$

If $d(y_{n-1}, y_n) \leq d(y_n, y_{n+1})$ for any n, then the above inequality becomes

$$d(y_n, y_{n+1}) < d(y_n, y_{n+1}),$$

a contradiction. Therefore

$$d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n)$$

for all n, so that $\{d(y_n, y_{n+1})\}$ is a positive non-decreasing sequence, and hence has a limit $r \geq 0$.

Using (2.8), and the fact that ψ and φ are continuous,

$$\begin{aligned} &\psi\left(\lim_n d(y_n, y_{n+1})\right) \leq\\ &\psi\left(\lim_n \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}\right)\\ &\qquad - \varphi(d(y_{n-1}, y_n)), \end{aligned}$$

or

$$\psi(r) \leq \psi(r) - \varphi(r),$$

which implies that $\varphi(r) = 0$, and hence

$$r = \lim_n d(y_n, y_{n+1}) = 0. \quad (2.9)$$

We shall now show that $\{y_n\}$ is a Cauchy sequence in X. Suppose, that $\{y_n\}$ is not a

Cauchy sequence. Then there exists an $\epsilon > 0$ and two subsequences $\{y_{m(k)}\}$ and $\{y_{n(k)}\}$ with $n(k) > m(k) \geq k$, such that

$$d(y_{m(k)}, y_{n(k)}) \geq \epsilon. \tag{2.10}$$

With $n(k)$ the smallest subsequence satisfying (2.10), one has

$$d(y_{m(k)}, y_{n(k)-1}) < \epsilon. \tag{2.11}$$

Using (2.10),(2.11) and the triangular inequality,

$$\begin{aligned} \epsilon \leq d(y_{m(k)}, y_{n(k)}) &\leq d(y_{m(k)}, y_{n(k)-1}) \\ &\quad + d(y_{n(k)-1}, y_{n(k)}) \\ &< \epsilon + d(y_{n(k)-1}, y_{n(k)}). \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.9), we obtain

$$\lim_n d(y_{m(k)}, y_{n(k)}) = \epsilon. \tag{2.12}$$

Again, using the triangular inequality we have

$$\begin{aligned} &\left| d(y_{m(k)-1}, y_{n(k)}) - d(y_{m(k)}, y_{n(k)}) \right| \leq \\ &\qquad d(y_{m(k)-1}, y_{m(k)}). \end{aligned}$$

Letting again $k \to \infty$ in the above inequality and using (2.9) and (2.12), we get

$$\lim_n d(y_{m(k)-1}, y_{n(k)}) = \epsilon. \tag{2.13}$$

On the other hand we have

$$\begin{aligned} d(y_{m(k)}, y_{n(k)}) &\leq d(y_{m(k)}, y_{n(k)+1}) \\ &\quad + d(y_{n(k)+1}, y_{n(k)}). \end{aligned}$$

Thanks to (2.9),(2.12), letting $k \to \infty$, we have from the above inequality that

$$\epsilon \leq \lim_n d(y_{m(k)}, y_{n(k)+1}). \tag{2.14}$$

Also, by the triangular inequality, we have

$$\begin{aligned} d(y_{m(k)}, y_{n(k)+1}) &\leq d(y_{m(k)-1}, y_{n(k)+1}) \\ &\quad + d(y_{m(k)-1}, y_{m(k)}) \\ &\leq d(y_{m(k)-1}, y_{n(k)}) \\ &\quad + d(y_{m(k)-1}, y_{m(k)}) \\ &\quad + d(y_{n(k)}, y_{n(k)+1}). \end{aligned}$$

Taking the limit as $k \to \infty$, and using (2.14), yieds

$$\epsilon \leq \lim_n d(y_{m(k)-1}, y_{n(k)+1}),$$

similarly, we can show that $\lim_n d(y_{m(k)-1}, y_{n(k)+1}) \leq \epsilon$, so

$$\lim_n d(y_{m(k)-1}, y_{n(k)+1}) = \epsilon. \tag{2.15}$$

Applying inequality (1.1) with $x = x_{m(k)}$ and $y = x_{n(k)+1}$, and using (2.7), we have

$$\begin{aligned} &\psi(d(y_{m(k)}, y_{n(k)+1})) \\ &\leq \alpha(y_{m(k)-1}, y_{n(k)})\psi(d(y_{m(k)}, y_{n(k)+1})) \\ &= \alpha(Tx_{m(k)}, Tx_{n(k)+1})\psi(d(fx_{m(k)}, fx_{n(k)+1})) \\ &\leq \psi\left(M(x_{m(k)}, x_{n(k)+1})\right) \\ &\qquad - \varphi(d(Tx_{m(k)}, Tx_{n(k)+1})) \\ &\leq \psi\left(M(x_{m(k)}, x_{n(k)+1})\right). \end{aligned} \tag{2.16}$$

Since ψ in nondecreasingn, (2.16) implies that

$$d(y_{m(k)}, y_{n(k)+1}) \leq M(x_{m(k)}, x_{n(k)+1}),$$

where

$$\begin{aligned} M(x_{m(k)}, x_{n(k)+1}) = \max\{ & \\ & d(Tx_{m(k)}, Tx_{n(k)+1}), \\ & d(Tx_{m(k)}, fx_{m(k)}), \\ & d(Tx_{n(k)+1}, fx_{n(k)+1}), \\ & \frac{1}{2}[d(Tx_{m(k)}, fx_{n(k)+1}) \\ & + d(fx_{m(k)}, Tx_{n(k)+1})]\} \\ = \max\{ & d(y_{m(k)-1}, y_{n(k)}), \\ & d(y_{m(k)-1}, y_{m(k)}), \\ & d(y_{n(k)}, y_{n(k)+1}), \\ & \frac{1}{2}[d(y_{m(k)-1}, y_{n(k)+1}) \\ & + d(y_{m(k)}, y_{n(k)})]\}. \end{aligned} \tag{2.17}$$

Using (2.13), (2.9), (2.15), and (2.11) in (2.17),

$$\lim_k M(x_{m(k)}, x_{n(k)+1}) = \epsilon. \tag{2.18}$$

Since

$$d(Tx_{m(k)}, Tx_{n(k)+1}) = d(y_{m(k)-1}, y_{n(k)}),$$

taking the limit of (2.16), using (2.14), (2.18), and the continuity of ψ and φ gives

$$\psi(\epsilon) \leq \psi(\epsilon) - \varphi(\epsilon),$$

which implies that $\varphi(\epsilon) = 0$ and hence that $\epsilon = 0$, which is a contradiction. Thus $\{y_n\}$ is a Cauchy sequence in X.

To prove that f and T have a coincidence point, from the completeness of (X, d), there exists a $u \in X$ such that

$$\lim_{n\to\infty} y_n = u. \tag{2.19}$$

From (2.5) and (2.19), we obtain

$$d(fx_n, u) \to 0, \quad d(Tx_n, u) \to 0. \tag{2.20}$$

Since the pair (f, T) is compatible,

$$d(T(fx_n), f(Tx_n)) \to 0. \tag{2.21}$$

Using the continuity of f, T and (2.20), we have

$$\begin{aligned} d(T(fx_n), Tu) &\to 0, \\ d(f(Tx_n), fu) &\to 0. \end{aligned} \tag{2.22}$$

The triangular inequality and (2.5) yield

$$\begin{aligned} d(Tu, fu) \leq\ & d(Tu, T(fx_n)) \\ & + d(T(fx_n), f(Tx_n)) \\ & + d(f(Tx_n), fu), \end{aligned} \tag{2.23}$$

combining (2.21) and (2.22) and letting $n \to \infty$ in (2.23), we obtain $d(Tu, fu) \leq 0$; that is, $d(Tu, fu) = 0$ and hence $Tu = fu$. Therefore u is a coincidence point of f and T. ■

Theorem 2.2 *Let (X, d) a complete metric space and $\alpha : X \times X \to [0, +\infty)$ be a function which satisfies (2.3). Let $f, T : X \to X$ satisfying α-(ψ, φ)-contractive condition, with $M(x, y)$ replaces by*

$$\begin{aligned} M(x, y) = \max\{& d(Tx, Ty), \\ & \frac{1}{2}[d(Tx, fx) + d(Ty, fy)], \\ & \frac{1}{2}[d(Tx, fy) + d(fx, Ty)]\}. \end{aligned}$$

We assume the following hypotheses:

i) $fX \subseteq TX$,

ii) f and T are continuous,

iii) the pair (f, T) is compatible,

iv) for all $x \in X$, we have

$$\alpha(fx, fy) \geq 1, \quad \forall\ y \in T^{-1}(fx).$$

Then f and T have a coincidence point $u \in X$, that is, $fu = Tu$.

Proof. Since

$$\begin{aligned} M(x, y) \leq \max\{& d(Tx, Ty) \\ & , d(Tx, fx), d(Ty, fy), \\ & \frac{1}{2}[d(Tx, fy) + d(fx, Ty)]\}, \end{aligned}$$

the result follows from Theorem 2.1. ■

In the next theorem we omit the continuity hypotheses and on f and T, and the compatibility of the pair (f, T).

Theorem 2.3 *Let (X, d) a complete metric space and $\alpha : X \times X \to [0, +\infty)$ be a function which satisfies (2.3). Let $f, T : X \to X$ satisfying α-(ψ, φ)-contractive condition, with $M(x, y)$ replaces by*

$$\begin{aligned} M(x, y) = \max\{& d(Tx, Ty), \\ & \frac{1}{2}[d(Tx, fx) + d(Ty, fy)], \\ & \frac{1}{2}[d(Tx, fy) + d(fx, Ty)]\}. \end{aligned}$$

We assume the following hypotheses:

i) $fX \subseteq TX$,

ii) TX is a closed subset of (X, d),

iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}_0$.

iv) for all $x \in X$, we have

$$\alpha(fx, fy) \geq 1, \quad \forall\ y \in T^{-1}(fx).$$

Then f and T have a coincidence point $u \in X$, that is, $fu = Tu$.

Proof. We take the same sequences $\{x_n\}$ and $\{y_n\}$ as in the proof of Theorem 2.1. Then $\{y_n\}$ is a Cauchy sequence in (X, d). Hence there exists a $v \in X$ such that

$$\lim_n y_n = v. \tag{2.24}$$

Since TX is a closed subset of (X,d), there exist a $u \in X$ such that

$$y_n = Tx_{n+1} \longrightarrow Tu.$$

Therefore $v = Tu$. On the other hand, from (2.6) and hypothesis (iii), we have

$$\alpha(y_n, v) \geq 1, \quad \forall n \in \mathbb{N}_0,$$

or

$$\alpha(Tx_n, Tu) \geq 1, \quad \forall n \in \mathbb{N}. \tag{2.25}$$

Applying inequality (1.1) with $x = x_n$ and $y = u$, and using (2.25), we obtain

$$\begin{aligned} \psi(d(y_n, fu)) &= \psi(d(fx_n, fu)) \quad (2.26)\\ &\leq \alpha(Tx_n, Tu)\psi(d(fx_n, fu))\\ &\leq \psi(M(x_n, u)) - \varphi(d(Tx_n, Tu))\\ &\leq \psi(M(x_n, u)), \end{aligned}$$

where

$$\begin{aligned} M(x_n, u) &= \max\{d(Tx_n, Tu),\\ &\quad \frac{1}{2}[d(Tx_n, fx_n) + d(Tu, fu)],\\ &\quad \frac{1}{2}[d(Tx_n, fu) + d(fx_n, Tu)]\}\\ &= \max\{d(y_{n-1}, v),\\ &\quad \frac{1}{2}[d(y_{n-1}, y_n) + d(v, fu)],\\ &\quad \frac{1}{2}[d(y_{n-1}, fu) + d(y_n, v)]\}. \end{aligned}$$

Since ψ is a nondecreasing function, (2.26) implies that

$$\begin{aligned} d(y_n, fu) &\leq M(x_n, u)\\ &= \max\{d(y_{n-1}, v),\\ &\quad \frac{1}{2}[d(y_{n-1}, y_n) + d(v, fu)],\\ &\quad \frac{1}{2}[d(y_{n-1}, fu) + d(y_n, v)]\}. \end{aligned}$$

Letting $n \to \infty$ in above inequality and using (2.9) and (2.24), we find

$$\begin{aligned} d(v, fu) &\leq \max\{0, \frac{1}{2}[0 + d(v, fu)],\\ &\quad \frac{1}{2}[d(v, fu) + 0]\}\\ &= \frac{1}{2}d(v, fu), \end{aligned}$$

which implies $\frac{1}{2}d(v, fu) \leq 0$; that is $d(v, fu) = 0$ and hence

$$Tu = v = fu.$$

Therefore u is a coincidence point of f and T. ■

The next result is an immediate consequence of Theorem 2.2 and 2.3 by taking $T = I_X$ (the identity mapping on X).

Corollary 2.1 *Let (X,d) be a complete metric space, $\alpha : X \times X \to [0, +\infty)$ be a function which satisfies (2.3) and let f be a self-map on X such that*

$$\alpha(fx, f(fx)) \geq 1, \quad \forall x \in X.$$

Suppose that, for all $x, y \in X$, we have

$$\begin{aligned} \alpha(x,y)\psi(d(fx, fy)) &\leq \psi(M(x,y))\\ &\quad - \varphi(d(x,y)), \end{aligned}$$

where

$$\begin{aligned} M(x,y) &= \max\{d(x,y),\\ &\quad \frac{1}{2}[d(x,fx) + d(y,fy)],\\ &\quad \frac{1}{2}[d(x,fy) + d(fx,y)]\}, \end{aligned}$$

ψ and φ are as in Definition 1.2.

Suppose that one of the following assertions holds:

a) f is continuous,

b) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}_0$.

Then f has a fixed point.

We shall now prove another existence and uniqueness theorem for a pair of maps.

Theorem 2.4 *In addition to the hypotheses of both Theorems 2.2 and 2.3, suppose that, for any $x, y \in X$, there exists $u \in X$ such that*

$$\alpha(fx, fu) \geq 1, \quad \alpha(fy, fu) \geq 1, \tag{2.27}$$

and, also in Theorem 2.3 that pair (f, T) is weakly compatible. Then f and T have a unique common fixed point; that is, there exists a unique $z \in X$ such that $fz = Tz = z$.

Proof. Referring to Theorem 2.2 or Theorem 2.3, the set of coincidence points of f and T is non-empty. We shall show that, if x and y are coincidence points of f and T; that is, if $fx = Tx$ and $fy = Ty$, then

$$Tx = Ty. \tag{2.28}$$

From (2.27) there exists a $u_0 \in X$ such that

$$\alpha(fx, fu_0) \geq 1, \quad \alpha(fy, fu_0) \geq 1. \tag{2.29}$$

Similar to the proof of Theorem 2.1, we define sequences $\{u_n\}$ and $\{p_n\}$ as follows:

$$p_n = fu_n = Tu_{n+1}, \ \forall n \in \mathbb{N}_0, \tag{2.30}$$

and

$$\alpha(p_m, p_n) \geq 1, \ \forall m, n \in \mathbb{N}_0 \text{with } m < n. \tag{2.31}$$

Using (2.29), (2.31) and (2.3), we have

$$\alpha(fx, p_n) \geq 1, \ \ \alpha(fy, p_n) \geq 1, \ \forall n \in \mathbb{N}_0. \tag{2.32}$$

Applying inequality (1.1) with $x = x$ and $y = u_n$, and using (2.32), we obtain

$$\begin{aligned} \psi(d(Tx, p_n)) &\leq \alpha(Tx, p_{n-1})\psi(d(Tx, p_n)) \\ &= \alpha(Tx, Tu_n)\psi(d(fx, fu_n)) \\ &\leq \psi(M(x, u_n)) \\ &\qquad - \varphi(d(Tx, Tu_n)) \\ &\leq \psi(M(x, u_n)), \end{aligned} \tag{2.33}$$

where

$$\begin{aligned} M(x, u_n) &= \max\{d(Tx, Tu_n), \\ &\quad \frac{1}{2}[d(Tx, fx) + d(Tu_n, fu_n)], \\ &\quad \frac{1}{2}[d(Tx, fu_n) + d(fx, Tu_n)]\} \\ &= \max\{d(Tx, p_{n-1}), \\ &\quad \frac{1}{2}[0 + d(p_{n-1}, p_n)], \\ &\quad \frac{1}{2}[d(Tx, p_n) + d(Tx, p_{n-1})]\} \\ &= \max\{d(Tx, p_{n-1}), \frac{1}{2}d(p_{n-1}, p_n), \\ &\quad \frac{1}{2}[d(Tx, p_n) + d(Tx, p_{n-1})]\}. \end{aligned}$$

Since

$$\frac{1}{2}d(p_{n-1}, p_n) \leq \frac{1}{2}\left[d(p_{n-1}, Tx) + d(Tx, p_n)\right],$$

it follows that

$$\begin{aligned} M(x, u_n) &= \max\{d(Tx, p_{n-1}), \\ &\quad \frac{1}{2}[d(Tx, p_n) + d(Tx, p_{n-1})]\}. \end{aligned} \tag{2.34}$$

Since ψ is a nondecreasing function, (2.33) implies that

$$\begin{aligned} d(Tx, p_n) &\leq M(x, u_n) \\ &= \max\{d(Tx, p_{n-1}), \\ &\quad \frac{1}{2}[d(Tx, p_n) + d(Tx, p_{n-1})]\}, \end{aligned} \tag{2.35}$$

Now, if $M(x, u_n) = d(Tx, p_{n-1})$, then $d(Tx, p_n) \leq d(Tx, p_{n-1})$, and, if $M(x, u_n) = \frac{1}{2}[d(Tx, p_n) + d(Tx, p_{n-1})]$, then

$$d(Tx, p_n) \leq \frac{1}{2}[d(Tx, p_n) + d(Tx, p_{n-1})],$$

which implies that $d(Tx, p_n) \leq d(Tx, p_{n-1})$.

Therefore, for any $n \in \mathbb{N}$,

$$d(Tx, p_n) \leq d(Tx, p_{n-1}).$$

It follows that the sequence $\{d(Tx, p_n)\}$ is monotonic non-increasing. Hence, there exists an $r \geq 0$ such that

$$\lim_n d(Tx, p_n) = r. \tag{2.36}$$

Letting $n \to \infty$ in (2.33) and using (2.34) and (2.36) and the continuity of ψ and φ, we get that

$$\begin{aligned} \psi(r) &\leq \psi\left(\max\{r, \frac{1}{2}(r + r)\}\right) - \varphi(r) \\ &\leq \psi\left(\max\{r, \frac{1}{2}(r + r)\}\right), \end{aligned}$$

which implies that $\varphi(r) = 0$ and hence that $r = 0$. Thus we have

$$\lim_{n\to\infty} d(Tx, p_n) = 0. \tag{2.37}$$

Similarly, one can show that

$$\lim_{n\to\infty} d(Ty, p_n) = 0. \tag{2.38}$$

Using (2.37) and (2.38), the uniqueness of the limit gives us $Tx = Ty$.

Let us denote

$$z := fx = Tx.$$

By the compatibility of the pair (f,T) in Theorem 2.2 or the weakly compatibility of the pair (f,T) in Theorem 2.3, we have

$$Tz = Tfx = fTx = fz,$$

which implies that z is a coincidence point of f and T. From (2.28) we get

$$Tz = Tx = z.$$

This proves that z is a common fixed point of f and T.

To prove uniqueness, suppose there is another common fixed point w; that is,

$$fw = Tw = w.$$

This implies that w is a coincidence point of f and T. From (2.28) this implies that

$$w = Tw = Tz = z,$$

which yields the uniqueness of the common fixed point. ■

3 Some examples

In this section we present some examples which illustrate our results.

Example 3.1 *Let $X = \mathbb{R}$ be endowed with the standard metric $d(x,y) = |x-y|$ for all $x, y \in X$ and let $\alpha : X \times X \to [0,\infty)$ be defined by*

$$\alpha(x,y) = \begin{cases} 1 & x \geq y \geq 0 \\ 0 & otherwise. \end{cases}$$

Obviously, (X,d) is a complete metric space and (2.3) is satisfied with α.

Define also $f,T : X \to X$ and $\psi, \varphi : [0,+\infty) \to [0,+\infty)$ by

$$f(x) = \begin{cases} \ln(1+\frac{x^2}{2}) & x \geq 0, \\ x^2 & x < 0, \end{cases}$$

and

$$T(x) = \begin{cases} x^2 & x \geq 0, \\ x & x < 0, \end{cases}$$

and $\psi(t) = t^2$ and $\varphi(t) = \dfrac{3}{4}t^2$. Then all the hypotheses of Theorem 2.1, 2.2 and 2.4 hold.

Proof. The proof of (i) and (ii) is clear. To prove (iii), let $\{x_n\}$ be any sequence in X such that

$$\lim_n fx_n = \lim_n Tx_n = t.$$

for some $t \in X$. Since $fx_n \geq 0$, we have $t \geq 0$. Since $Tx_n \to t$ as $n \to \infty$, we have $\{x_n\}$ has at most only finitely many elements lower than 0. Thus, $fx_n = \ln(1+\frac{x_n^2}{2})$ and $Tx_n = x_n^2$ for all $n \in \mathbb{N}$ except at most for finitely many elements, and we have $x_n^2 \to t$ and $x_n^2 \to 2(e^t - 1)$ as $n \to \infty$. By uniqueness of limit $2(e^t - 1) = t$ and hence $t = 0$. Thus, $fx_n \to 0$ and $Tx_n \to 0$ as $n \to \infty$. Since f and T are continuous, we have

$$\begin{aligned} \lim_n d(T(fx_n), f(Tx_n)) &= d(T0, f0) \\ &= d(0,0) = 0. \end{aligned}$$

Thus, the pair (f,S) is compatible.

To prove (iv), let $x, y \in X$ be such that $y \in T^{-1}(fx)$. If $x \geq 0$, then

$$Ty = fx = \ln(1+\frac{x^2}{2}) \geq 0.$$

In this case, we must have $Ty = y^2$. Thus, $y^2 = \ln(1+\frac{x^2}{2})$. Hence,

$$fy = \ln(1+\frac{y^2}{2}) \leq \frac{y^2}{2} \leq y^2 = \ln(1+\frac{x^2}{2}) = fx,$$

also, $fx \geq 0$ and $fy \geq 0$. Therefore, we have $\alpha(fx,fy) = 1$.

If $x < 0$, then

$$Ty = fx = x^2 \geq 0.$$

In this case, we must have $Ty = y^2$. Thus, $y = -x$. Hence,

$$fy = \ln(1+\frac{y^2}{2}) \leq \frac{y^2}{2} \leq y^2 = (-x)^2 = fx,$$

also, $fx \geq 0$ and $fy \geq 0$. Therefore, we have $\alpha(fx,fy) = 1$.

In order to show that f and T do satisfy the α-(ψ,φ)-contractive condition (1.1), let $x, y \in X$,

Case 1. $x \geq 0$, $y \geq 0$ and $x \geq y$.

If we use mean value theorem, then we have

$$\begin{aligned} d(fx,fy) &= \left|\ln\left(1+\frac{x^2}{2}\right) - \ln\left(1+\frac{y^2}{2}\right)\right| \\ &\leq \frac{1}{2}|x^2 - y^2| = \frac{1}{2}d(Tx,Ty) \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} d(fx,fy)^2 &\leq \frac{1}{4}d(Tx,Ty)^2 \\ &= d(Tx,Ty)^2 - \frac{3}{4}d(Tx,Ty)^2 \\ &\leq M(x,y)^2 - \frac{3}{4}d(Tx,Ty)^2. \end{aligned}$$

In this case, we have $\alpha(Tx, Ty) = \alpha(x^2, y^2) = 1$. Therefore, we obtain

$$\begin{aligned} \alpha(Tx,Ty)\psi(d(fx,fy)) &\leq \psi(M(x,y)) \\ &\quad -\varphi(d(Tx,Ty)). \end{aligned}$$

Case 2. ($x \geq 0$, $y \geq 0$ and $x < y$) or $x < 0$ or $y < 0$.

In this case, we have $\alpha(Tx, Ty) = 0$. Therefore, we have

$$\begin{aligned} \alpha(Tx,Ty)\psi(d(fx,fy)) &= 0 \\ &\leq \frac{1}{4}d(Tx,Ty)^2 \\ &= d(Tx,Ty)^2 \\ &\quad -\frac{3}{4}d(Tx,Ty)^2 \\ &\leq M(x,y)^2 \\ &\quad -\frac{3}{4}d(Tx,Ty)^2 \\ &= \psi(M(x,y)) \\ &\quad -\varphi(d(Tx,Ty)). \end{aligned}$$

Thus, f and T satisfy all the hypotheses of Theorem 2.1 and 2.2. Therefore, f and T have a coincidence point.

Finally, if $x, y \in X$ then by definition of f, we have $fx \geq 0$ and $fy \geq 0$. Now, if

$$u := \begin{cases} x & fx \geq fy, \\ y & fx < fy, \end{cases}$$

then $\alpha(fx, fu) = 1$ and $\alpha(fy, fu) = 1$. Thus, by applying Theorem 2.4, we conclude that f and T have a unique common fixed point. In fact, 0 is the unique common fixed poin of f and T. ■

We now give an example involving a function f that is not continuous.

Example 3.2 *Let $X = [0, +\infty)$ be endowed with the standard metric $d(x, y) = |x-y|$ for all $x, y \in X$ and let $\alpha : X \times X \to [0, \infty)$ be defined by*

$$\alpha(x,y) = \begin{cases} 1 & 0 \leq x, y \leq \frac{1}{2} \\ 0 & \textit{otherwise}. \end{cases}$$

Obviously, (X, d) is a complete metric space and (2.3) is satisfied with α.

Define also $f, T : X \to X$ and $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ by

$$f(x) = \begin{cases} \frac{x}{3} & 0 \leq x \leq 1, \\ \frac{1}{x+1} & x > 1, \end{cases},$$

$T(x) = \frac{x}{2}$, $\psi(t) = t$ and $\varphi(t) = \dfrac{1}{3}t$. Then all the hypotheses of Theorem 2.3 and 2.4 hold.

Proof. The proof of (i) and (ii) is clear. To prove (iii), if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$ and $x_n \to x$ as $n \to \infty$, then $\{x_n\} \subseteq [0, \frac{1}{2}]$ and hence $x \in [0, \frac{1}{2}]$. This implies that $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}_0$.

To prove (iv), let $x, y \in X$ be such that $y \in T^{-1}(fx)$. If $0 \leq x \leq 1$, then

$$Ty = fx = \frac{x}{3} < \frac{1}{2},$$

since $Ty = \frac{y}{2}$, we must have $y = \frac{2}{3}x$. Hence,

$$fy = f(\frac{2}{3}x) = \frac{2}{9}x < \frac{1}{2}.$$

Therefore, we have $\alpha(fx, fy) = 1$. If $x > 1$, then

$$Ty = fx = \frac{1}{x+1} < \frac{1}{2},$$

since $Ty = \frac{y}{2}$, we must have $y = \frac{2}{x+1} < 1$. Hence,

$$fy = f(\frac{2}{x+1}) = \frac{1}{3}\frac{2}{x+1} < \frac{1}{2}.$$

Therefore, we have $\alpha(fx, fy) = 1$.

In order to show that f and T do satisfy the α-(ψ, φ)-contractive condition (1.1), let $x, y \in X$,

Case 1. $0 \leq x, y \leq 1$.

$$\begin{aligned} d(fx,fy) &= \left|\frac{x}{3} - \frac{y}{3}\right| = \frac{2}{3}\left|\frac{x}{2} - \frac{y}{2}\right| \\ &= \frac{2}{3}d(Tx,Ty) \\ &\leq d(Tx,Ty) - \frac{1}{3}d(Tx,Ty) \\ &\leq M(x,y) - \frac{1}{3}d(Tx,Ty). \end{aligned}$$

In this case, we have $\alpha(Tx, Ty) = \alpha(\frac{x}{2}, \frac{y}{2}) = 1$. Therefore, we obtain

$$\begin{aligned} \alpha(Tx,Ty)\psi(d(fx,fy)) &\leq \psi(M(x,y)) \\ &\quad -\varphi(d(Tx,Ty)). \end{aligned}$$

Case 2. $x > 1$ or $y > 1$.

In this case, we have $\alpha(Tx, Ty) = 0$. Therefore, we have

$$\begin{aligned}\alpha(Tx,Ty)\psi(d(fx,fy)) &= 0\\ &\leq \frac{2}{3}d(Tx,Ty)\\ &= d(Tx,Ty)\\ &\quad -\frac{1}{3}d(Tx,Ty)\\ &\leq M(x,y)\\ &\quad -\frac{1}{3}d(Tx,Ty)\\ &= \psi(M(x,y))\\ &\quad -\varphi(d(Tx,Ty)).\end{aligned}$$

Thus, f, T, α, ψ and φ satisfy all the hypotheses of Theorem 2.3. Therefore, f and T have a coincidence point.

Finally, if $x, y \in X$ then by definition of f, we have $0 \leq fx, fy \leq \frac{1}{2}$. Now, if $u := x$ then $\alpha(fx, fu) = 1$ and $\alpha(fy, fu) = 1$. Thus, by applying Theorem 2.4, we conclude that f and T have a unique common fixed point. In fact, 0 is the unique common fixed poin of f and T. ■

Example 3.3 *Let $X = \{1, 2, 3, 4\}$ endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$ and let $\alpha : X \times X \to [0, +\infty)$ be defined by*

$$\alpha(x,y) = \begin{cases} 1 & (x,y) = (1,1),(2,2),(3,3),\\ & \qquad\quad (4,4),(4,1),(4,2),\\ 0 & \qquad\qquad\qquad \textit{otherwise}.\end{cases}$$

Clearly, (X, d) is a complete metric space and (2.3) is satisfied with α.

Now, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$ and $x_n \to x$ as $n \to \infty$, then there exists $k \in \mathbb{N}$ such that $x_n = x$ for all $n \geq k$. Therefore, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}_0$.

Let $\psi, \varphi : [0, \infty) \to [0, \infty)$ be defined by $\psi(t) = t^3 + t$ and $\varphi(t) = \frac{t}{1+t^2}$ and selfmaps f and T on X be given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4\\ 2 & 1 & 2 & 4\end{pmatrix}$$

and

$$T = \begin{pmatrix} 1 & 2 & 3 & 4\\ 4 & 1 & 2 & 3\end{pmatrix}.$$

It is easy to see that, f and T satisfy all the conditions given in Theorem 2.3, and so f and T have a coincidence point. In fact, 2 and 3 are coincidence points of f and T.

4 Application to integral equations

In this section we study an existence and uniqueness of solutions to a class of integral equations.

Consider the integral equation

$$x(t) = \int_a^b K(t, s, x(s))ds, \forall t \in [a, b], \quad (4.39)$$

where $b > a \geq 0$ and $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$. The purpose of this section is to give an existence theorem for a solution of (4.39) using Corollary 2.1.

Consider the space $X := \mathcal{C}[a, b]$ of real continuous functions defined on $[a, b]$. Obviously this space, with the metric given by

$$\begin{aligned} d(x,y) &= \|x-y\|_\infty\\ &= \sup_{a\leq t\leq b}|x(t)-y(t)|, \quad \forall\, x,y\in X,\end{aligned}$$

is a complete metric space.

Let $f : X \to X$ be the mapping defined by

$$fx(t) = \int_a^b K(t, s, x(s))ds, \qquad \forall\ t \in [a, b].$$

for all $x \in X$. Then the existence of a solution to (4.39) is equivalent to the existence of a fixed point of f.

We will now prove the following result.

Theorem 4.1 *Suppose that the following hypotheses hold:*

(i) $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous;

(ii) for all $t, s \in [a, b]$ and $x \in X$, we have $|K(t, s, x(s))| \leq \dfrac{1}{b-a}$;

(iii) for all $s, t \in [a, b]$ and $x, y \in X$ with $\|x\|_\infty, \|y\|_\infty \leq 1$, we have

$$|K(t, s, x(s)) - K(t, s, y(s))|^2 \leq p(t, s)\ln(1 + |x(s) - y(s)|^3),$$

where $p : [a, b] \times [a, b] \to [0, +\infty)$ is a continuous function satisfying

$$\sup_{a \leq t \leq b} \int_a^b p(s, t)ds < \frac{1}{b - a}.$$

Then, the integral equation (4.39) has a unique solution $w \in X$.

Proof. From condition (iii), for all $t \in [a, b]$ and $x, y \in X$ such that $\|x\|_\infty, \|y\|_\infty \leq 1$, we have

$$\begin{aligned}
&|fx(t) - fy(t)|^2 \\
&\leq \left(\int_a^b |K(t, s, x(s)) - K(t, s, y(s))|ds\right)^2 \\
&\leq \int_a^b 1^2 ds \int_a^b |K(t, s, x(s)) - K(t, s, y(s))|^2 ds \\
&\leq (b - a) \int_a^b p(t, s)\ln(1 + |x(s) - y(s)|^3)ds \\
&\leq (b - a) \int_a^b p(t, s)\ln(1 + d(x, y)^3)ds \\
&= (b - a) \left(\int_a^b p(t, s)ds\right) \ln(1 + d(x, y)^3) \\
&< \ln(1 + d(x, y)^3) \\
&= d(x, y)^2 - \left(d(x, y)^2 - \ln(1 + d(x, y)^3)\right) \\
&\leq M(x, y)^2 - \left(d(x, y)^2 - \ln(1 + d(x, y)^3)\right).
\end{aligned}$$

Therefore

$$\left(\sup_{a \leq t \leq b} |fx(t) - fy(t)|\right)^2 \leq M(x, y)^2 - \left(d(x, y)^2 - \ln\left(1 + d(x, y)^3\right)\right).$$

Set $\psi(t) = t^2$ and $\varphi(t) = t^2 - \ln\left(1 + t^3\right)$. Then

$$\psi(d(fx, fy)) \leq \psi\left(M(x, y)\right) - \varphi\left(d(x, y)\right)$$

for all $x, y \in X$ such that $\|x\|_\infty, \|y\|_\infty \leq 1$. Defne the function $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \|x\|_\infty, \|y\|_\infty \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

For all $x, y \in X$ we have

$$\alpha(x, y)\psi(d(fx, fy)) \leq \psi\left(M(x, y)\right) - \varphi\left(d(x, y)\right).$$

Then, f is an α-(ψ, φ)-contractive mapping

Clearly, (2.3) is satisfied with α. From condition (ii), for all $x \in X$, we get $\|fx\|_\infty \leq 1$. Therefore

$$\alpha(fx, f(fx)) = 1, \quad \forall\, x \in X.$$

Moreover, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$ and $x_n \to x$ as $n \to \infty$, then $\|x_n\|_\infty \leq 1$ for all $n \in \mathbb{N}_0$, and hence $\|x\|_\infty \leq 1$. This implies that $\alpha(x_n, x) = 1$ for all $n \in \mathbb{N}_0$.

All of the hypotheses of Corollary 2.1 are satisfied. On the other hand, since $\|fx\|_\infty, \|fy\|_\infty \leq 1$ for all $x, y \in X$, we have $\alpha(fx, fy) = 1$. Therefore, condition (2.27) of Theorem 2.4 is also satisfied with $u = x$. Thus f has a unique fixed point $w \in X$; that is, w is the unique solution of the integral equation (4.39). ■

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