Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 11, No. 2, 2019 Article ID IJIM-001062, 9 pages
Research Article

Numerical Solution of Second Kind Volterra and Fredholm Integral Equations Based on a Direct Method Via Triangular Functions

S. Hatamzadeh-Varmazyar *†, Z. Masouri ‡

Received Date: 2017-04-27 Revised Date: 2018-07-31 Accepted Date: 2018-12-02

Abstract

A numerical method for solving linear Volterra and Fredholm integral equations of the second kind is formulated. Based on a special representation of vector forms of triangular functions (TFs) and the related operational matrix of integration, the integral equation reduces to a linear system of algebraic equations. The generation of this system needs no integration, so all calculations can easily be implemented. Numerical results for some examples show that the method has a good accuracy.

Keywords : Integral equations of the second kind; Direct method; Vector forms; Triangular functions; Approximate solution.

1 Introduction

Numerical methods are widely used for solving integral and integro-differential equations, because a great number of problems in physical science and engineering are modeled by such equations [6, 5, 2, 15, 11, 13, 16, 17, 9, 3, 10, 4, 18].

This paper uses the TFs as a set of orthogonal basis functions for formulation of a direct method for solving both Volterra and Fredholm integral equations of the second kind. For this purpose, we review the TFs and a special representation of their vector forms as well as the related operational matrix of integration. Then, the direct method is formulated for numerical solution of the second kind integral equations. Finally, some examples are solved by the method. The obtained results are compared with those of other methods to illustrate the efficiency and accuracy of the direct method for solving the mentioned integral equations.

*Corresponding author. s.hatamzadeh@yahoo.com.
†Department of Electrical Engineering, Islamshahr Branch, Islamic Azad University, Tehran, Iran.
‡Department of Mathematics, Islamshahr Branch, Islamic Azad University, Tehran, Iran.

2 Review of triangular functions

2.1 Definition

Two m-sets of TFs are defined over the interval $[0, T)$ as [7, 6]

$$T1_i(t) = \begin{cases} 1 - \frac{t-ih}{h}, & ih \le t < (i+1)h, \\ 0, & \text{otherwise}, \end{cases}$$
$$T2_i(t) = \begin{cases} \frac{t-ih}{h}, & ih \le t < (i+1)h, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $i = 0, 1, \ldots, m-1$, with a positive integer value for m. Also, consider $h = T/m$, and $T1_i$ as the ith left-handed TF and $T2_i$ as the ith right-handed TF.

In this paper, it is assumed that $T = 1$, so TFs are defined over $[0, 1)$, and $h = 1/m$.

From the definition of TFs, it is clear that they are disjoint, orthogonal, and complete [7]. Therefore, we can write

$$\int_0^1 T1_i(t)T1_j(t)dt = \int_0^1 T2_i(t)T2_j(t)dt = \begin{cases} \frac{h}{3}, & i = j, \\ 0, & i \neq j. \end{cases} \tag{2.2}$$

Also,

$$\varphi_i(t) = T1_i(t) + T2_i(t), \quad i = 0, 1, \ldots, m-1, \tag{2.3}$$

where $\varphi_i(t)$ is the ith BPF defined as

$$\varphi_i(t) = \begin{cases} 1, & ih \leqslant t < (i+1)h, \\ 0, & \text{otherwise}, \end{cases} \tag{2.4}$$

where $i = 0, 1, ..., m-1$.

2.2 Vector forms

Consider the first m terms of left-handed TFs and the first m terms of right-handed TFs and write them concisely as m-vectors:

$$\mathbf{T1}(t) = [T1_0(t), T1_1(t), ..., T1_{m-1}(t)]^T,$$
$$\mathbf{T2}(t) = [T2_0(t), T2_1(t), ..., T2_{m-1}(t)]^T, \tag{2.5}$$

where $\mathbf{T1}(t)$ and $\mathbf{T2}(t)$ are called left-handed triangular functions (LHTF) vector and right-handed triangular functions (RHTF) vector, respectively.

The following properties of the product of two TFs vectors may be obtained [6]:

$$\mathbf{T1}(t)\mathbf{T1}^T(t) \simeq \begin{pmatrix} T1_0(t) & 0 & \ldots & 0 \\ 0 & T1_1(t) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & T1_{m-1}(t) \end{pmatrix},$$
$$\mathbf{T2}(t)\mathbf{T2}^T(t) \simeq \begin{pmatrix} T2_0(t) & 0 & \ldots & 0 \\ 0 & T2_1(t) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & T2_{m-1}(t) \end{pmatrix}, \tag{2.6}$$

and

$$\mathbf{T1}(t)\mathbf{T2}^T(t) \simeq \mathbf{0},$$
$$\mathbf{T2}(t)\mathbf{T1}^T(t) \simeq \mathbf{0}, \tag{2.7}$$

where $\mathbf{0}$ is the zero $m \times m$ matrix. Also,

$$\int_0^1 \mathbf{T1}(t)\mathbf{T1}^T(t)dt = \int_0^1 \mathbf{T2}(t)\mathbf{T2}^T(t)dt \simeq \frac{h}{3}I,$$
$$\int_0^1 \mathbf{T1}(t)\mathbf{T2}^T(t)dt = \int_0^1 \mathbf{T2}(t)\mathbf{T1}^T(t)dt \simeq \frac{h}{6}I, \tag{2.8}$$

in which I is $m \times m$ identity matrix.

2.3 TFs expansion

The expansion of a function $f(t)$ over $[0, 1)$ with respect to TFs, may be compactly written as

$$f(t) \simeq \sum_{i=0}^{m-1} c_i T1_i(t) + \sum_{i=0}^{m-1} d_i T2_i(t) = \mathbf{c}^T\mathbf{T1}(t) + \mathbf{d}^T\mathbf{T2}(t), \tag{2.9}$$

where we may put $c_i = f(ih)$ and $d_i = f((i+1)h)$ for $i = 0, 1, \ldots, m-1$. So, approximating a known function by TFs needs no integration to evaluate the coefficients.

2.4 Operational matrix of integration

Expressing $\int_0^s \mathbf{T1}(\tau)d\tau$ and $\int_0^s \mathbf{T2}(\tau)d\tau$ in terms of TFs follows [7]:

$$\int_0^s \mathbf{T1}(\tau)d\tau \simeq P1\mathbf{T1}(s) + P2\mathbf{T2}(s),$$
$$\int_0^s \mathbf{T2}(\tau)d\tau \simeq P1\mathbf{T1}(s) + P2\mathbf{T2}(s), \tag{2.10}$$

where $P1_{m\times m}$ and $P2_{m\times m}$ are called operational matrices of integration in TFs domain and represented as follows:

$$P1 = \frac{h}{2}\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \tag{2.11}$$

$$P2 = \frac{h}{2}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

So, the integral of any function $f(t)$ can be approximated as

$$\int_0^s f(\tau)d\tau \simeq \int_0^s \left[\mathbf{c}^T\mathbf{T1}(\tau) + \mathbf{d}^T\mathbf{T2}(\tau)\right] d\tau$$
$$\simeq (\mathbf{c}+\mathbf{d})^T P1\mathbf{T1}(s) + (\mathbf{c}+\mathbf{d})^T P2\mathbf{T2}(s). \tag{2.12}$$

3 A special representation of TFs vector forms and other properties

In this section, we review a special representation of TFs vector forms that has originally been introduced in [6].

3.1 Definition and expansion

Let $\mathbf{T}(t)$ be a $2m$-vector defined as [6]

$$\mathbf{T}(t) = \begin{pmatrix} \mathbf{T1}(t) \\ \mathbf{T2}(t) \end{pmatrix}, \quad 0 \leqslant t < 1 \tag{3.13}$$

where $\mathbf{T1}(t)$ and $\mathbf{T2}(t)$ have been defined in (2.5). Now, the expansion of $f(t)$ with respect to TFs can be written as

$$\begin{aligned} f(t) &\simeq F1^T\mathbf{T1}(t) + F2^T\mathbf{T2}(t) \\ &= F^T\mathbf{T}(t) \\ &= \mathbf{T}^T(t)F, \end{aligned} \tag{3.14}$$

where $F1$ and $F2$ are TFs coefficients with $F1_i = f(ih)$ and $F2_i = f((i+1)h)$, for $i = 0, 1, \dots, m-1$. Also, $2m$-vector F is defined as

$$F = \begin{pmatrix} F1 \\ F2 \end{pmatrix}. \tag{3.15}$$

Now, assume that $k(s,t)$ is a function of two variables. It can be expanded with respect to TFs as follows:

$$k(s,t) \simeq \mathbf{T}^T(s)\ K\ \mathbf{T}(t), \tag{3.16}$$

where $\mathbf{T}(s)$ and $\mathbf{T}(t)$ are $2m_1$- and $2m_2$-dimensional TFs respectively, and K is a $2m_1 \times 2m_2$ TFs coefficient matrix. For convenience, we put $m_1 = m_2 = m$. So, matrix K can be written as

$$K = \begin{pmatrix} (K11)_{m\times m} & (K12)_{m\times m} \\ (K21)_{m\times m} & (K22)_{m\times m} \end{pmatrix}, \tag{3.17}$$

where $K11$, $K12$, $K21$, and $K22$ can be computed by sampling of function $k(s,t)$ at points s_i and t_i such that $s_i = t_i = ih$, for $i = 0, 1, \dots, m$. Therefore,

$$\begin{aligned} (K11)_{i,j} &= k(s_i,t_j), && i = 0,1,\dots,m-1, \\ & && j = 0,1,\dots,m-1, \\ (K12)_{i,j} &= k(s_i,t_j), && i = 0,1,\dots,m-1, \\ & && j = 1,2,\dots,m, \\ (K21)_{i,j} &= k(s_i,t_j), && i = 1,2,\dots,m, \\ & && j = 0,1,\dots,m-1, \\ (K22)_{i,j} &= k(s_i,t_j), && i = 1,2,\dots,m, \\ & && j = 1,2,\dots,m. \end{aligned} \tag{3.18}$$

3.2 Product properties

Let X be a $2m$-vector which can be written as $X^T = (X1^T \quad X2^T)$ such that $X1$ and $X2$ are m-vectors. Now, it can be concluded from Eqs. (2.6) and (2.7) that [6]:

$$\begin{aligned} &\mathbf{T}(t)\mathbf{T}^T(t)X \\ &= \begin{pmatrix} \mathbf{T1}(t) \\ \mathbf{T2}(t) \end{pmatrix} \left(\mathbf{T1}^T(t) \quad \mathbf{T2}^T(t)\right) \begin{pmatrix} X1 \\ X2 \end{pmatrix} \\ &\simeq \begin{pmatrix} diag(\mathbf{T1}(t)) & \mathbf{0}_{m\times m} \\ \mathbf{0}_{m\times m} & diag(\mathbf{T2}(t)) \end{pmatrix} \begin{pmatrix} X1 \\ X2 \end{pmatrix} \\ &= diag(\mathbf{T}(t))\ X \\ &= diag(X)\ \mathbf{T}(t). \end{aligned} \tag{3.19}$$

Therefore,

$$\mathbf{T}(t)\mathbf{T}^T(t)X \simeq \tilde{X}\mathbf{T}(t), \tag{3.20}$$

where $\tilde{X} = diag(X)$ is a $2m\times 2m$ diagonal matrix.

Now, let B be a $2m \times 2m$ matrix as:

$$B = \begin{pmatrix} (B11)_{m\times m} & (B12)_{m\times m} \\ (B21)_{m\times m} & (B22)_{m\times m} \end{pmatrix}. \tag{3.21}$$

So, it can be similarly concluded from Eqs. (2.6) and (2.7) that:

$$\begin{aligned} &\mathbf{T}^T(t)B\mathbf{T}(t) \\ &= \begin{pmatrix}\mathbf{T1}^T(t) & \mathbf{T2}^T(t)\end{pmatrix} \begin{pmatrix} B11 & B12 \\ B21 & B22 \end{pmatrix} \begin{pmatrix}\mathbf{T1}(t) \\ \mathbf{T2}(t)\end{pmatrix} \\ &\simeq \mathbf{T1}^T(t)B11\ \mathbf{T1}(t) + \mathbf{T2}^T(t)B22\ \mathbf{T2}(t) \\ &\simeq \hat{B}11^T\ \mathbf{T1}(t) + \hat{B}22^T\ \mathbf{T2}(t), \end{aligned} \tag{3.22}$$

where $\hat{B}11$ and $\hat{B}22$ are m-vectors with elements equal to the diagonal entries of matrices $B11$ and $B22$, respectively. Therefore,

$$\mathbf{T}^T(t)B\mathbf{T}(t) \simeq \hat{B}^T\mathbf{T}(t), \tag{3.23}$$

in which $\hat{B}$ is a $2m$-vector with elements equal to the diagonal entries of matrix B. Also, it is immediately concluded from Eqs. (2.8):

$$\begin{aligned} &\int_0^1 \mathbf{T}(t)\mathbf{T}^T(t)\ dt \\ &= \int_0^1 \begin{pmatrix}\mathbf{T1}(t) \\ \mathbf{T2}(t)\end{pmatrix} \begin{pmatrix}\mathbf{T1}^T(t) & \mathbf{T2}^T(t)\end{pmatrix} dt \\ &= \int_0^1 \begin{pmatrix} \mathbf{T1}(t)\mathbf{T1}^T(t) & \mathbf{T1}(t)\mathbf{T2}^T(t) \\ \mathbf{T2}(t)\mathbf{T1}^T(t) & \mathbf{T2}(t)\mathbf{T2}^T(t) \end{pmatrix} dt \\ &\simeq \begin{pmatrix} \frac{h}{3}I_{m\times m} & \frac{h}{6}I_{m\times m} \\ \frac{h}{6}I_{m\times m} & \frac{h}{3}I_{m\times m} \end{pmatrix}. \end{aligned} \tag{3.24}$$

Therefore,

$$\int_0^1 \mathbf{T}(t)\mathbf{T}^T(t)\ dt \simeq D, \tag{3.25}$$

where D is the following $2m \times 2m$ matrix:

$$D = \frac{h}{3}\begin{pmatrix} 1 & 0 & \dots & 0 & 1/2 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 1/2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 1/2 \\ 1/2 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 1/2 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1/2 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{3.26}$$

3.3 Operational matrix

Expressing $\int_0^s \mathbf{T}(\tau)d\tau$ in terms of $\mathbf{T}(s)$, and from Eqs. (2.10), we can write [6]

$$\begin{aligned} \int_0^s \mathbf{T}(\tau)d\tau &= \int_0^s \begin{pmatrix}\mathbf{T1}(\tau) \\ \mathbf{T2}(\tau)\end{pmatrix} d\tau \\ &\simeq \begin{pmatrix} P1\mathbf{T1}(s) + P2\mathbf{T2}(s) \\ P1\mathbf{T1}(s) + P2\mathbf{T2}(s) \end{pmatrix} \\ &= \begin{pmatrix} P1 & P2 \\ P1 & P2 \end{pmatrix} \begin{pmatrix}\mathbf{T1}(s) \\ \mathbf{T2}(s)\end{pmatrix}, \end{aligned} \tag{3.27}$$

so,

$$\int_0^s \mathbf{T}(\tau)d\tau \simeq P\mathbf{T}(s), \tag{3.28}$$

where $P_{2m\times 2m}$, operational matrix of $\mathbf{T}(s)$, is:

$$P = \begin{pmatrix} P1 & P2 \\ P1 & P2 \end{pmatrix}, \tag{3.29}$$

where $P1$ and $P2$ are given by (2.11).

Now, the integral of any function $f(t)$ can be approximated as

$$\begin{aligned} \int_0^s f(\tau)d\tau &\simeq \int_0^s F^T\mathbf{T}(\tau)d\tau \\ &\simeq F^TP\mathbf{T}(s). \end{aligned} \tag{3.30}$$

4 Direct method for solving linear second kind integral equations

Here, by using the results obtained in the previous sections as to the TFs, a numerical direct method for solving second kind integral equations is formulated. The formulation is given for both Volterra and Fredholm integral equations.

4.1 Volterra integral equation

Let us consider the following linear Volterra integral equation of the second kind:

$$x(s) + \lambda\int_0^s k(s,t)x(t)t = y(s), \quad 0 \leqslant s < 1, \tag{4.31}$$

where the parameter λ and the functions $y(s)$ and $k(s,t)$ are known but $x(s)$ is not. Moreover, $k(s,t) \in L^2([0,1)\times[0,1))$ and $y(s) \in L^2([0,1))$.

Approximating the functions $x(s)$, $y(s)$, and $k(s,t)$ with respect to TFs, using (3.14) and (3.16), gives

$$\begin{aligned} x(s) &\simeq X^T\mathbf{T}(s) = \mathbf{T}^T(s)X, \\ y(s) &\simeq Y^T\mathbf{T}(s) = \mathbf{T}^T(s)Y, \\ k(s,t) &\simeq \mathbf{T}^T(s)K\mathbf{T}(t), \end{aligned} \tag{4.32}$$

where $2m$-vectors X and Y, and $2m\times 2m$ matrix K are TFs coefficients of $x(s)$, $y(s)$, and $k(s,t)$, respectively. Note that in (4.32), X is the unknown vector and should be computed.

Substituting (4.32) into (4.31) gives

$$\begin{aligned} Y^T\mathbf{T}(s) &\simeq X^T\mathbf{T}(s) \\ &+ \lambda\int_0^s \mathbf{T}^T(s)K\mathbf{T}(t)\mathbf{T}^T(t)Xt. \end{aligned} \tag{4.33}$$

Using Eq. (3.20) follows

$$\begin{aligned} Y^T\mathbf{T}(s) &\simeq X^T\mathbf{T}(s) + \lambda\mathbf{T}^T(s)K\int_0^s \tilde{X}\mathbf{T}(t)t \\ &\simeq X^T\mathbf{T}(s) + \lambda\mathbf{T}^T(s)K\tilde{X}\int_0^s \mathbf{T}(t)t. \end{aligned} \tag{4.34}$$

Using operational matrix P, in Eq. (3.28), results in

$$Y^T\mathbf{T}(s) \simeq X^T\mathbf{T}(s) + \lambda\mathbf{T}^T(s)K\tilde{X}P\mathbf{T}(s), \tag{4.35}$$

in which $\lambda K\tilde{X}P$ is a $2m\times 2m$ matrix. Using Eq. (3.23) follows

$$\mathbf{T}^T(s)\lambda K\tilde{X}P\mathbf{T}(s) \simeq \hat{X}^T\mathbf{T}(s), \tag{4.36}$$

where $\hat{X}$ is a $2m$-vector with components equal to the diagonal entries of matrix $\lambda K\tilde{X}P$.

Now, combining (4.35) and (4.36) and replacing $\simeq$ with $=$, we obtain

$$X + \hat{X} = Y. \tag{4.37}$$

Equation (4.37) is a linear system of $2m$ algebraic equations for the $2m$ unknowns $X1_0, X1_1, \dots, X1_{m-1}, X2_0, X2_1, \dots, X2_{m-1}$, components of $X^T = (X1^T \;\; X2^T)$. Hence, an approximate solution $x(s) \simeq X^T\mathbf{T}(s)$, or $x(s) \simeq X1^T\mathbf{T1}(s) + X2^T\mathbf{T2}(s)$ can be computed for integral equation (4.31) without using any projection method.

4.2 Fredholm integral equation

Let us consider the following linear Fredholm integral equation of the second kind:

$$x(s) + \lambda\int_0^1 k(s,t)x(t)t = y(s), \quad 0 \leqslant s < 1, \tag{4.38}$$

where the parameter λ and the functions $y(s)$ and $k(s,t)$ are known and $x(s)$ is the unknown function to be determined. Moreover, $k(s,t) \in L^2([0,1)\times[0,1))$ and $y(s) \in L^2([0,1))$. Without loss of generality, it is supposed that the interval of integration in Eq. (4.38) is $[0,1)$, since any finite interval $[a,b)$ can be transformed to interval $[0,1)$ by linear maps [8].

Similar to the direct method for Volterra integral equation, substituting (4.32) into (4.38) follows

$$\begin{aligned} Y^T\mathbf{T}(s) &\simeq X^T\mathbf{T}(s) \\ &+ \lambda\mathbf{T}^T(s)K\int_0^1 \mathbf{T}(t)\mathbf{T}^T(t)Xt. \end{aligned} \tag{4.39}$$

Using Eq. (3.25) gives

$$Y^T\mathbf{T}(s) \simeq X^T\mathbf{T}(s) + (\lambda KDX)^T\mathbf{T}(s). \tag{4.40}$$

Now, replacing $\simeq$ with $=$ results in

$$(I + \lambda KD)X = Y. \tag{4.41}$$

Equation (4.41) is a linear system of algebraic equations. So, an approximate solution $x(s) \simeq X^T\mathbf{T}(s) = X1^T\mathbf{T1}(s) + X2^T\mathbf{T2}(s)$, is obtained for Eq. (4.38). Note that, this approach does not use any projection method such as collocation, Galerkin, etc.

5 Test examples

Here, the given direct method is applied to solve some examples. The numerical results obtained by the method are compared with both the exact solution and those obtained by some other methods such as block-pulse functions (BPFs) method, rationalized Haar wavelet method [19], Legendre wavelet method [21], Adomian decomposition method [1], and expansion-iterative method [14].

5.1 Numerical results

Example 5.1 *Consider the following Fredholm integral equation [19, 8]:*

$$x(s) - \int_0^1 e^{st}x(t)t = e^s - \frac{e^{s+1}-1}{s+1}, \tag{5.42}$$

Table 1: Numerical results for Example 5.1

s	Exact Solution	Direct method ($m = 16$)	Direct method ($m = 32$)	BPFs method ($m = 32$)	Rationalized Haar wavelet method [19] ($k = 32$)
0	1	0.997376	0.999344	1.016236	1.01642
0.1	1.105171	1.102930	1.104568	1.116091	1.11627
0.2	1.221403	1.218903	1.220824	1.225752	1.22593
0.3	1.349859	1.347264	1.349264	1.346191	1.34637
0.4	1.491825	1.489399	1.491158	1.478465	1.47864
0.5	1.648721	1.645485	1.647912	1.675268	1.62391
0.6	1.822119	1.819638	1.821429	1.839883	1.84004
0.7	2.013753	2.010978	2.013136	2.020674	2.02082
0.8	2.225541	2.222754	2.224933	2.219234	2.21936
0.9	2.459603	2.457250	2.458916	2.437307	2.43742

Table 2: Numerical results for Example 5.2

.

s	Exact Solution	Direct method ($m = 16$)	Direct method ($m = 32$)	BPFs method ($m = 32$)	Legendre wavelet method [21]
0	1	0.999374	0.999844	1.031832	1.012990
0.1	1.221403	1.222909	1.221598	1.244627	—
0.2	1.491825	1.492803	1.492294	1.501307	1.487708
0.3	1.822119	1.823200	1.822684	1.810922	—
0.4	2.225541	2.228355	2.225880	2.184388	2.230965
0.5	2.718282	2.716581	2.717857	2.804810	—
0.6	3.320117	3.324211	3.320648	3.383247	3.307555
0.7	4.055200	4.057859	4.056475	4.080975	—
0.8	4.953032	4.955970	4.954570	4.922595	4.962956
0.9	6.049647	6.057297	6.050568	5.937783	—

Table 3: Numerical results for Example 5.3

.

s	Exact Solution	Direct method ($m = 8$)	Direct method ($m = 16$)	BPFs method ($m = 16$)	Adomian decomposition method [1]
0	0	0	0	0.031250	0
0.1	0.099833	0.100000	0.099854	0.093628	0.09983333
0.2	0.198669	0.198828	0.198732	0.217044	0.19866958
0.3	0.295520	0.295715	0.295623	0.277601	0.29552231
0.4	0.389418	0.389905	0.389484	0.395228	0.38942488
0.5	0.479426	0.480651	0.479731	0.506686	0.47944013
0.6	0.564642	0.565390	0.564736	0.559553	0.56466968
0.7	0.644218	0.644629	0.644415	0.658532	0.64426292
0.8	0.717356	0.717765	0.717576	0.704258	0.71742550
0.9	0.783327	0.784225	0.783432	0.787288	0.78342727

with exact solution $x(s) = e^s$. The numerical results are shown in Table 1.

Example 5.2 *For the following Fredholm integral equation of the second kind [21, 20]:*

$$x(s) + \int_0^1 \frac{1}{3} e^{2s-\frac{5}{3}t} x(t)t = e^{2s+\frac{1}{3}}, \qquad (5.43)$$

with exact solution $x(s) = e^{2s}$, Table 2 shows the numerical results.

Table 4: Numerical results for Example 5.4

s	Exact solution	Direct method ($m = 16$)	Direct method ($m = 32$)	Expansion-iterative method [14] ($m = 32$)
0	0	0	0	0.015625
0.1	0.100000	0.099998	0.100000	0.109375
0.2	0.200000	0.199986	0.199997	0.203125
0.3	0.300000	0.299955	0.299989	0.296875
0.4	0.400000	0.399895	0.399974	0.390625
0.5	0.500000	0.499801	0.499950	0.515625
0.6	0.600000	0.599663	0.599916	0.609375
0.7	0.700000	0.699488	0.699872	0.703125
0.8	0.800000	0.799282	0.799820	0.796875
0.9	0.900000	0.899065	0.899766	0.890625

Table 5: Mean-absolute errors, for Examples 5.1-5.4, in terms of m.

m	Example 5.1	Example 5.2	Example 5.3	Example 5.4
2	1.6 e −1	1.5 e −1	7.8 e −3	1.9 e −2
4	4.1 e −2	4.1 e −2	1.9 e −3	4.6 e −3
8	1.0 e −2	1.0 e −2	4.7 e −4	1.1 e −3
16	2.6 e −3	2.6 e −3	1.2 e −4	2.9 e −4
32	6.5 e −4	6.4 e −4	2.9 e −5	7.2 e −5
64	1.6 e −4	1.6 e −4	7.3 e −6	1.8 e −5
128	4.1 e −5	4.0 e −5	1.8 e −6	4.5 e −6
256	1.0 e −5	1.0 e −5	4.6 e −7	1.1 e −6
512	2.5 e −6	2.5 e −6	1.1 e −7	2.8 e −7
1024	6.4 e −7	6.4 e −7	2.9 e −8	7.0 e −8

Example 5.3 *For the following second kind Volterra integral equation [1]:*

$$x(s) + \int_0^s (s-t)x(t)t = s, \tag{5.44}$$

with exact solution $x(s) = \sin(s)$*, Table 3 shows the numerical results.*

Example 5.4 *Consider the following second kind Volterra integral equation [14]:*

$$x(s) + \int_0^s (st^2 + s^2t)x(t)t = s + \frac{7}{12}s^5, \tag{5.45}$$

with exact solution $x(s) = s$*. Table 4 gives the results.*

5.2 Convergence rate

We give here the mean-absolute errors associated with the direct method. The errors are calculated for all the mentioned examples. Table 5 shows the results for some different values of m. We see that the direct method has a reasonable convergence rate.

6 Conclusion

A direct method was formulated based on a special representation of TFs vector forms. This approach, without applying any projection method, transforms a Volterra or Fredholm integral equation of the second kind to a set of algebraic equations. Its efficiency was checked on some examples. The results confirmed the applicability of the method for solving second kind integral equations.

References

[1] E. Babolian, A. Davari, Numerical implementation of Adomian decomposition method for linear Volterra integral equations of the second kind, *Applied Mathematics and Computation* 165 (2005) 223-227.

[2] E. Babolian, Z. Masouri, S. Hatamzadeh-Varmazyar, A direct method for numerically

solving integral equations system using orthogonal triangular functions, *International Journal of Industrial Mathematics* 1 (2009) 135-145.

[3] E. Babolian, Z. Masouri, S. Hatamzadeh-Varmazyar, A set of multi-dimensional orthogonal basis functions and its application to solve integral equations, *International Journal of Applied Mathematics and Computation* 2 (2010) 032-049.

[4] E. Babolian, Z. Masouri, S. Hatamzadeh-Varmazyar, Introducing a direct method to solve nonlinear Volterra and Fredholm integral equations using orthogonal triangular functions, *Mathematics Scientific Journal* 5 (2009) 11-26.

[5] E. Babolian, Z. Masouri, S. Hatamzadeh-Varmazyar, New direct method to solve nonlinear Volterra-Fredholm integral and integro-differential equations using operational matrix with block-pulse functions, *Progress In Electromagnetics Research B* 8 (2008) 59-76.

[6] E. Babolian, Z. Masouri, S. Hatamzadeh-Varmazyar, Numerical solution of nonlinear Volterra-Fredholm integro-differential equations via direct method using triangular functions, *Computers & Mathematics with Applications* 58 (2009) 239-247.

[7] A. Deb, A. Dasgupta, G. Sarkar, A new set of orthogonal functions and its application to the analysis of dynamic systems, *Journal of the Franklin Institute* 343 (2006) 1-26.

[8] L. M. Delves, J. L. Mohamed, Computational Methods for Integral Equations, *Cambridge University Press, Cambridge*, 1985.

[9] S. Hatamzadeh-Varmazyar, Z. Masouri, A fast numerical method for analysis of one- and two-dimensional electromagnetic scattering using a set of cardinal functions, Engineering Analysis with Boundary Elements 36 (2012) 1631-1639.

[10] S. Hatamzadeh-Varmazyar, Z. Masouri, Determining the electromagnetic fields scattered from PEC cylinders, *International Journal of Mathematics & Computation* 28 (2017) 1-8.

[11] S. Hatamzadeh-Varmazyar, Z. Masouri, Numerical expansion-iterative method for analysis of integral equation models arising in one-and two-dimensional electromagnetic scattering, *Engineering Analysis with Boundary Elements* 36 (2012) 416-422.

[12] S. Hatamzadeh-Varmazyar, M. Naser-Moghadasi, E. Babolian, Z. Masouri, Calculating the radar cross section of the resistive targets using the Haar wavelets, *Progress In Electromagnetics Research* 83 (2008) 55-80.

[13] S. Hatamzadeh-Varmazyar, M. Naser Moghadasi, R. Sadeghzadeh-Sheikhan, Numerical method for analysis of radiation from thin wire dipole antenna, *International Journal of Industrial Mathematics* 3 (2011) 135-142.

[14] Z. Masouri, Numerical expansion-iterative method for solving second kind Volterra and Fredholm integral equations using block-pulse functions, *Advanced Computational Techniques in Electromagnetics* 2012.

[15] Z. Masouri, E. Babolian, S. Hatamzadeh-Varmazyar, An expansion-iterative method for numerically solving Volterra integral equation of the first kind, *Computers & Mathematics with Applications* 59 (2010) 1491-1499.

[16] Z. Masouri, S. Hatamzadeh-Varmazyar, An analysis of electromagnetic scattering from finite-width strips, *International Journal of Industrial Mathematics* 5 (2013) 199-204.

[17] Z. Masouri, S. Hatamzadeh-Varmazyar, Evaluation of current distribution induced on perfect electrically conducting scatterers, *International Journal of Industrial Mathematics* 5 (2013) 167-173.

[18] Z. Masouri, S. Hatamzadeh-Varmazyar, Numerical solution of Fredholm integral equations of the first kind with real or complex kernel using triangular functions, *Proceedings of 38th Annual Iranian Mathematics Conference, Zanjan University, Zanjan, Iran* (2007) 284-286.

[19] K. Maleknejad, F. Mirzaee, Using rationalized Haar wavelets for solving linear integral

equations, *Applied Mathematics and Computation* 160 (2005) 579-587.

[20] K. Maleknejad, M. Karami, Using the WPG method for solving integral equations of the second kind, *Applied Mathematics and Computation* 166 (2005) 123-130.

[21] K. Maleknejad, M. Tavassoli Kajani, Y. Mahmoudi, Numerical solution of linear Fredholm and Volterra integral equation of the second kind by using Legendre wavelets, *Kybernetes* 32 (2003) 1530-1539.

Saeed Hatamzadeh-Varmazyar, Ph.D. in Electrical Engineering, is an Assistant Professor at Islamshar Branch, Islamic Azad University, Tehran, Iran. His research interests include computational Electromagnetics, numerical methods for solving integral equations, electromagnetic theory, singular integral equations, electromagnetic radiation and antenna, and propagation of electromagnetic waves, including scattering, diffraction, and etc. Dr. Hatamzadeh-Varmazyar is the author of many research articles published in scientific journals. More details may be found on his official website available online at http://www.hatamzadeh.ir/ and http://www.hatamzadeh.org/.

Zahra Masouri received her B.Sc., M.Sc., and Ph.D. degrees in Applied Mathematics, respectively, in 1998, 2000, and 2010, as the first rank graduate at all the mentioned educational levels. She is currently an Assistant Professor at Islamshar Branch, Islamic Azad University, Tehran, Iran. Her research interest is in numerical analysis; numerical solution of linear and nonlinear integral, integro-differential, and differential equations; numerical linear algebra; and computational Electromagnetics. Dr. Masouri was the research manager and the research vice-chancellor at Khorramabad Branch of Islamic Azad University for some years. She also is the author of many research articles published in scientific journals or presented at various conferences and member of the editorial board of some international journals.