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Research Article

Stochastic Two-stage Network-structures Under P-models: A DEA Based Approach

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Abstract

Regular network data envelopment analysis (NDEA) models for calculating the efficiency of decision-making units (DMUs) with two-stage construction requires the data set to be deterministic. In the real world, however, there are many observations that have stochastic behavior. This paper proposes a network DEA with two-stage structures models by the attention to stochastic data. The stochastic two-stage network DEA models are formulated based on the P-models of chance constrained programming and leader-follower concepts. On the basis of probability distribution properties and by the assumption of stochastic single factor components of data set, the probabilistic form of models is transformed to the equivalent deterministic linear programming model. In addition, the relationship between each stage as leader or follower is discussed. A real case on 16 commercial banks in China has been clarified to confirm the applicability of the proposed approaches under different levels.

Keywords : Stochastic DEA; Chance constraint model; Two-stage system; Performance evaluation.

1 Introduction

In the non-parametric approach, applied for evaluating the performance of some decision-making units (DMUs), data envelopment analysis (DEA) is a method that consumes similar inputs to produce similar outputs, without knowing any function of input/output and weights. Many studies extended DEA from the view of theoretical and empirical, in several fields of science, and engineering such as healthcare, agriculture, banking and supply chains. For more studies, the reader can refer to [24], [10], [21], [6], [31], [7], [30]. In many situations, we confront DMUs with two process structures and intermediate measures. Namely outputs of initial stage use as inputs for next stage. At first, standard DEA approach used by Seiford and Zhu [22] to evaluate the efficiency of commercial banks of US without considering the intermediate measures [28]. For example, to perform an efficient status, the inputs of the second stage should be reduced, while these inputs are outputs of the first stage. Kao and Hwang [13] Considered each stage as an in-

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dependent system and the efficiency of the total system as the product of the efficiencies of every stage. Liang et al. [16, 17], introduced several two-stage network models according to the concepts of non-cooperative and cooperative games theory. Optimal system design (OSD) network DEA models proposed by Wei and Chang [25] to help DMUs to find the optimal portfolios as inputs and outputs in terms of profit maximization.
In the mentioned network models the variability and uncertainty of the data set of a production process have ignored. Although, some researchers have been done to consider the uncertainty data in two-stage DEA models based on fuzzy theory ([18], [15], [1], [9], [19] and [11]), there is a little work on two-stage network DEA models in terms of stochastic data. Zhou et al. [29] proposed a stochastic network DEA model in terms of centralized control organization mechanism. Their model is a chance-constrained programming where optimize expected value where fall in the class of E-Model [4].
In this study, we focus on the class of P-Model where introduced by Charnes and Cooper [4] and propose a stochastic two-stage DEA model according to the concept of the non-cooperative game theory (or Leader-Follower). As matter of fact, by this methodology, we can replace deterministic concepts such as efficient and inefficient, with concepts of α-stochastic efficient and α-stochastic inefficient.
The structure of the paper is arranged as follows. In section 2, we remember the version of P-model of stochastic DEA models and stochastic efficiency based on it. Next, in section 3, the stochastic two-stage network DEA models according to the concept of the non-cooperative game theory are proposed and transformation to deterministic and linear model is explained. A numerical case of 16 commercial banks in China is illustrated to justify the new model in section 4. Finally, the last section summarizes and concludes.

2 Stochastic CCR Efficiency

Cooper et al. [8], introduced the following version of P-Model where evidently has built on the CCR model of DEA.

$$\begin{aligned} max \quad & \mathbb{P}\left(\frac{\sum_{r=1}^{S} U_r\tilde{y}_{ro}}{\sum_{i=1}^{M} V_i\tilde{x}_{io}} \geq 1\right) \\ s.t. \quad & \\ & \mathbb{P}\left(\frac{\sum_{r=1}^{S} U_r\tilde{y}_{rj}}{\sum_{i=1}^{M} V_i\tilde{x}_{ij}} \leq 1\right) \geq 1-\alpha_j \\ & U_r, V_i \geq 0, \quad \forall r, i \end{aligned} \tag{2.1}$$

In this model, each DMU, uses M inputs $\tilde{x}_j = (\tilde{x}_{1j}, \tilde{x}_{2j}, ..., \tilde{x}_{Mj})^T$ to obtains D outputs, $\tilde{y}_j = (\tilde{y}_{1j}, \tilde{y}_{2j}, ..., \tilde{y}_{Sj})^T$. Here $\mathbb{P}$ means Probability and the symbol $\sim$ is used to represent the random inputs and outputs. Furthermore, by solving this model for each DMU, non-negative virtual multipliers, $U \in \mathbb{R}_+^S$ and $V \in \mathbb{R}_+^M$, are determined. Also, $\alpha_j \in (0,1)$ is pre-selected and it is the minimum possibility required to match the related chance constraint [25]. By choosing $U_r = 0$, and $V_i > 0$ for all r and i, in constraints of (2.1), we can see that the model is feasible [4].
As we know, for each continuous distribution, it is not mindless to write

$$\mathbb{P}\left(\frac{\sum_{r=1}^{S} U_r^*\tilde{y}_{rj}}{\sum_{i=1}^{M} V_i^*\tilde{x}_{ij}} \leq 1\right) + \mathbb{P}\left(\frac{\sum_{r=1}^{S} U_r^*\tilde{y}_{rj}}{\sum_{i=1}^{M} V_i^*\tilde{x}_{ij}} \geq 1\right) = 1$$

or

$$\begin{aligned} & \mathbb{P}\left(\frac{\sum_{r=1}^{S} U_r^*\tilde{y}_{rj}}{\sum_{i=1}^{M} V_i^*\tilde{x}_{ij}} \leq 1\right) = 1-\alpha^* \geq 1-\alpha_o \\ & U_r, V_i \geq 0, \quad \forall r, i. \end{aligned} \tag{2.2}$$

In this paper, $*$ refers to an optimal value, hence α^* is the probability of obtaining a value of at least unity with this choice of weights and therefore $1-\alpha^*$ is the probability of failing to reach this value [8]. In addition, $1-\alpha_o$ is the probability of inefficiency of DMU_o. Clearly, from 2.2 we have $\alpha_o \geq \alpha^*$.

Definition 2.1 *([8]) DMU_o is "stochastic efficient" if and only if $\alpha_o = \alpha^*$.*

3 Stochastic two-stage network DEA models

Figure 1 describes a two-stage process, where each DMU is composed of two sub-DMU in

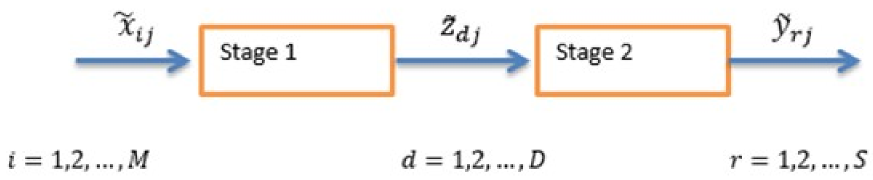

Figure 1: Two-stage process

series, and the sub-DMU in stage 2 consume the products of the sub-DMUs in stage 1 as inputs. For each $\mathrm{DMU}_{j\in 1,2,\ldots,n}$, $\tilde{x}_j = (\tilde{x}_{1j}, \tilde{x}_{2j}, \ldots, \tilde{x}_{Mj})^T$ represent random input vector and $\tilde{z}_j = (\tilde{z}_{1j}, \tilde{z}_{2j}, \ldots, \tilde{z}_{Dj})^T$ represent random output vector of the first stage. Since this random output vector is used as random input by the second stage, are introduced as intermediate products. The second stage $\tilde{y}_j = (\tilde{y}_{1j}, \tilde{y}_{2j}, \ldots, \tilde{y}_{Sj})^T$ produces a random output vector.

3.1 The Non-cooperative model

In this section, we present the stochastic two-stage network DEA model according to the concept of the non-cooperative game theory (or Leader-Follower), which was utilized to expand models for performance evaluation by Liang et al.[16].

3.1.1 Stage 1 is the leader, while Stage 2 is the follower

Here we assume that the first stage is the leader. Hence, the importance of the stage 1 is more than the second stage (follower). Moreover, the leader's efficiency is known and the efficiency of the follower is computed based on it.
Model (3.3) evaluates the efficiency of the Sub-DMU in stage 1 as a leader.

$$\begin{aligned} & max \quad \mathbb{P}\left(\frac{W\tilde{z}_o}{V\tilde{x}_o} \geq 1\right) \\ & s.t. \qquad\qquad\qquad\qquad (3.3) \\ & \mathbb{P}\left(\frac{W\tilde{z}_j}{V\tilde{x}_j} \leq 1\right) \geq 1-\alpha_j \\ & W, V \geq 0, \end{aligned}$$

By using the Cooper et al. [8], transformation in stochastic programming, deterministic model can obtain to the following model:

$$\begin{aligned} & e_{1l}^{*(o)} = max\ (wz_o - vx_o) \\ & s.t. \qquad\qquad\qquad\qquad (3.4) \\ & w^t\Sigma_{z_oz_o}w + v^t\Sigma_{x_ox_o} - 2w^t\Sigma_{z_ox_o}v \geq 1, \\ & (wz_j - vx_j) - \Phi^{-1}(\alpha_j)\xi_j \leq 0, \quad j = 1, \ldots, n, \\ & C_{\alpha_j}[w^t\Sigma_{zz}w + v^t\Sigma_{xx}v - 2w^t\Sigma_{zx}v - \xi_j^2] \leq o, \\ & \qquad\qquad j = 1, \ldots, n, \\ & w, v, \xi \geq 0, \end{aligned}$$

The relation between optimal solution of (3.3) and (3.4) is exhibited as follow.

Theorem 3.1

$$\Phi(w^*z_o - v^*x_o) = \mathbb{P}\left(\frac{W^*\tilde{z}_o}{V^*\tilde{x}_o} \geq 1\right).$$

Proof: See [8].
According to the optimal solution of the first stage, model (3.5) calculates the efficiency of the sub-DMU in stage 2 as a follower:

$$\begin{aligned} & e_{2f}^{*(o)} = max\ \mathbb{P}\left(\frac{U\tilde{y}_o}{W\tilde{z}_o} \geq 1\right) \\ & s.t. \qquad\qquad\qquad (3.5) \\ & \mathbb{P}\left(\frac{W\tilde{z}_j}{V\tilde{x}_j} \leq 1\right) \geq 1-\alpha_j, \quad j = 1, \ldots, n, \qquad (3.5a) \\ & \mathbb{P}\left(\frac{U\tilde{y}_j}{W\tilde{z}_j} \leq 1\right) \geq 1-\alpha_j \quad j = 1, \ldots, n, \qquad (3.5b) \\ & \mathbb{P}\left(\frac{W\tilde{z}_o}{V\tilde{x}_o} \geq 1\right) \geq \Phi(e_{1l}^{*(o)}) \qquad (3.5c) \\ & W, U, V \geq 0, \end{aligned}$$

where, $e_{1l}^{*(o)}$ is the efficiency of stage 1 calculated by (3.4). As it is seen, the objective function and the constraints of the first stage are as constraints in the model (3.5).
In what follows, for computation and implementation in applicable circumstances and by using Cooper et al. [8] approach, we will achieve the deterministic equivalent of the model (3.5) as follow.
From constraint (3.5a) we have

$$\begin{aligned}&\mathbb{P}\Big(W\tilde{z}_j - V\tilde{x}_j \le 0\Big) \ge 1-\alpha_j\\ &\Rightarrow \mathbb{P}\Big(\frac{W\tilde{z}_j - V\tilde{x}_j - (Wz_j - Vx_j)}{\sigma_{W\tilde{z}_j - V\tilde{x}_j}}\\ &\le \frac{-(Wz_j - Vx_j)}{\sigma_{W\tilde{z}_j - V\tilde{x}_j}}\Big) \ge 1-\alpha_j\end{aligned} \tag{3.6}$$

where $z_j = \mathbb{E}[\tilde{z}_j]$, $x_j = \mathbb{E}[\tilde{x}_j]$ and $\sigma_{W\tilde{z}_j - V\tilde{x}_j} = \sqrt{var(W\tilde{z}_j - V\tilde{x}_j)}$.
Next, by introducing some new variables $\tilde{\xi}_j$, defined by

$$\tilde{\xi}_j = \frac{W\tilde{z}_j - V\tilde{x}_j - (Wz_j - Vx_j)}{\sigma_{W\tilde{z}_j - V\tilde{x}_j}}, \quad j = 1, ..., n, \tag{3.7}$$

so that $\tilde{\xi}_j \sim N(0,1)$ (the standard normal probability distribution with zero mean and unit variance) and by direct substitution in (3.6) we obtain

$$\mathbb{P}\Big(\tilde{\xi}_j \le \frac{-(Wz_j - Vx_j)}{\sigma_{W\tilde{z}_j - V\tilde{x}_j}}\Big) \ge 1-\alpha_j, \quad j = 1, ..., n. \tag{3.8}$$

Since $\tilde{\xi}_j$ follows the standard normal probability distribution, we have

$$\Phi\Big(\frac{-(Wz_j - Vx_j)}{\sigma_{W\tilde{z}_j - V\tilde{x}_j}}\Big) \ge 1-\alpha_j, \ j = 1, ..., n. \tag{3.9}$$

According to the property of invertibility of Φ, (3.9) can rewrite as

$$\frac{-(Wz_j - Vx_j)}{\sigma_{W\tilde{z}_j - V\tilde{x}_j}} \ge \Phi^{-1}(1-\alpha_j), \ j = 1, ..., n. \tag{3.10}$$

Now, by introducing nonnegative "spacer variables", η_j, as are introduced by Charnes and Cooper [4], replace (3.10) with the following two inequalities:

$$(Wz_j - Vx_j) - \Phi^{-1}(\alpha_j)\eta_j \le 0, \ j = 1, ..., n, \tag{3.11}$$

$$\begin{aligned}&C_{\alpha_j}[W^t\Sigma_{zz}W + V^t\Sigma_{xx}V\\ &\quad - 2W^t\Sigma_{zx}V - \eta_j^2] \le 0, \quad j = 1, ..., n,\end{aligned}$$

where,

$$C_{\alpha_j} = \begin{cases} 1 & if \ \alpha_j < 0.5, \\ 0 & if \ \alpha_j = 0.5, \\ -1 & if \ \alpha_j > 0.5. \end{cases}$$

Similarly, constraint (3.5b) in Model (3.5) could be transformed to the next inequality,

$$(Uy_j - Wz_j) - \Phi^{-1}(\alpha_j)\zeta_j \le 0, \ j = 1, ..., n, \tag{3.12}$$

$$\begin{aligned}&C_{\alpha_j}[U^t\Sigma_{yy}U + W^t\Sigma_{zz}W\\ &\quad - 2W^t\Sigma_{zy}U - \zeta_j^2] \le 0, \quad j = 1, ..., n,\end{aligned}$$

For constraint (3.5c) in (3.5),

$$\begin{aligned}&\mathbb{P}\left(\frac{W\tilde{z}_o}{V\tilde{x}_o} \ge 1\right) \ge \Phi(e_{1l}^{*(o)})\\ &\Rightarrow \mathbb{P}\Big(V\tilde{x}_o - W\tilde{z}_o \le 0\Big) \ge \Phi(e_{1l}^{*(o)})\\ &\Rightarrow \mathbb{P}\left(\frac{V\tilde{x}_o - W\tilde{z}_o - (Vx_o - Wz_o)}{\sigma_{V\tilde{x}_o - W\tilde{z}_o}}\right.\\ &\left.\le \frac{-(Vx_o - Wz_o)}{\sigma_{V\tilde{x}_0 - W\tilde{z}_o}}\right) \ge \Phi(e_{1l}^{*(o)}),\end{aligned} \tag{3.13}$$

where $z_o = \mathbb{E}[\tilde{z}_o]$, $x_o = \mathbb{E}[\tilde{x}_o]$. Now, by introducing a new variable $\tilde{\gamma}_o$, defined by $\tilde{\gamma}_o = \frac{V\tilde{x}_o - W\tilde{z}_o - (Vx_o - Wz_o)}{\sigma_{V\tilde{x}_o - W\tilde{z}_o}}$, so that $\tilde{\gamma}_o \sim N(0,1)$, and by direct substitution in (3.13) we get

$$\begin{aligned}&\mathbb{P}\left(\tilde{\gamma}_o \le \frac{-(Vx_o - Wz_o)}{\sigma_{V\tilde{x}_0 - W\tilde{z}_o}}\right) \ge \Phi(e_{1l}^{*(o)})\\ &\Rightarrow \Phi\left(\frac{-(Vx_o - Wz_o)}{\sigma_{V\tilde{x}_0 - W\tilde{z}_o}}\right) \ge \Phi(e_{1l}^{*(o)})\end{aligned}$$

Since Φ is a strictly increasing function, then

$$\frac{-(Vx_o - Wz_o)}{\sigma_{V\tilde{x}_0 - W\tilde{z}_o}} \ge e_{1l}^{*(o)}.$$

Thus by using Charnes and Cooper [4] approach and introducing a non-negative variable, ς_o, constraint (3.5c) could be converted to the following two inequalities

$$\begin{aligned}&(Wz_o - Vx_o) - e_1^{*(o)}\varsigma_o \leq 0,\\ &C_{e_{1l}^{*(o)}}[V^t\Sigma_{x_ox_o}V + W^t\Sigma_{z_oz_o}W\\ &\qquad - 2W^t\Sigma_{z_ox_o}V - \varsigma_o^2] \leq 0,\end{aligned} \tag{3.14}$$

where

$$C_{e_{1l}^{*(o)}} = \begin{cases} 1 & if\ e_{1l}^{*(o)} > 0,\\ 0 & if\ e_{1l}^{*(o)} = 0,\\ -1 & if\ e_{1l}^{*(o)} < 0.\end{cases}$$

Now by replacing (3.5a), (3.5b), (3.5c) with (3.11), (3.12) and (3.14) respectively, we then have

$$\begin{aligned}&e_{2f}^{*(o)} = max\ \mathbb{P}\left(\frac{U\tilde{y}_o}{W\tilde{z}_o} \geq 1\right)\\ &s.t.\\ &(Uy_j - Wz_j) - \Phi^{-1}(\alpha_j)\zeta_j \leq 0,\ j=1,...,n,\\ &C_{\alpha_j}[U^t\Sigma_{yy}U + W^t\Sigma_{zz}W\\ &\qquad - 2W^t\Sigma_{zy}U - \zeta_j^2] \leq 0,\ j=1,...,n,\\ &(Wz_j - Vx_j) - \Phi^{-1}(\alpha_j)\eta_j \leq 0,\ j=1,...,n,\\ &C_{\alpha_j}[W^t\Sigma_{zz}W + V^t\Sigma_{xx}V\\ &\qquad - 2W^t\Sigma_{zx}V - \eta_j^2] \leq 0,\ j=1,...,n,\\ &(Wz_o - Vx_o) - e_{1l}^{*(o)}\varsigma_o \leq 0,\\ &C_{e_{1l}^{*(o)}}[V^t\Sigma_{x_ox_o}V + W^t\Sigma_{z_oz_o}W\\ &\qquad - 2W^t\Sigma_{z_ox_o}V - \varsigma_o^2] \leq 0,\\ &W,U,V,\zeta,\eta,\varsigma_o \geq 0,\end{aligned} \tag{3.15}$$

All constraints in (3.15) are deterministic; but since the vectors $\tilde{y}_o$ and $\tilde{z}_o$ as random variables are included in the objective function; this problem is still uncertainty. Clearly, it can be seen that (3.15) is equivalent to

$$\begin{aligned}&e_{2f}^{*(o)} = Max\ \gamma\\ &s.t.\\ &\mathbb{P}\left(\frac{U\tilde{y}_o}{W\tilde{z}_o} \geq 1\right) \geq \gamma,\\ &(Uy_j - Wz_j) - \Phi^{-1}(\alpha_j)\zeta_j \leq 0,\ j=1,...,n,\\ &C_{\alpha_j}[U^t\Sigma_{yy}U + W^t\Sigma_{zz}W\\ &\qquad - 2W^t\Sigma_{zy}U - \zeta_j^2] \leq 0,\ j=1,...,n,\\ &(Wz_j - Vx_j) - \Phi^{-1}(\alpha_j)\eta_j \leq 0,\ j=1,...,n,\\ &C_{\alpha_j}[W^t\Sigma_{zz}W + V^t\Sigma_{xx}V\\ &\qquad - 2W^t\Sigma_{zx}V - \eta_j^2] \leq 0,\ j=1,...,n,\\ &(Wz_o - Vx_o) - e_{1l}^{*(o)}\varsigma_o \leq 0,\\ &C_{e_{1l}^{*(o)}}[V^t\Sigma_{x_ox_o}V + W^t\Sigma_{z_oz_o}W\\ &\qquad - 2W^t\Sigma_{z_ox_o}V - \varsigma_o^2] \leq 0,\\ &W,U,V,\zeta,\eta,\varsigma_o \geq 0,\end{aligned} \tag{3.16}$$

Proceeding as in Cooper et al.[8], the deterministic equivalent of the first constraint in (3.16) is obtained as follows:

$$\begin{aligned}&\mathbb{P}\left(\frac{U\tilde{y}_o}{W\tilde{z}_o} \geq 1\right) \geq \gamma\\ &\Rightarrow \mathbb{P}\Big(U\tilde{y}_o \geq W\tilde{z}_o \geq 1\Big) \geq \gamma\\ &\Rightarrow \mathbb{P}\Big(W\tilde{z}_o - U\tilde{y}_o \leq 0\Big) \geq \gamma\\ &\Rightarrow \mathbb{P}\Big(\frac{W\tilde{z}_o - U\tilde{y}_o - (Wz_o - Uy_o)}{\sigma_{W\tilde{z}_o - U\tilde{y}_o}}\\ &\leq \frac{-(Wz_o - Uy_o)}{\sigma_{W\tilde{z}_o - U\tilde{y}_o}}\Big) \geq \gamma\\ &\Rightarrow \Phi\left(\frac{Uy_o - Wz_o}{\sigma_{W\tilde{z}_o - U\tilde{y}_o}}\right) \geq \gamma\\ &\Rightarrow \frac{Uy_o - Wz_o}{\sigma_{W\tilde{z}_o - U\tilde{y}_o}} \geq \Phi^{-1}(\gamma).\end{aligned}$$

Hence, the problem (3.16) is equivalent to

$$
\begin{aligned}
e_{2f}^{*(o)} &= Max\ \gamma \\
& s.t. \\
& \frac{Uy_o - Wz_o}{\sigma_{W\tilde{z}_o - U\tilde{y}_o}} \geq \Phi^{-1}(\gamma), \\
& (Uy_j - Wz_j) - \Phi^{-1}(\alpha_j)\zeta_j \leq 0, \quad j = 1, ..., n, \\
& C_{\alpha_j}[U^t\Sigma_{yy}U + W^t\Sigma_{zz}W - 2W^t\Sigma_{zy}U - \zeta_j^2] \leq 0, \ j = 1, ..., n, \\
& (Wz_j - Vx_j) - \Phi^{-1}(\alpha_j)\eta_j \leq 0, \quad j = 1, ..., n, \\
& C_{\alpha_j}[W^t\Sigma_{zz}W + V^t\Sigma_{xx}V - 2W^t\Sigma_{zx}V - \eta_j^2] \leq 0, \ j = 1, ..., n, \\
& (Wz_o - Vx_o) - e_{1l}^{*(o)}\varsigma_o \leq 0, \\
& C_{e_{1l}^{*(o)}}[V^t\Sigma_{x_ox_o}V + W^t\Sigma_{z_oz_o}W - 2W^t\Sigma_{z_ox_o}V - \varsigma_o^2] \leq 0, \\
& W, U, V, \zeta, \eta, \varsigma_o \geq 0,
\end{aligned}
\tag{3.17}
$$

This problem is deterministic but because of the existence of denominator in the first constraint, is non-convex [8]. To remove this difficulty, since $\Phi^{-1}(\gamma)$ is strictly increasing the function of γ, (3.17) converts to the following problem

$$
\begin{aligned}
e_{2f}^{*(o)} &= Max\ \theta \\
& s.t. \\
& \frac{Uy_o - Wz_o}{\sigma_{W\tilde{z}_o - U\tilde{y}_o}} \geq \theta, \\
& (Uy_j - Wz_j) - \Phi^{-1}(\alpha_j)\zeta_j \leq 0, \quad j = 1, ..., n, \\
& C_{\alpha_j}[U^t\Sigma_{yy}U + W^t\Sigma_{zz}W - 2W^t\Sigma_{zy}U - \zeta_j^2] \leq 0, \ j = 1, ..., n, \\
& (Wz_j - Vx_j) - \Phi^{-1}(\alpha_j)\eta_j \leq 0, \quad j = 1, ..., n, \\
& C_{\alpha_j}[W^t\Sigma_{zz}W + V^t\Sigma_{xx}V - 2W^t\Sigma_{zx}V - \eta_j^2] \leq 0, \ j = 1, ..., n, \\
& (Wz_o - Vx_o) - e_{1l}^{*(o)}\varsigma_o \leq 0, \\
& C_{e_{1l}^{*(o)}}[V^t\Sigma_{x_ox_o}V + W^t\Sigma_{z_oz_o}W - 2W^t\Sigma_{z_ox_o}V - \varsigma_o^2] \leq 0, \\
& W, U, V, \zeta, \eta, \varsigma_o \geq 0,
\end{aligned}
\tag{3.18}
$$

The models (3.17) and (3.18) have the same solution structure. Therefore, we have:

$$
\theta^* = \Phi^{-1}(\gamma^*) \tag{3.19}
$$

Clearly, we can see that (3.18) is equivalent to the next problem:

$$
\begin{aligned}
e_{2f}^{*(o)} &= Max\ \frac{Uy_o - Wz_o}{\sigma_{W\tilde{z}_o - U\tilde{y}_o}} \\
& s.t. \\
& (Uy_j - Wz_j) - \Phi^{-1}(\alpha_j)\zeta_j \leq 0, \quad j = 1, ..., n, \\
& C_{\alpha_j}[U^t\Sigma_{yy}U + W^t\Sigma_{zz}W - 2W^t\Sigma_{zy}U - \zeta_j^2] \leq 0, \ j = 1, ..., n, \\
& (Wz_j - Vx_j) - \Phi^{-1}(\alpha_j)\eta_j \leq 0, \quad j = 1, ..., n, \\
& C_{\alpha_j}[W^t\Sigma_{zz}W + V^t\Sigma_{xx}V - 2W^t\Sigma_{zx}V - \eta_j^2] \leq 0, \ j = 1, ..., n, \\
& (Wz_o - Vx_o) - e_{1l}^{*(o)}\varsigma_o \leq 0, \\
& C_{e_{1l}^{*(o)}}[V^t\Sigma_{x_ox_o}V + W^t\Sigma_{z_oz_o}W - 2W^t\Sigma_{z_ox_o}V - \varsigma_o^2] \leq 0, \\
& W, U, V, \zeta, \eta, \varsigma_o \geq 0.
\end{aligned}
\tag{3.20}
$$

Since $\frac{Uy_o - Wz_o}{\sigma_{W\tilde{z}_o - U\tilde{y}_o}}$ is bounded from above by $\Phi^{-1}(\alpha_o)$, by introducing a positive variable, ϖ , (3.20) is equivalent to

$$
\begin{aligned}
& e_{2f}^{*(o)} = Max\ \frac{Uy_o - Wz_o}{\varpi} \\
& s.t. \qquad\qquad\qquad\qquad\qquad\qquad (3.21)\\
& U^t\Sigma_{y_oy_o}U + W^t\Sigma_{z_oz_o}W \\
& \quad - 2W^t\Sigma_{z_oy_o}U - \varpi^2 \geq 0, \\
& (Uy_j - Wz_j) - \Phi^{-1}(\alpha_j)\zeta_j \leq 0, \\
& \qquad\qquad j = 1, ..., n, \\
& C_{\alpha_j}[U^t\Sigma_{yy}U + W^t\Sigma_{zz}W \\
& \quad - 2W^t\Sigma_{zy}U - \zeta_j^2] \leq 0,\ j = 1, ..., n, \\
& (Wz_j - Vx_j) - \Phi^{-1}(\alpha_j)\eta_j \leq 0, \\
& \qquad\qquad j = 1, ..., n, \\
& C_{\alpha_j}[W^t\Sigma_{zz}W + V^t\Sigma_{xx}V \\
& \quad - 2W^t\Sigma_{zx}V - \eta_j^2] \leq 0,\ j = 1, ..., n, \\
& (Wz_o - Vx_o) - e_{1l}^{*(o)}\varsigma_o \leq 0, \\
& C_{e_{1l}^{*(o)}}[V^t\Sigma_{x_ox_o}V + W^t\Sigma_{z_oz_o}W \\
& \quad - 2W^t\Sigma_{z_ox_o}V - \varsigma_o^2] \leq 0, \\
& W, U, V, \zeta, \eta, \varsigma_o, \varpi \geq 0.
\end{aligned}
$$

The problem (3.21) involves a fractional objective function. Hence, by using the Charnes-Cooper transformation of linear fractional programming (see, [3]), considering $t := \frac{1}{\varpi}, \lambda := tW, \mu := tU, \nu := tV, \tau_j := t\zeta_j, \rho_j = t\xi_j, \pi_o := t\varsigma_o$, and replacing them in (3.21), the following quadratic programming problem is made:

$$
\begin{aligned}
& e_{2f}^{*(o)} = Max\ \mu y_o - \lambda z_o \\
& s.t. \qquad\qquad\qquad\qquad\qquad\qquad (3.22)\\
& \mu^t\Sigma_{y_oy_o}\mu + \lambda^t\Sigma_{z_oz_o}\lambda - 2\lambda^t\Sigma_{z_oy_o}\mu \geq 1, \\
& (\mu y_j - \lambda z_j) - \Phi^{-1}(\alpha_j)\tau_j \leq 0,\ j = 1, ..., n, \\
& C_{\alpha_j}[\mu^t\Sigma_{yy}\mu + \lambda^t\Sigma_{zz}\lambda - 2\lambda^t\Sigma_{zy}\mu - \tau_j^2] \leq 0, \\
& \qquad\qquad j = 1, ..., n, \\
& (\lambda z_j - \nu x_j) - \Phi^{-1}(\alpha_j)\rho_j \leq 0,\ \ j = 1, ..., n, \\
& C_{\alpha_j}[\lambda^t\Sigma_{zz}\lambda + \nu^t\Sigma_{xx}\nu - 2\lambda^t\Sigma_{zx}\nu - \rho_j^2] \leq 0, \\
& \qquad\qquad j = 1, ..., n, \\
& (\lambda z_o - \nu x_o) - e_{1l}^{*(o)}\pi_o \leq 0, \\
& C_{e_{1l}^{*(o)}}[\nu^t\Sigma_{x_ox_o}\nu + \lambda^t\Sigma_{z_oz_o}\lambda - 2\lambda^t\Sigma_{z_ox_o}\nu \\
& \qquad\qquad - \pi_o^2] \leq 0, \\
& \lambda_d, \mu_r, \nu, \tau_j, \rho_j, \pi_o \geq 0.
\end{aligned}
$$

Now for concluding of this section, Theorem (3.2) present to show the relationship between solutions of problems (3.5) and (3.22).

Theorem 3.2 *Let $(\lambda^*, \mu^*, \nu^*, \tau^*, \rho^*, \pi^*)$ and (W^*, U^*, V^*) be optimal solutions of (3.22) and (3.5), respectively; then*

$$\Phi(\mu^* y_o - \lambda^* z_o) = \mathbb{P}\left(\frac{U^*\tilde{y}_o}{W^*\tilde{z}_o} \geq 1\right)$$

Furthermore, is stochastically efficient if and only if $\Phi(\mu^* y_o - \lambda^* z_o) = \alpha_o$.

Proof: From (3.19), we have $\Phi(\mu^* y_o - \lambda^* z_o) = \gamma^*$, where γ^* is the optimal solution of problem (3.17). Because of the problem (3.14) is equivalent to problem (3.5),

$$\gamma^* = \mathbb{P}\left(\frac{U^*\tilde{y}_o}{W^*\tilde{z}_o} \geq 1\right)$$

and, consequently

$$\Phi(\mu^* y_o - \lambda^* z_o) = \mathbb{P}\left(\frac{U^*\tilde{y}_o}{W^*\tilde{z}_o} \geq 1\right).\blacksquare$$

3.1.2 Transformation to linear programming

Now, further, we adopt the single factor symmetric disturbance assumption, i.e., following Cooper et al. [8] suppose all components of any input, intermediate and output are determined only by a single random factor. Earlier, Sharpe [23] and Kahane [12] examined the applications of the single underlying factor assumption in finance and economics. More explicitly,

$$
\begin{aligned}
\tilde{x}_{ij} &= x_{ij} + a_{ij}\xi, \\
\tilde{z}_{dj} &= z_{dj} + b_{dj}\eta, \\
\tilde{y}_{rj} &= y_{rj} + c_{rj}\epsilon,
\end{aligned}
$$

where $x_{ij}, a_{ij}, z_{dj}, b_{dj}, y_{rj}, c_{rj}$, are non-negative and ξ, η, ϵ are independent standard normal distributed random variables. Also, x_{ij}, z_{dj}, and y_{rj} are the expected values for $\tilde{x}_{ij}, \tilde{z}_{dj}$, and $\tilde{y}_{rj}$ respectively, and a_{ij}, b_{dj}, and c_{rj} are standard deviations for $\tilde{x}_{ij}, \tilde{z}_{dj}$, and $\tilde{y}_{rj}$. Meanwhile, having finite realizations and nonnegative distributions, are reasonable restrictions for all $\tilde{x}_{ij}, \tilde{z}_{dj}$, and $\tilde{y}_{rj}$

[29]. Similar to those analysis used in the previous section, the problem (3.3) transform to the following problem,

$$
\begin{aligned}
& e_{1l}^{*(o)} = Max\ (wz_o - vx_o) \\
& s.t. \qquad\qquad\qquad\qquad\qquad (3.23) \\
& |va_o - wb_o| \geq 1, \\
& (vx_j - wz_j) \geq \Phi^{-1}(1-\alpha_j)|va_j - wb_j|, \\
& \qquad\qquad j = 1, ..., n, \\
& w, v \geq 0.
\end{aligned}
$$

Since there is the absolute value in the model then, model (3.23) is a non-linear programming problem. However, by using the goal programming theory introduced by Charnes and Cooper [2, 5], the problem (3.23) can be transformed to a quadratic programming problem [8].
By considering

$$
\begin{aligned}
|va_j - wb_j| &= p_j^+ + p_j^-, \\
va_j - wb_j &= p_j^+ - p_j^-, \\
p_j^+ p_j^- &= 0, \\
p_j^+, p_j^- &\geq 0,
\end{aligned}
$$

and, replace them in (3.23) we have

$$
\begin{aligned}
& e_{1l}^{*(o)} = Max\ (wz_o - vx_o) \\
& s.t. \qquad\qquad\qquad\qquad\qquad (3.24) \\
& p_j^+ + p_j^- \geq 1, \\
& (vx_j - wz_j) \geq \Phi^{-1}(1-\alpha_j)(p_j^+ + p_j^-), \\
& \qquad\qquad j = 1, ..., n, \\
& va_j - wb_j = p_j^+ - p_j^-, \quad j = 1, ..., n, \\
& p_j^+ p_j^- = 0, \\
& w, v, p^+, p^- \geq 0.
\end{aligned}
$$

Due to the existence of constraint $p_j^+ p_j^- = 0$, the problem (3.24) is a non-linear programming problem. Without this constraint, the problem is linear. Moreover, if linear programming problem has an optimal solution, then has an optimal extreme point and at this point at least one of values of p_j^+ or p_j^- is zero. Consequently, if we use the simplex algorithm to solve this linear programming, we can find this optimal extreme point and we can avoid constraint $p_j^+ p_j^- = 0$.

$$
\begin{aligned}
& e_{1l}^{*(o)} = Max\ (wz_o - vx_o) \\
& s.t. \qquad\qquad\qquad\qquad\qquad (3.25) \\
& p_j^+ + p_j^- \geq 1, \\
& (vx_j - wz_j) \geq \Phi^{-1}(1-\alpha_j)(p_j^+ + p_j^-), \\
& \qquad\qquad j = 1, ..., n, \\
& va_j - wb_j = p_j^+ - p_j^-, \quad j = 1, ..., n, \\
& w, v, p^+, p^- \geq 0.
\end{aligned}
$$

Also, for stage 2 as a follower, by employing analysis similar to those used in the previous section, the problem (3.5) transform to the following problem,

$$
\begin{aligned}
& e_{2f}^{*(o)} = Max\ (uy_o - wz_o) \\
& s.t. \qquad\qquad\qquad\qquad\qquad (3.26) \\
& |wb_o - uc_o| \geq 1, \\
& (wz_j - uy_j) \geq \Phi^{-1}(1-\alpha_j)|wb_j - uc_j|, \\
& \qquad\qquad j = 1, ..., n, \\
& (vx_j - wz_j) \geq \Phi^{-1}(1-\alpha_j)|va_j - wb_j|, \\
& \qquad\qquad j = 1, ..., n, \\
& (wz_o - vx_o) \geq e_{1l}^{*(o)}|va_o - wb_o|, \\
& w, v, u \geq 0.
\end{aligned}
$$

By considering

$$
\begin{aligned}
|wb_j - uc_j| &= q_j^+ + q_j^-, \\
wb_j - uc_j &= q_j^+ - q_j^-, \\
q_j^+ q_j^- &= 0, \\
q_j^+, q_j^- &\geq 0, \\
|va_j - wb_j| &= p_j^+ + p_j^-, \\
va_j - wb_j &= p_j^+ - p_j^-, \\
p_j^+ p_j^- &= 0, \\
p_j^+, p_j^- &\geq 0.
\end{aligned}
$$

and, replace them in (3.26) we have the following non-linear programming:

$$
\begin{aligned}
& e_{2f}^{*(o)} = Max\ (uy_o - wz_o) \\
& s.t. \qquad (3.27) \\
& q_o^+ + q_o^- \geq 1, \\
& (wz_j - uy_j) \geq \Phi^{-1}(1-\alpha_j)(q_j^+ + q_j^-), \quad j = 1, ..., n, \\
& wb_j - uc_j = q_j^+ - q_j^-, \quad j = 1, ..., n, \\
& (vx_j - wz_j) \geq \Phi^{-1}(1-\alpha_j)(p_j^+ + p_j^-), \quad j = 1, ..., n, \\
& va_j - wb_j = p_j^+ - p_j^-, \quad j = 1, ..., n, \\
& (wz_o - vx_o) \geq e_{1l}^{*(o)}(p_o^+ + p_o^-), \\
& q_j^+ q_j^- = 0, \quad j = 1, ..., n, \\
& p_j^+ p_j^- = 0, \quad j = 1, ..., n, \\
& w, v, u, p^+, p^-, q^+, q^- \geq 0.
\end{aligned}
$$

Similar to what was discussed before, linear form of (3.27) is as follow

$$
\begin{aligned}
& e_{2f}^{*(o)} = Max\ (uy_o - wz_o) \\
& s.t. \qquad (3.28) \\
& q_o^+ + q_o^- \geq 1, \\
& (wz_j - uy_j) \geq \Phi^{-1}(1-\alpha_j)(q_j^+ + q_j^-), \quad j = 1, ..., n, \\
& wb_j - uc_j = q_j^+ - q_j^-, \quad j = 1, ..., n, \\
& (vx_j - wz_j) \geq \Phi^{-1}(1-\alpha_j)(p_j^+ + p_j^-), \quad j = 1, ..., n, \\
& va_j - wb_j = p_j^+ - p_j^-, \quad j = 1, ..., n, \\
& (wz_o - vx_o) \geq e_{1l}^{*(o)}(p_o^+ + p_o^-), \\
& w, v, u, p^+, p^-, q^+, q^- \geq 0.
\end{aligned}
$$

3.1.3 Stage 2 is the leader, while Stage 1 is the follower

With the similar manner, we suppose the stage 2 is the leader. Therefore, the importance of the second stage is more than the first stage (follower). Hence, the leaders efficiency is known and the efficiency of follower is computed based on it. Model (3.29), present efficiency of the Sub-DMU in stage 2, as follow:

$$
\begin{aligned}
& max\ \mathbb{P}\left(\frac{U\tilde{y}_o}{W\tilde{z}_o} \geq 1\right) \\
& s.t. \qquad (3.29) \\
& \mathbb{P}\left(\frac{U\tilde{y}_j}{W\tilde{z}_j} \leq 1\right) \geq 1 - \alpha_j, \\
& W, U \geq 0.
\end{aligned}
$$

By using the Cooper et al. [8] transformation in stochastic programming, the deterministic model can be obtained to the following model:

$$
\begin{aligned}
& e_{2l}^{*(o)} = max\ (uy_o - wz_o) \\
& s.t. \qquad (3.30) \\
& w^t\Sigma_{z_oz_o}w + u^t\Sigma_{y_oy_o}u - 2w^t\Sigma_{z_oy_o}u \geq 1, \\
& (uy_j - wz_j) - \Phi^{-1}(\alpha_j)\xi_j \leq 0, \ j = 1, ..., n, \\
& C_{\alpha_j}[w^t\Sigma_{zz}w + u^t\Sigma_{yy}u - 2w^t\Sigma_{zy}u - \xi_j^2] \leq o, \quad j = 1, ..., n, \\
& w, u, \xi \geq 0,
\end{aligned}
$$

The relation between optimal solutions of (3.29) and (3.30) is exhibited as follow:

Theorem 3.3 *Let (w^*, u^*, ν^*, ξ^*) and (W^*, U^*) be optimal solutions of (3.30) and (3.29), respectively; then*

$$\Phi(u^*y_o - w^*z_o) = \mathbb{P}\left(\frac{U^*\tilde{y}_o}{W^*\tilde{z}_o} \geq 1\right).$$

Furthermore, is stochastically efficient if and only if $\Phi(u^*y_o - w^*z_o) = \alpha_o$.

In addition, by assuming single factor components of any input, intermediate and output, the deterministic linear programming model when $\alpha \leq 0.5$ could be obtained as follows

$$
\begin{aligned}
& e_{2l}^{*(o)} = Max\ (uy_o - wz_o) \\
& s.t. \qquad (3.31) \\
& |wb_o - uc_o| \geq 1, \\
& (wz_j - uy_j) \geq \Phi^{-1}(1-\alpha_j)|wb_j - uc_j|, \quad j = 1, ..., n, \\
& w, u \geq 0.
\end{aligned}
$$

Since there is an absolute value in the model (3.31) then this model is a non-linear programming problem. However, by using the goal programming theory introduced by Charnes and

Cooper [2, 5], the problem (3.31) can become a quadratic programming problem [8].
By considering

$$|wb_j - uc_j| = q_j^+ + q_j^-,$$
$$wb_j - uc_j = q_j^+ - q_j^-,$$
$$q_j^+ q_j^- = 0,$$
$$q_j^+, q_j^- \geq 0,$$

and, replace them in (3.31) we have

$$\begin{aligned}
& e_{2l}^{*(o)} = Max\ (uy_o - wz_o) \\
& s.t. \qquad\qquad (3.32) \\
& q_j^+ + q_j^- \geq 1, \\
& (wz_j - uy_j) \geq \Phi^{-1}(1-\alpha_j)(q_j^+ + q_j^-), \\
& \qquad j = 1, ..., n, \\
& wb_j - uc_j = q_j^+ - q_j^-, \quad j = 1, ..., n, \\
& q_j^+ q_j^- = 0, \\
& w, u, q^+, q^- \geq 0.
\end{aligned}$$

And the linear form of (3.32) is as follow

$$\begin{aligned}
& e_{2l}^{*(o)} = Max\ (uy_o - wz_o) \\
& s.t. \qquad\qquad (3.33) \\
& q_j^+ + q_j^- \geq 1, \\
& (wz_j - uy_j) \geq \Phi^{-1}(1-\alpha_j)(q_j^+ + q_j^-), \\
& \qquad j = 1, ..., n, \\
& wb_j - uc_j = q_j^+ - q_j^-, \quad j = 1, ..., n, \\
& w, u, q^+, q^- \geq 0.
\end{aligned}$$

According to the optimal solution of the second stage, the following model calculates the efficiency of stage 1:

$$\begin{aligned}
& e_{2f}^{*(o)} = max\ \mathbb{P}\left(\frac{W\tilde{z}_o}{V\tilde{x}_o} \geq 1\right) \\
& s.t. \qquad\qquad (3.34) \\
& \mathbb{P}\left(\frac{W\tilde{z}_j}{V\tilde{x}_j} \leq 1\right) \geq 1-\alpha_j, \quad j = 1, ..., n, \\
& \mathbb{P}\left(\frac{U\tilde{y}_j}{W\tilde{z}_j} \leq 1\right) \geq 1-\alpha_j, \quad j = 1, ..., n, \\
& \mathbb{P}\left(\frac{U\tilde{y}_o}{W\tilde{z}_o} \geq 1\right) \geq \Phi(e_{2l}^{*(o)}), \\
& W, U, V \geq 0,
\end{aligned}$$

and equivalent deterministic model of (3.34) is

$$\begin{aligned}
& e_{1f}^{*(o)} = Max\ wz_o - vx_o \\
& s.t. \qquad\qquad (3.35) \\
& v^t\Sigma_{x_ox_o}v + w^t\Sigma_{z_oz_o}w - 2w^t\Sigma_{x_oz_o}v \geq 1, \\
& (wz_j - vx_j) - \Phi^{-1}(\alpha_j)\tau_j \leq 0, \quad j = 1, ..., n, \\
& C_{\alpha_j}[v^t\Sigma_{xx}v + w^t\Sigma_{zz}w - 2v^t\Sigma_{zx}w - \tau_j^2] \leq 0, \\
& \qquad j = 1, ..., n, \\
& (uy_j - wz_j) - \Phi^{-1}(\alpha_j)\rho_j \leq 0, \quad j = 1, ..., n, \\
& C_{\alpha_j}[w^t\Sigma_{zz}w + u^t\Sigma_{yy}u - 2w^t\Sigma_{zy}u - \rho_j^2] \leq 0, \\
& \qquad j = 1, ..., n, \\
& (uy_o - wz_o) - e_1^{*(o)}\pi_o \leq 0, \\
& C_{e_1^{*(o)}}[u^t\Sigma_{y_oy_o}u + w^t\Sigma_{z_oz_o}w - 2w^t\Sigma_{z_oy_o}\nu \\
& \qquad - \pi_o^2] \leq 0, \\
& w, u, v, \rho, \pi, \tau \geq 0.
\end{aligned}$$

where

$$C_{e_{2l}^{*(o)}} = \begin{cases} 1 & if\ e_{2l}^{*(o)} > 0, \\ 0 & if\ e_{2l}^{*(o)} = 0, \\ -1 & if\ e_{2l}^{*(o)} < 0. \end{cases}$$

Theorem 3.4 *Let* $(w^*, u^*, v^*, \tau^*, \rho^*, \pi^*)$ *and* (W^*, U^*, V^*) *be optimal solutions of (3.35) and (3.34), respectively; then*

$$\Phi(w^*z_o - v^*x_o) = \mathbb{P}\left(\frac{W^*\tilde{z}_o}{V^*\tilde{x}_o} \geq 1\right)$$

Furthermore, is stochastically efficient if and only if $\Phi(w^*z_o - v^*x_o) = \alpha_o$.

Finally, by assuming the single factor of components of input, intermediate and output, the deterministic linear programming model when $\alpha \leq 0.5$ could be obtained as follows

$$\begin{aligned}
& e_{1f}^{*(o)} = Max\ (wz_o - vx_o) \\
& s.t. \qquad\qquad (3.36) \\
& |va_o - wb_o| \geq 1, \\
& (wz_j - uy_j) \geq \Phi^{-1}(1-\alpha_j)|wb_j - uc_j|, \\
& \qquad j = 1, ..., n, \\
& (vx_j - wz_j) \geq \Phi^{-1}(1-\alpha_j)|va_j - wb_j|, \\
& \qquad j = 1, ..., n, \\
& (uy_o - wz_o) \geq e_{2l}^{*(o)}|wb_o - uc_o|, \\
& w, v, u \geq 0.
\end{aligned}$$

By considering

$$
\begin{aligned}
|wb_j - uc_j| &= q_j^+ + q_j^-,\\
wb_j - uc_j &= q_j^+ - q_j^-,\\
q_j^+ q_j^- &= 0,\\
q_j^+, q_j^- &\geq 0,\\
|va_j - wb_j| &= p_j^+ + p_j^-,\\
va_j - wb_j &= p_j^+ - p_j^-,\\
p_j^+ p_j^- &= 0,\\
p_j^+, p_j^- &\geq 0.
\end{aligned}
$$

And, replace them in (3.36) we have the following non-linear programming:

$$
\begin{aligned}
& e_{1f}^{*(o)} = Max\ (wz_o - vx_o)\\
& s.t. \qquad\qquad (3.37)\\
& p_o^+ + p_o^- \geq 1,\\
& (wz_j - uy_j) \geq \Phi^{-1}(1-\alpha_j)(q_j^+ + q_j^-), \quad j = 1, ..., n,\\
& wb_j - uc_j = q_j^+ - q_j^-, \quad j = 1, ..., n,\\
& (vx_j - wz_j) \geq \Phi^{-1}(1-\alpha_j)(p_j^+ + p_j^-), \quad j = 1, ..., n,\\
& va_j - wb_j = p_j^+ - p_j^-, \quad j = 1, ..., n,\\
& (uy_o - wz_o) \geq e_{2l}^{*(o)}(q_o^+ + q_o^-),\\
& p_j^+ p_j^- = 0, \quad j = 1, ..., n,\\
& q_j^+ q_j^- = 0, \quad j = 1, ..., n,\\
& w, v, u, q^+, q^-, q^+, q^- \geq 0.
\end{aligned}
$$

Finally, the following model is the linear form of (3.37):

$$
\begin{aligned}
& e_{1f}^{*(o)} = Max\ (wz_o - vx_o)\\
& s.t. \qquad\qquad (3.38)\\
& p_o^+ + p_o^- \geq 1,\\
& (wz_j - uy_j) \geq \Phi^{-1}(1-\alpha_j)(q_j^+ + q_j^-), \quad j = 1, ..., n,\\
& wb_j - uc_j = q_j^+ - q_j^-, \quad j = 1, ..., n,\\
& (vx_j - wz_j) \geq \Phi^{-1}(1-\alpha_j)(p_j^+ + p_j^-), \quad j = 1, ..., n,\\
& va_j - wb_j = p_j^+ - p_j^-, \quad j = 1, ..., n,\\
& (uy_o - wz_o) \geq e_{2l}^{*(o)}(q_o^+ + q_o^-),\\
& w, v, u, q^+, q^-, q^+, q^- \geq 0.
\end{aligned}
$$

Next theorem compares the efficiency of one stage when it is follower and leader.

Theorem 3.5 *For every optimal solution of models (3.25) and (3.38), we have* $e_{1f}^{*(o)} \leq e_{1l}^{*(o)}$.
Proof:
Suppose $(w^*, v^*, u^*, q^{+*}, q^{-*}, q^{+*}, q^{-*})$ *is optimal solution of model (3.38). Since this solution is a feasible solution of (3.25), therefore,* $e_{1f}^{*(o)} \leq e_{1l}^{*(o)}$.■

Theorem 3.6 *For every optimal solution of models (3.28) and (3.33), we have* $e_{2f}^{*(o)} \leq e_{2l}^{*(o)}$.
Proof: *Similar to the proof of Theorem 3.5.*■

Corollary 3.1 If one stage is efficient under level α as a follower, then it is efficient under level α as a leader. Also, if one stage is not efficient under α level as a leader, then it is not efficient under level α as a follower.

Theorem 3.7 *For every level* $\alpha \leq \alpha'$, $e_{1f}^{*(o)}(\alpha) \geq e_{1f}^{*(o)}(\alpha')$.

Proof: Let $(w^*, v^*, u^*, q^{+*}, q^{-*}, q^{+*}, q^{-*})$ is an optimal solution of model (3.38) to evaluating performance of DMU_o in level α. Since $\Phi^{-1}(\alpha)$, is a strictly increasing function if $\alpha \leq \alpha'$ then $\Phi^{-1}(\alpha) \leq \Phi^{-1}(\alpha')$. Therefore

$$
\begin{aligned}
(\lambda^* z_j - v^* x_j) - \Phi^{-1}(\alpha')\tau_j^* &\leq (\lambda^* z_j - v^* x_j) - \Phi^{-1}(\alpha')\tau_j^*\\
&\leq 0,\\
(\mu^* y_j - \lambda^* z_j) - \Phi^{-1}(\alpha')\rho_j^* &\leq (\mu^* y_j - \lambda^* z_j) - \Phi^{-1}(\alpha)\rho_j^*\\
&\leq 0,
\end{aligned}
$$

Hence, $(w^*, v^*, u^*, q^{+*}, q^{-*}, q^{+*}, q^{-*})$ is an optimal solution of model (3.38) to evaluating performance of in level α'. According to maximization model, $e_{1f}^{*(o)}(\alpha) \geq e_{1f}^{*(o)}(\alpha')$.■

Theorem 3.8 *For every level* $\alpha < \alpha'$,

$$
\begin{aligned}
e_{1l}^{*(o)}(\alpha) &\geq e_{1l}^{*(o)}(\alpha'),\\
e_{2l}^{*(o)}(\alpha) &\geq e_{2l}^{*(o)}(\alpha'),\\
e_{2f}^{*(o)}(\alpha) &\geq e_{2f}^{*(o)}(\alpha').
\end{aligned}
$$

Corollary 3.2 If one stage either as a leader or follower is stochastically efficient under level $\alpha'(\alpha' > \alpha)$, then is stochastically efficient under level α. In addition, if one stage either as a leader or follower is not stochastically efficient under level α, then it is not stochastically efficient under level α'.

4 An illustrative application

After formulating our theoretical framework, we apply our proposed model to a real case. In performance evaluation studies, the commercial banks efficiency has been an interesting research.([29], [27], [20], [22]). The most of existing studies on bank benchmarking consider deterministic data.

We now apply stochastic data set of commercial banks taken from Zhou et al.[29] to illustrate models proposed in the preceding sections. In the computational studies, GAMS 23.5 and MATLAB 2017a solve nonlinear problems.

Zhou et al. [29] used a dataset of 16 commercial banks and assumed banking process as a kind of network with the two-stage process. Stage 1, consume Employee, Fixed assets, Expenses as inputs to product Deposits and Interbank Deposits as outputs. Then Stage 2, uses the productions of stage 1 as inputs to produce Loan and Profit as final outputs. Furthermore, all inputs and outputs are supposed has the normal distribution. Real data set of annual reports and internal databases cover the period from 2000 to 2010 from the Almanac of Chinas Finance and Banking [29]. Table 1 listed the approximated mean values and standard derivations of 16 banks.

Table 2 shows the overall efficiency scores using stochastic CCR DEA model where only considers the inputs of stage 1 and outputs of stage 2 of the process and avoid intermediate products, under levels $\alpha = 0.1, ..., 0.5$. It can be seen, DMUs 15 has the best performance at all levels. Since this approach, do not have any information about the performance of each stage, thus it is not appropriate to ignore the internal structure and consider it as a black box.

The efficiency scores using models (3.25) and (3.28) under levels $\alpha = 0.1, ..., 0.5$ are listed in Table 3. It means stage 1 is as a leader and stage 2 as a follower. As we can see, DMU 15 have the best performance, which indicates their both stages are stochastically DEA efficient at all levels. DMUs 2, 3, 9, 11, 14 and 16 perform the worst, which indicates their both stages are not stochastically DEA efficient under all levels. The first stage of some DMUs is not stochastically efficient under all levels, while stage 2 is stochastically efficient under some levels. For example, in DMU 1, under levels $\alpha = 0.1$ and $\alpha = 0.2$, stage 1 is not efficient while stage 2 is efficient. On the contrary, the first stage of some DMUs such as DMU 12 is stochastically efficient under some levels, while the second stage is not stochastically efficient. By comparing efficiencies under various levels, we can find out about the importance of selection of the decision makers confidence level in evaluating the performance [29].

The efficiency scores under models (3.33) and (3.38) are shown in Table 4. It means, stage 1 is as a follower and stage 2 is as a leader. As mentioned, DMU 15 has the best performance, which indicates in both stages are stochastically DEA efficient under all levels. In addition, DMUs 2, 3, 9, 11, 14 and 16 perform the worst, which indicates their both stages are not stochastically DEA efficient under all levels.

From Table 3 and Table 4, we can understand that, if one stage is efficient under level α as a follower, then it is efficient under level α as a leader, and if one stage is not efficient under level α as a leader, then it is not efficient under level α as a follower. For instance, the second stage in DMU 1 is stochastically efficient under level $\alpha = 0.2$ as a follower. It is stochastically efficient under level $\alpha = 0.2$ as a leader. While, the first stage of DMU 1, is not stochastically efficient under level $\alpha = 0.2$ as a leader, it is not stochastically efficient under level $\alpha = 0.2$ as a follower.

Finally, we can conclude that the DMU 15 has the best performance since they are stochastically efficient for all models under all levels. This indicates that one process is efficient if and only if both sub-processes is efficient and the best performance of only one sub-process cannot assure the efficiency of the total process.

Table 1: Estimated mean values and standard deviations of 16 banks

Bank	Employee (10^3)	Fixed assets (RMB 10^8)	Expenses (RMB 10^8)	Deposits (RMB 10^8)	Interbank Deposits (RMB 10^8)	Loan (RMB 10^8)	Profit (RMB 10^8)
1	(21.39,5.39)	(8.65,0.92)	(13.24,3.74)	(935.41,310.39)	(108.61,103.81)	(651.35,259.78)	(13.32,4.97)
2	(249.28,13.38)	(90.05,12.14)	(66.07,9.17)	(4925.35,1058.49)	(603.39,320.35)	(3189.65,1076.21)	(65.07,17.79)
3	(441.88,5.40)	(104.29,23.62)	(94.26,5.81)	(5989.71,1139.97)	(289.77,170.88)	(3014.98,451.41)	(51.45,29.15)
4	(49.29,0.57)	(9.91,0.06)	(8.54,1.07)	(183.29,53.73)	(119.83,56.46)	(1206.90,244.25)	(1.63,0.77)
5	(19.85,3.50)	(3.98,0.53)	(14.80,4.15)	(785.79,238.77)	(120.24,37.66)	(646.48,181.90)	(7.89,3.47)
6	(300.30,1.75)	(64.48,9.76)	(99.19,16.94)	(6375.92,1462.90)	(447.46,230.58)	(3683.58,821.06)	(92.64,26.62)
7	(16.99,2.54)	(8.18,2.66)	(8.30,1.86)	(523.90,115.67)	(121.69,78.56)	(450.52,128.52)	(7.32,2.28)
8	(385.61,17.29)	(80.30,12.33)	(91.51,15.73)	(8223.45,1558.40)	(592.61,236.65)	(4436.01,884.11)	(111.15,35.27)
9	(36.92,7.74)	(11.68,2.66)	(20.34,6.08)	(1250.65,367.00)	(115.79,67.66)	(852.75,269.32)	(20.95,5.99)
10	(19.54,5.89)	(3.39,0.30)	(10.35,2.84)	(558.34,80.84)	(182.91,54.44)	(489.99,161.84)	(11.39,4.12)
11	(10.38,1.63)	(1.68,0.19)	(5.22,1.32)	(254.97,50.46)	(36.06,24.25)	(281.71,78.97)	(0.61,1.95)
12	(17.60,4.39)	(6.26,0.70)	(12.68,2.74)	(805.30,269.91)	(222.44,98.13)	(681.27,147.24)	(12.52,4.96)
13	(77.73,8.60)	(22.74,2.07)	(23.56,2.65)	(1674.17,337.22)	(483.08,178.37)	(1298.78,386.33)	(28.52,8.23)
14	(11.11,1.86)	(3.64,0.20)	(7.29,1.62)	(394.35,72.21)	(87.01,29.12)	(345.67,70.81)	(3.07,1.02)
15	(6.22,1.05)	(12.44,7.93)	(5.50,2.05)	(242.99,108.07)	(115.45,78.89)	(2840.68,734.78)	(20.76,4.79)
16	(14.19,1.04)	(3.29,0.21)	(6.53,1.18)	(337.37,65.30)	(51.05,9.38)	(298.34,71.92)	(2.78,1.81)

* For (a,b), a is the mean value, and b is the standard derivation.

Table 2: Efficiency scores of stochastic CCR model

BANK	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$
1	0.0120	0.0000	0.2773	0.0273	0.0018
2	0.0000	0.0051	0.0743	0.1944	0.3656
3	0.0000	0.0000	0.0000	0.0014	0.1264
4	0.0343	0.0895	0.1655	0.2645	0.3760
5	0.0209	0.1127	0.1733	0.3373	0.4589
6	0.0000	0.0000	0.0037	0.1115	0.3841
7	0.0935	0.1799	0.2690	0.3624	0.4494
8	0.0000	0.0000	0.0052	0.1147	0.3888
9	0.0000	0.0000	0.0299	0.2185	0.4446
10	0.0380	0.1815	0.3000	0.4000	0.5000
11	0.0562	0.1465	0.2241	0.3460	0.4299
12	0.1000	0.2000	0.3000	0.3679	0.4893
13	0.0887	0.1020	0.2591	0.1498	0.4584
14	0.0013	0.0328	0.1133	0.2517	0.4160
15	0.1000	0.2000	0.3000	0.4000	0.5000
16	0.0000	0.0000	0.0000	0.0042	0.2356

Table 3: The efficiency scores of stochastic two-stage network DEA model (stage 1 as the leader, stage 2 as the follower)

DMU	Confidence level (α)									
	0.1		0.2		0.3		0.4		0.5	
	$e^{*(o)}_{1l}$	$e^{*(o)}_{2f}$	$e^{*(o)}_{1l}$	$e^{*(o)}_{2f}$	$e^{*(o)}_{1l}$	$e^{*(o)}_{2f}$	$e^{*(o)}_{1l}$	$e^{*(o)}_{2f}$	$e^{*(o)}_{1l}$	$e^{*(o)}_{2f}$
1	0.1	0.1	0.001472	0.2	0.013532	0.2572	0.001464	0.2775	0.056033	0.2924
2	0.010079	0.01238	0.012691	0.05256	0.021416	0.08579	0	0.09823	0.083719	0.1096
3	4.84E-06	0.005834	2.09E-05	0.08168	3.84E-05	0.1031	6.97E-05	0.1205	0.000183	0.1688
4	1.29E-26	0.01626	0.001724	0.03402	0.009976	0.04464	4.27E-06	0.05446	0.05381	0.06412
5	0.000102	0.000639	8.5E-05	0.000817	3.38E-08	0.000871	8.64E-05	0.000914	8.92E-05	0.000945
6	1.02E-12	0.006552	6.33E-16	0.08444	2.38E-09	0.2801	2.38E-09	0.4	2.38E-09	0.5
7	0.00014	0.04169	0.033537	0.07758	1.83E-42	0.09224	0.002149	0.09823	0.007521	0.1028
8	2.99E-09	0.000837	2.95E-17	0.0169	0.3	0.06509	1.33E-12	0.1286	0.5	0.2141
9	0.000525	0.005137	2.67E-05	0.1126	0.000639	0.2391	0.000525	0.3129	0.000646	0.375
10	0.020091	0.000169	0.075583	0.00039	0.024828	0.00039	0.011232	0.00039	0.023714	0.00039
11	0.000691	0.0377	0.000473	0.09244	0.000807	0.1038	0.000824	0.1129	0.000767	0.1211
12	0.1	0.01005	0.2	0.01441	0.3	0.01559	0.4	0.0165	0.5	0.01728
13	0.1	0.001946	0.2	0.004431	0.3	0.005641	0.33414	0.006071	0.5	0.006383
14	0.000105	0.000616	0.000105	0.001985	0.000196	0.002269	3.48E-05	0.002504	0.000792	0.002715
15	0.051956	0.1	0.06285	0.2	0.063944	0.3	0.029965	0.4	0.055146	0.5
16	5.92E-73	1.19E-07	1.46E-08	1.19E-07	2.57E-39	1.19E-07	1.34E-08	7.24E-08	1.80E-25	1.19E-07

Table 4: The efficiency scores of stochastic two-stage network DEA model (stage 2 as the leader, stage 1 as the follower)

DMU	Confidence level (α)									
	0.1		0.2		0.3		0.4		0.5	
	$e^{*(o)}_{2l}$	$e^{*(o)}_{1f}$	$e^{*(o)}_{2l}$	$e^{*(o)}_{1f}$	$e^{*(o)}_{2l}$	$e^{*(o)}_{1f}$	$e^{*(o)}_{2l}$	$e^{*(o)}_{1f}$	$e^{*(o)}_{2l}$	$e^{*(o)}_{1f}$
1	0.1	0.0002	0.2	0	0.25717	0.0002	0.27749	0.0002	0.29236	2.68E-05
2	0.012382	2.90E-13	0.052558	1.16E-131	0.085788	2.90E-13	0.099254	2.90E-13	0.10964	2.90E-13
3	5.83E-03	7.99E-06	0.082695	0	0.10308	6.28E-06	0.12047	6.28E-06	0.16877	1.16E-05
4	1.97E-02	7.24E-16	0.034017	0	0.044637	7.19E-16	5.45E-02	7.24E-16	0.064123	7.24E-16
5	0.000678	0.000181	0.000817	3.94E-05	8.71E-04	0.000181	0.000914	0.000181	0.000953	0.000181
6	6.55E-03	2.00E-09	8.44E-02	7.31E-22	2.80E-01	2.38E-09	4.00E-01	2.38E-09	5.00E-01	2.38E-09
7	0.041692	0.004517	0.077578	0.001876	9.22E-02	0.000929	0.098233	0.006999	0.10281	0.002317
8	8.37E-04	0.1	1.69E-02	2.04E-13	0.065183	2.99E-09	1.29E-01	0.4	0.21407	0
9	0.022329	0.000631	0.11261	0.000525	0.23908	0.000454	0.31289	0.000642	0.37504	3.08E-11
10	0.000169	0.01964	0.00039	0.06152	0.00039	1.72E-137	0.00039	0.08341	0.00039	0.5
11	0.0377	0.000771	0.092438	0.000791	0.10377	0.000807	0.11293	0.000824	0.12106	0.005847
12	0.010052	0.1	0.01441	0.2	0.015586	0.3	0.016496	0.4	0.017278	0.5
13	0.001946	0.1	0.004431	0.2	0.005642	0.1125	0.006071	0.1294	0.006383	0.5
14	0.000616	0.00018	0.001985	0.000189	0.00227	0.000531	0.002504	0.000202	0.002715	0.000965
15	0.1	0.06341	0.2	0.000818	0.3	1.80E-34	0.4	0.06419	0.5	0.04033
16	1.19E-07	1.35E-08	1.19E-07	5.86E-51	1.19E-07	1.18E-08	1.19E-07	1.86E-31	1.19E-07	5.39E-13

5 Conclusion

The present paper analyses the performance evaluation of network structures with the two-stage process when the data are uncertainty. The stochastic network DEA for performance evaluation of two-stage processes according to non-cooperative game theory is proposed. We focus on a more general class of stochastic models that Charnes and Cooper [4] refer to as P-Model. Then, the chance-constrained and objective function of models are converted to deterministic by using the Cooper et al. [8] approach. The relationship between the optimal solution of the stochastic model and deterministic model, efficiency of one stage as leader or follower, are discussed. The efficiency of one stage when is the follower is less than or equal to the efficiency of it when is the leader. Furthermore, the stochastically efficiency of the entire process depends on the stochastically efficiency of both sub-process under non-cooperative stochastic network model. We pint out this paper uses the concept of a non-cooperative game theory. For further research, one can examine the behavior of a network process with two-stage structure by using the concept of the cooperative game theory.
An empirical example of 16 commercial banks in China to demonstrate the theoretical contributions of the current paper has been used. Within an application, the procedure of banks is divided into two processes, namely down payment producing process and income making process [29]. Finally, the current model is according to concepts of non-cooperative game theory and P-model, how to modify this model by using concepts of cooperative game theory and E-model is left for future research.

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