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A New Two-stage Iterative Method for Linear Systems and its Application in Solving the Poisson's Equation

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Abstract

In the current study, we investigate the two-stage iterative method for solving linear systems. In this regard, a new comparison theorem for the spectral radius of two-stage iterative method with different inner and outer splitting matrices under suitable conditions is presented. Our new theorem shows which splitting generates fast convergence in iterative methods. Finally, based on the Poisson's equation, we solve the Poisson-Block tridiagonal matrix by two different splittings, which is an issue that is frequently encountered in mechanical engineering and theoretical physics. Based on a particular linear system, numerical computations are presented, which clearly show the reliability and efficiency of the presented algorithm.

Keywords : Two stage iterative method; Splitting; Comparison theorem; Spectral radius.

1 Introduction

Consider the following system of linear equations

$$Ax = b, \qquad (1.1)$$

where $A \in \mathbb{R}^{n \times n}$ and $b, x \in \mathbb{R}^n$. Linear systems 1.1 occur in a wide variety of areas, including numerical differential equations, eigenvalue problems, economics models, design and computer analysis of circuits, power system networks, chemical engineering processes, physical and biological sciences. See [16, 17, 21, 26, 30, 31, 14], for discussion of such applications. For any splitting,

$A = M - N$ with $det(M) \neq 0$ the basic iterative methods for solving 1.1 is

$$x_i = M^{-1}Nx_{i-1} + M^{-1}b, \ i = 1, 2, \dots \qquad (1.2)$$

The *spectral radius* of a real square matrix A is the maximum moduli of the eigenvalues of A, which is denoted by $\rho(A)$. For a splitting $A = M - N$ the iteration scheme 1.2 converges to the unique solution $x = A^{-1}b$ for any initial vector value, if and only if $\rho(M^{-1}N) < 1$, where $T = M^{-1}N$ is called the iteration matrix. There are some popular iterative methods for solving linear systems 1.1 based on 1.2; for instance, Jacobi, Gauss-Seidel, SOR, etc. [13, 15, 19, 20, 22, 25, 28, 29, 32]. The methods of the above-mentioned models, proceed by solving at each step a simpler system; however, when this system is itself solved by an inner iterative method, the global method is called a two-stage iterative method. This method is one of

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the drastic choices for getting the numerical solutions of linear systems. The two-stage iterative method was first proposed by Nichols [12] in 1973 for solving systems of linear equations; but then has been extensively studied by many authors [3, 5, 6, 7, 11, 18, 23, 24, 1]. This method, also called the inner/outer method, consists of approximating the linear system 1.2 by using inner iterations; i.e., let $A = M - N$ and the splitting $M = F - G$ perform, say $(s(k))$ inner iterations. In this regard, some comparison theorems for splitting $A = F_1 - G_1 - N = F_2 - G_2 - N$ have been proposed in the literature [2, 4, 8, 9, 27]. In this paper, On the basis of nonnegative matrix theory, we present a new comparison theorem for the convergent splittings of two iteration matrices, induced by our proposed method. But up to now, no study has discussed comparison theorems when both inner and outer iterations are different splittings (i.e $N_1 \neq N_2$). So, this paper is planning to fill in this gap and reach a comparison theorem for two-stage iterative methods under different conditions. Our new comparison theorem shows that which splitting produces less error, which is considered the better one then.

This paper is organized as follows. After reviewing the two-stage iterative method and introducing some related essential concepts and results in Section 2, we further set up our new results in Section 3. The convergence analysis and error bounds of our method will be presented in this Section. In Section 4, we examine the advantages of our results by carrying out numerical computations. For example, we solved the Poisson-Block tridiagonal matrix from the Poisson's equation perspective, which arises in mechanical engineering and theoretical physics. Finally, conclusions are presented in Section 5.

2   Preliminaries

Consider the linear system 1.1 with outer splitting $A = M - N$ and inner splitting $M = F - G$. Then the algorithm of two-stage iterative method is as follows. Algorithm1.

Step 1.

Choose an initial vector x_0, tol, number of outer iteration m and sequence of number of inner iterations, $s(k), k = 1, ..., m$.

Step 2.

For $i = 1, ..., m$ do
$y_0 = x_{i-1}$
For $j = 1, ..., s(k)$
$Fy_j = Gy_{j-1} + Ny_{i-1} + b$
$x_i = y_{s(k)}$.

Step 3.

If $b - Ax_i \leq tol$, then stop; otherwise, set $i = i+1$ and go to Step 2.

When the number of inner iterations is fixed in each outer step, i.e., $s(k) = s$, $s \geq 1$, it is said that the method is stationary, while a non-stationary two-stage method is such that the number of inner iterations may change with the outer iterations. Throughout the current paper, it is assumed that $s(k) = s$, $s \geq 1$.

By replacing the loop over j and by 1.2, the two-stage iterative methods for solving the system of linear equations 1.1 have the following form

$$x_i = (F^{-1}G)^s x_{i-1} + \\ + \sum_{j=0}^{s-1}(F^{-1}G)^j F^{-1}(Nx_{i-1} + b), \qquad (2.3) \\ i = 1, 2, ...$$

Clearly, the iteration matrix corresponding to 2.3 is

$$T_s = (F^{-1}G)^s + \sum_{j=0}^{s-1}(F^{-1}G)^j F^{-1}N \\ = I - (I - (F^{-1}G)^s)(I - M^{-1}N) \qquad (2.4)$$

where I denotes the $n \times n$ identity matrix.
If $\rho(F^{-1}G) < 1$, then $I - (F^{-1}G)^s$ is nonsingular. Then there exists a unique pair of matrices [10], B_s and C_s, such that $M = B_s - C_s$ and $R = (F^{-1}G)^s = B_s^{-1}C_s$ where

$$B_s = M(I - R)^{-1}, \qquad (2.5)$$

$$C_s = M(I - R)^{-1}R, \qquad (2.6)$$

$$T_s = B_s^{-1}(C_s + N). \qquad (2.7)$$

In this manner we are ready to establish our new theorem in the next section.

3 Main results

In this section we present a new theorem under suitable conditions. Based on this theorem, we can find which double splitting is more efficient compared with other ones. We should begin with some basic notations and preliminary results first.

Definition 3.1 *Let A be a real matrix. The splitting $A = M - N$ is called*
(a) convergent if $\rho(M^{-1}N) < 1$,
(b) regular if $M^{-1} \geq 0$ and $N \geq 0$, and
(c) weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.

Lemma 3.1 *Let $A = M - N$ be a convergent regular splitting, and let $R \geq 0$, $\rho(R) < 1$. If the unique splitting is as $M = B_s - C_s$ such that $R = B_s^{-1}C_s$ is a weak regular splitting, then the two-stage iterative method for any nonnegative s of inner iterations will be convergent.*

Proof. see [10].

Lemma 3.2 *Let $A = M - N$ be regular or a weak regular splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq 0$.*

Proof. see [2].

Lemma 3.3 *Let $A = M_1 - N_1 = M_2 - N_2$ be two weak regular splitting of A, where $A^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$, then $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2)$.*

Proof. see [12].
Now, we establish new results in the following theorem.

Theorem 3.1 *Let $A^{-1} \geq 0$, $A = M_1 - N_1 = M_2 - N_2$ be regular splitting and let $M_1 = F_1 - G_1$, $M_2 = F_2 - G_2$ be weak regular splitting. If $M_2^{-1} \geq \alpha M_1^{-1}$ then*

$$\rho(T_s(M2 - N_2)) \leq \rho(T_s(M1 - N_1)) < 1$$

where $\alpha = \frac{1-\rho_1}{1-\rho_2}$ with $\rho_i = \rho(F_i^{-1}G_i)$ for $i = 1, 2$.

Proof. By Lemma 3.1, it is easy to show that $\rho(T_s(M_1 - N_1)) < 1$. But by two-stage iterative method (Algorithm1), for $i = 1, 2$ we have

$$A = M_i - N_i,$$

$$M_i = F_i - G_i = B_{i,s} - C_{i,s}$$

and

$$R_i = (F_i^{-1}G_i)^s = B_{i,s}^{-1}C_{i,s},$$

so we get that

$$A = M_{i,T} - N_{i,T},$$

where

$$M_{i,T} = B_{i,s} = M_i(I - R_i)^{-1}$$

and

$$N_{i,T} = (C_{i,s} + N_i) = M_i(I - R_i)^{-1}R_i + N_i.$$

Since $A = M_i - N_i$ are regular splittings, $M_i = F_i - G_i$ is the weak regular splitting. By Lemma 3.2, $\rho_1, \rho_2 < 1$. Furthermore, it can be shown that $A = M_{(i,T)} - N_{(i,T)}$ is the weak regular splitting. Therefore, to apply Lemma 3.3, it is only necessary to show that $M_{2,T}^{-1} \geq M_{1,T}^{-1}$. Since $M_2^{-1} \geq \alpha M_1^{-1}$, we have

$$(1 - \rho(F_2^{-1}G_2))M_1 \geq M_2(1 - \rho(F_1^{-1}G_1)).$$

Since
$$GF^{-1} = F(F^{-1}G)F^{-1},$$

we have

$$\rho(GF^{-1}) = \rho(F^{-1}G),$$

so

$$(I - (G_2F_2^{-1}))M_1 \geq M_2(I - (F_1^{-1}G_1))$$

$$\Rightarrow (I - (G_2F_2^{-1})^s)M_1 \geq M_2(I - (F_1^{-1}G_1)^s)$$

$$\Rightarrow M_2^{-1}(I - (G_2F_2^{-1})^s) \geq (I - (F_1^{-1}G_1)^s)M_1^{-1}$$

$$\Rightarrow M_2^{-1} - M_2^{-1}(G_2F_2^{-1})^s \geq (I - R_1)M_1^{-1}.$$

Since

$$M(F^{-1}G)^s = (GF^{-1})^sM,$$

we have

$$(I - (F_2^{-1}G_2)^s)M_2^{-1} \geq (I - R_1)M_1^{-1},$$

and the proof is completed.∎

4   Examples

In this section, as an application of Theorem 3.1, we give some examples to illustrate the results that have been obtained in the previous sections. We use the number of inner iteration as $s(k) = s$, $s \geq 1$. Moreover, we denote the number of outer iteration by m.

Example 4.1 Let $A = \begin{pmatrix} 1 & -1 \\ -1.5 & 1.9 \end{pmatrix}$ where

$$M_1 = \begin{pmatrix} 1.1 & -0.9 \\ -0.9 & 2.1 \end{pmatrix}, F_1 = \begin{pmatrix} 2.1 & -1.8 \\ -1.6 & 5.1 \end{pmatrix},$$

$$G_1 = \begin{pmatrix} 1 & -0.9 \\ -0.7 & 3 \end{pmatrix}, N_1 = \begin{pmatrix} 0.1 & 0.1 \\ 0.6 & 0.2 \end{pmatrix}.$$

$$M_2 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, F_2 = \begin{pmatrix} 1.25 & -1.8 \\ -1.25 & 3.6 \end{pmatrix},$$

$$G_2 = \begin{pmatrix} 0.25 & -0.8 \\ -0.25 & 1.6 \end{pmatrix}, N_2 = \begin{pmatrix} 0 & 0 \\ 0.5 & 0.1 \end{pmatrix}.$$

It is not difficult to examine splittings as defined in Theorem 3.1 Furthermore

$$\alpha = 0.6609$$

$$\alpha M_1^{-1} = \begin{pmatrix} 0.9253 & 0.3965 \\ 0.3965 & 0.4847 \end{pmatrix}$$
$$< \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = M_2^{-1}.$$

Hence, Theorem 3.1 implies the following inequality

$$\rho(T_s(M_2 - N_2)) \leq \rho(T_s(M_1 - N_1)) \leq 1, \tag{4.8}$$

$$\forall s \geq 1.$$

In fact, by computations, Table. 1 shows that our theorem holds for the above matrix. In this example $A^{-1} = \begin{pmatrix} 4.75 & 2.5 \\ 3.75 & 2.5 \end{pmatrix}$, so the exact solution of the system $Ax = b$ with $b = (1,2)^T$ is $(\frac{39}{4}, \frac{35}{47})^T$. We report the numerical solution errors of this system in Table. 2 with initial vector value $v_0 = (8,8)^T$. From Table. 2 we can see that errors generated by second splitting are less than errors generated by first splitting with the same inner and outer iterations.

Remark 4.1 Moreover if the matrix is large then we can use the Krylov subspace methods.

With this method we can the N-dimensional problem onto nested Krylov subspaces of increasing dimension. We now consider methods for improving the accuracy. Let x be approximated solution with Krylov subspaces method to the linear system of equations $Ax = b$ and let $r = b - Ax$ be the corresponding residual vector. Then one can attempt to improve the solution by solving the system $A\delta = r$ for a correction δ and taking $x_c = x + \delta$ as a new approximation. If no further rounding errors are performed in the computation of δ this is the exact solution. Otherwise this refinement process can be iterated.

Example 4.2 In this example we consider linear system $Ax = b$ with $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{pmatrix}$ and $b = (-1, 2, 0.5)^T$.

The exact solution of this system is $(2.5, 3.5, 1)^T$. It is easy to see that $A^{-1} \geq 0$. Some regular splitting of $A = M - N$ and weak regular splitting of $M = F - G$ are given below.

$$M_1 = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{pmatrix}, F_1 = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{pmatrix},$$

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, N_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}, F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix},$$

$$G_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, N_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

$$M_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}, F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, N_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In this example we denote $\frac{1-\rho_i}{1-\rho_j}$ with $\alpha_{i,j}$. For the above splitting of A we have

$$\alpha_{2,1} = 0.6667, \alpha_{2,3} = 1, \alpha_{3,1} = 0.667,$$

Table 1: Spectral radii of iteration matrices in example 4.1.

s	$\rho(T_s(M_1 - N_1))$	T˙s(M˙2-N˙2)
1	0.9002	0.7072
2	0.8392	0.6295
3	0.8019	0.6092
4	0.7791	0.6033
5	0.7652	0.6013

Table 2: Numerical solution errors in example 4.1.

s	m	by first splitting	by second splitting
10	20	0.0381	5.5871×10^{-4}
20	30	7.7560×10^{-4}	4.7439×10^{-7}
30	40	6.6980×10^{-5}	2.0870×10^{-9}
40	50	3.6971×10^{-6}	1.2394×10^{-11}
50	60	1.9139×10^{-7}	6.9095×10^{-14}

Table 3: Numerical solution errors in example 4.2.

s	m	by first splitting	by second splitting	by third splitting
5	10	0.0072	0.2434	0.0116
6	15	2.2426×10^{-4}	0.0432	0.0012
7	20	7.0080×10^{-6}	0.0077	3.2259×10^{-4}
8	25	2.1900×10^{-7}	0.0014	1.0500×10^{-4}
9	30	6.8438×10^{-9}	2.4983×10^{-4}	3.4921×10^{-5}

Table 4: Results of example 4.3 with s=50.

n	no_{GJ}	error of GJ	no_{JG}	error of JG
4	55	6.8978×10^{-10}	103	2.0283×10^{-9}
8	171	2.0555×10^{-8}	327	4.6248×10^{-8}
16	575	5.6102×10^{-7}	1106	1.1536×10^{-6}

and

$$M_1^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0.5 & 1.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{pmatrix} \geq$$

$$\begin{pmatrix} 0.6667 & 0 & 0 \\ 0 & 0.6667 & 0.3333 \\ 0 & 0 & 0.3333 \end{pmatrix} = \alpha_{2,1} M_2^{-1},$$

$$M_3^{-1} = \begin{pmatrix} 1 & 1 & 0.5 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0.5 \end{pmatrix} \geq$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0.5 \end{pmatrix} = \alpha_{2,3} M_2^{-1}.$$

Hence, Theorem 3.1 implies the following inequalities for all s

$$\rho(T_s(M_1 - N_1)) \leq \rho(T_s(M_2 - N_2)) \leq 1, \quad (4.9)$$

$$\rho(T_s(M_3 - N_3)) \leq \rho(T_s(M_2 - N_2)) \leq 1. \quad (4.10)$$

For initial vector value $(0, 0, 0)^T$ the errors generated by above spilittings are shown in Table. 3 with the same inner and outer iterations.

Figs. 1 and 2 show the errors of the above splittings of A for inner iterations $m = 1, 2, 3, , 50$ and outer iterations $s = 10, 25$, respectively. From

Figs. 1, 2, and Table. 3 we can find that the first splitting converges to the exact solution faster than the third splitting, and the same happens for the third splitting compared with the second splitting. This is supported by inequalities 4.9 and 4.10.

Example 4.3 *Poisson Block tridiagonal matrix from Poisson's equation.*

The Poisson equation is a very powerful tool for modeling the behavior of electrostatic systems, although it might be only solved analytically for very simplified models. Consequently, a numerical simulation must be utilized in order to model the behavior of complex geometries with practical values. In two-dimension spaces, the Poisson equation can be written as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho,$$

where u is the unknown function and ρ is a given function.

The finite difference method is based on local approximations of the partial derivatives in a Partial Differential Equation, which are derived by low order Taylor series expansions. The method is quite simple to define and rather easy to implement. Also, it is particularly appealing for simple regions, such as rectangles, and when uniform meshes are used. The matrices that result from these discretizations are often well structured, which means that they typically consist of a few nonzero diagonals. Today, Finite-difference methods are the dominant approach to numerical solutions of partial differential equations. Using the finite difference numerical method, the discretization of the two dimensional Poisson equation can be written as

$$Ax = b, \qquad (4.11)$$

where matrix A is a block tridiagonal matrix of order n^2, and it called the poisson matrix. To produce a poisson matrix of dimension n, one may use the MATLAB command

$$A = gallery('poisson', n).$$

For solving system 4.11 through using the two-stage iterative method with initial vector zeroes,

we consider the classical iterative methods such as Jacobi and Gauss-Seidel for inner and outer iteration.

First splitting.

In this case, we apply Gauss-Seidel's method to outer iteration and Jacobi's method for inner iteration, i.e., $A = M_G - N_G$, $M_G = F_J - G_J$.

Second splitting.

We use Jacobi and Gauss-Seidel method to outer and inner iterations, respectively, i.e. $A = M_J - N_J$, $M_J = F_G - G_G$.

It is not difficult to examine that splittings are defined as in Theorem 3.1. Furthermore:

$$\alpha = 1,$$

and

$$M_G^{-1} \geq M_J^{-1},$$

hence Theorem 3.1 implies that the first splitting performs much better than second splitting. The results of the numerical solution of system 4.11 with $(1, 1, 1, ..., 1)^T$ are shown in Table 4, where in Table 4, no_{GJ} and no_{JG} denote number of outer splitting of first and second splitting respectively. The stopping criterion for outer iteration m is $\frac{\|x_i - x_{i-1}\|_2}{\|x_i\|_2} < 10^{-10}$, where x_i is the numerical solution in the i'th iteration.

In Table 4, we report the number of iterations for the corresponding splitting of A with different n for $s = 50$ (number of inner iterations) and the numerical errors as $\|exact - numerical\|_2$.

The results of Table 4 were supported by theorem 3.1, and the first splitting performs much better than the second splitting.

5   Conclusions

In this paper, we have studied the two-stage iteration method for solving systems of linear equations. Moreover, we have discussed comparison theorems when both inner and outer iterations are different splittings (i.e $N_1 \neq N_2$). By this theorem we can find that which splitting performs much better than the other. The numerical results also show our claim in this new theorem.
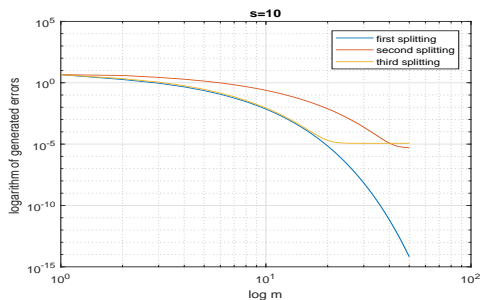
Figure 1: Comparison of error of spltttings with inner iteration $s = 10$ and outer iteration $m = 1, 2, 3, ..., 50$ in Example 4.2.
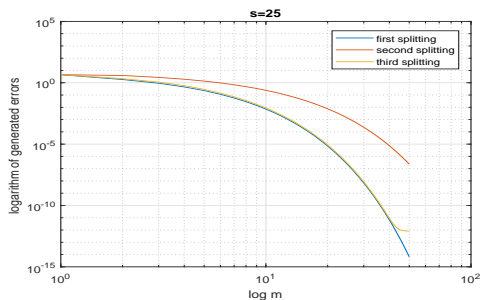

Figure 2: Comparison of error of spltttings with inner iteration $s = 25$ and outer iteration $m = 1, 2, 3, ..., 50$ in example 4.2.

In Example 4.2 we consider three different splittings. The results in Table. 3, Figs. 1, and 2 show that which splitting converges fast and works with high accuracy. This result was supported by our new theorem. Moreover, in Example 4.3 we solved the Poisson-Block tridiagonal matrix from Poisson's equation by two different splittings, which arises in mechanical engineering and theoretical physics. In this regard, Table 4 shows that it is important to choose the appropriate splitting.

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