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Research Article

Local Annihilation Method and Some Stiff Problems

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Abstract

In this article, a new scheme inspired from collocation method is presented for numerical solution of stiff initial-value problems and Fredholm integral equations of the first kind based on the derivatives of residual function. Then, the error analysis of this method is investigated by presenting an error bound. The efficiency of the new method is compared with the efficiency of the collocation method by presenting some stiff and ill-posed test problems. Numerical comparisons indicate that the presented method yields accurate approximations in many cases in which the collocation method is failed.

Keywords : Local annihilation method; Residual function; Stiff problems; Ill-posed problems.

1 Introduction

The Galerkin, collocation and least squares methods are efficient techniques for numerical solutions of most functional equations [7, 18, 19, 20]. However, obtaining an accurate solution by these methods fails for ill-posed problems [4, 13, 27]. In many recent numerical approaches, some accurate approximations are obtained by modifying the previous methods [3, 4, 15, 21, 22, 1]. L-stable [26], composite Runge-Kutta [2, 12], finite volume approximation [14] and the approximations based on wavelet bases [5, 8, 17] are among the most successful methods for the numerical solutions of such stiff problems in recent years. In this article, the attempt is to show that some simple techniques can sometimes be efficient. therefore, it is tried to get an approximate solution by local annihilating the residual function and its derivatives. The presented method has a simple structure, suitable error bound and significant numerical results. To describe this method, we consider the functional equation

$$Ax = f, \tag{1.1}$$

where A is a linear operator and f is a known function. Let $\{p_j\}_{j=0}^{\infty}$ be a basis for solution space of the Eq. (1.1). Then we may approximate the unknown solution x by

$$x_n(t) = \sum_{j=0}^{n} a_j p_j(t). \tag{1.2}$$

By substituting (1.2) into (1.1), we find that

$$\sum_{j=0}^{n} a_j q_j(s) = f(s) + r_n(s), \quad a \le s \le b, \tag{1.3}$$

where

$$q_j(s) = (Ap_j)(s), \quad j = 0, \cdots, n,$$

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and r_n is the residual function. For determining the unknown coefficients $\{a_j\}_{j=0}^n$ by collocation method(CM), we should solve the following linear system of equations [9, 19]

$$r_n(s_i) = 0 \quad \Longrightarrow \quad \sum_{j=0}^{n} a_j q_j(s_i) = f(s_i),$$

$$i = 0, \cdots, n,$$

where the collocation points $\{s_i\}_{i=0}^n$ are chosen arbitrarily from $[a, b]$.

In ill-posed functional equations (for example, Fredholm integral equations of the first kind), the condition number of the coefficient matrix of the above system is greatly enlarged by increasing n. Therefore, any approximate solution of Eq. (1.1) is determined by very large error [7, 10]. In the new method, the derivatives of the residual function r_n have a main role in reducing the ill-posedness of the problem and the approximate solutions are determined in the form of multi-rule functions. The presented error analysis and numerical examples show that this method is efficient and applicable for numerical solution of many ill-posed and stiff functional equations.

2 Description of the method

2.1 The main idea

Suppose that the exact solution of the functional equation (1.1) is approximated by (1.2). According to the Eq. (1.3), the residual function can be written as

$$r_n(s) = \sum_{j=0}^{n} a_j q_j(s) - f(s), \quad a \le s \le b. \tag{2.4}$$

Let $\{s_i\}_{i=0}^m$ be arbitrary selected points from $[a, b]$. Suppose that the residual function r_n is infinitely differentiable on the set $\{s_i\}_{i=0}^m$ (it is assumed that the left and right derivatives exist at the boundary points). To determine the approximate solution of (1.1) in a neighborhood of $s_i, i = 0, \cdots, m$, we set

$$r_n^{(k)}(s_i) = 0, \quad k = 0, \cdots, n, \tag{2.5}$$

where $r_n^{(k)}$ denotes kth derivative of r_n. It follows then from (2.4) that (for fixed i)

$$\sum_{j=0}^{n} a_{ij} q_j^{(k)}(s_i) = f^{(k)}(s_i), \tag{2.6}$$

$$k = 0, \cdots, n \quad i = 0, \cdots, m.$$

The index i in the coefficients $\{a_{ij}\}_{j=0}^n$ denotes that the approximate solution $x_n(t)$ is determined in a neighborhood of s_i. We denote this solution by $x_{n,i}(t)$. The matrix form of the linear system of equations (2.6) is as follows.

$$Q^{(i)} a^{(i)} = F^{(i)}, \quad i = 0, \cdots, m \tag{2.7}$$

or, in the equivalent block matrix form

$$\begin{pmatrix} Q^{(0)} & 0 & \cdots & 0 \\ 0 & Q^{(1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q^{(m)} \end{pmatrix} \begin{pmatrix} a^{(0)} \\ a^{(1)} \\ \vdots \\ a^{(m)} \end{pmatrix} = \begin{pmatrix} F^{(0)} \\ F^{(1)} \\ \vdots \\ F^{(m)} \end{pmatrix},$$

where

$$Q^{(i)} = \begin{pmatrix} q_0(s_i) & q_1(s_i) & \cdots & q_n(s_i) \\ q_0'(s_i) & q_1'(s_i) & \cdots & q_n'(s_i) \\ \vdots & \vdots & \ddots & \vdots \\ q_0^{(n)}(s_i) & q_1^{(n)}(s_i) & \cdots & q_n^{(n)}(s_i) \end{pmatrix},$$

$$a^{(i)} = \begin{pmatrix} a_{i0} & a_{i1} & \cdots & a_{in} \end{pmatrix}^T,$$

,

$$F^{(i)} = \begin{pmatrix} f(s_i) & f'(s_i) & \cdots & f^{(n)}(s_i) \end{pmatrix}^T,$$

for $i = 0, \cdots, m$ and $0 \in \mathbb{R}^{(m+1)\times(m+1)}$ is the zero matrix.

By solving the above linear system, the approximate solution of the functional equation (1.1) around the point s_i is determined by

$$x_{n,i}(t) = \sum_{j=0}^{n} a_{ij} p_j(t), \quad i = 0, \cdots, m. \tag{2.8}$$

By setting

$$t_0 = s_0, \quad t_i = \frac{s_{i-1} + s_i}{2}, \quad i = 1, \cdots, m,$$

$$t_{m+1} = s_m,$$

Table 1: Maximum absolute error, $\|e\|_\infty$, for the Example 3.1.

	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-5}$	
n	CM	LAM	CM	LAM
1	$7.7178e-03$	$8.8784e-02$	$7.7254e-05$	$8.9575e-04$
2	$9.9593e+00$	$1.5909e-03$	$9.9996e+00$	$1.6195e-07$
3	$9.8806e+00$	$4.2570e-05$	$9.9988e+00$	$4.3728e-11$
4	$9.7824e+00$	$1.5185e-06$	$9.9978e+00$	$1.5741e-14$
5	$9.6722e+00$	$6.7700e-08$	$9.9967e+00$	$7.0827e-18$
6	$9.5536e+00$	$3.6217e-09$	$9.9954e+00$	$3.8243e-21$
7	$9.4290e+00$	$2.2602e-10$	$9.9941e+00$	$2.4091e-24$
8	$9.2997e+00$	$1.6119e-11$	$9.9927e+00$	$1.7344e-27$
9	$9.1671e+00$	$1.2931e-12$	$9.9913e+00$	$1.4047e-30$
10	$9.0317e+00$	$1.1525e-13$	$9.9898e+00$	$1.2641e-33$
11	$8.8945e+00$	$1.1298e-14$	$9.9883e+00$	$1.2514e-36$
12	$8.7558e+00$	$1.2082e-15$	$9.9868e+00$	$1.3514e-39$
13	$8.6161e+00$	$1.3995e-16$	$9.9851e+00$	$1.5810e-42$
14	$8.4758e+00$	$1.7456e-17$	$9.9841e+00$	$1.9918e-45$
15	$8.3352e+00$	$2.3326e-18$	$9.9838e+00$	$2.6888e-48$

Table 2: Maximum absolute error, $\|e\|_\infty$, for the Example 3.2.

	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-5}$	
n	CM	LAM	CM	LAM
1	$1.9960e-03$	$8.5698e-03$	$2.0000e-05$	$9.9990e-05$
2	$5.9010e-02$	$1.3350e-04$	$6.2843e-03$	$1.9996e-08$
3	$5.4490e-02$	$2.7068e-06$	$6.2437e-03$	$5.9979e-12$
4	$4.8966e-02$	$6.3505e-08$	$6.1929e-03$	$4.4409e-15$
5	$4.2878e-02$	$1.6287e-09$	$6.1354e-03$	$1.7764e-15$
6	$3.6515e-02$	$4.4275e-11$	$6.0728e-03$	$3.5527e-15$
7	$3.0151e-02$	$1.2550e-12$	$6.0062e-03$	$1.4211e-14$
8	$2.4070e-02$	$3.8192e-14$	$5.9362e-03$	$3.0531e-14$
9	$1.8531e-02$	$4.2633e-14$	$5.8634e-03$	$2.8422e-14$
10	$1.3733e-02$	$2.8422e-14$	$5.7881e-03$	$5.4512e-14$
11	$9.7829e-03$	$1.9185e-13$	$5.7105e-03$	$1.1369e-13$
12	$6.6928e-03$	$2.8422e-13$	$5.6309e-03$	$2.5580e-13$
13	$4.3950e-03$	$4.5475e-13$	$5.5496e-03$	$6.8212e-13$
14	$2.7699e-03$	$2.2737e-12$	$5.4666e-03$	$1.1369e-12$
15	$1.6758e-03$	$3.1832e-12$	$5.3830e-03$	$4.5475e-12$

, the approximate solution of Eq.(1.1) takes the piecewise function form

$$x_n(t) = \begin{cases} \sum_{j=0}^{n} a_{0j} p_j(t) & t_0 \le t < t_1 \\ \sum_{j=0}^{n} a_{1j} p_j(t) & t_1 \le t < t_2 \\ \vdots & \vdots \\ \sum_{j=0}^{n} a_{mj} p_j(t) & t_m \le t \le t_{m+1}. \end{cases} \tag{2.9}$$

We call this approach the local annihilation method(LAM).

2.2 Error analysis

In this section, the convergence of the local annihilation method is proved by presenting an error bound. Let $\{s_i\}_{i=0}^m$ be a set of partition points in $[a, b]$ and $x_n(t)$ given by (2.9) be the related approximate solution. We define

$$|\Delta| = \max_{1\le i\le m} |s_i - s_{i-1}|. \tag{2.10}$$

and prove that the convergence order of the presented method is $O(\frac{1}{n!n})$.

Lemma 2.1 *Let*

$$\max_{0\le i\le m} |r_n^{(k)}(s_i)| \le M, \quad k = 0, 1, 2, \cdots,$$

Table 3: Maximum absolute error, $\|e\|_\infty$, for the Example 3.3**A**.

	$\varepsilon = 10^{-12}$		$\varepsilon = 10^{-14}$	
n	CM	LAM	CM	LAM
1	$1.4234e-01$	$1.4183e-01$	$1.4234e-01$	$1.4183e-01$
2	$1.4622e-02$	$1.4586e-02$	$1.4622e-02$	$1.4586e-02$
3	$1.0596e-03$	$1.0557e-03$	$1.0598e-03$	$1.0577e-03$
4	$9.2471e-05$	$1.5272e-04$	$2.0686e-04$	$5.3245e-05$
5	$2.3014e-04$	$1.3046e-04$	$6.5265e-05$	$1.1055e-01$
6	$8.8760e-04$	$6.6836e-06$	$1.2366e-02$	$9.7149e-06$
7	$8.1003e-04$	$5.7086e-06$	$3.9647e+00$	$2.0703e-06$
8	$2.0861e-02$	$4.8482e-06$	$1.3408e-02$	$7.6022e-07$
9	$3.7981e-03$	$4.9232e-06$	$6.9882e-02$	$7.2029e-07$
10	$1.8363e-02$	$4.9175e-06$	$4.9028e-02$	$7.2342e-07$
11	$2.8237e-03$	$4.9179e-06$	$5.4459e-02$	$7.2375e-07$
12	$9.5276e-04$	$4.9178e-06$	$3.5002e+00$	$7.2312e-07$
13	$2.1355e-03$	$4.9178e-06$	$4.7860e-02$	$7.2375e-07$
14	$4.3927e-04$	$4.9178e-06$	$9.6935e-02$	$7.2342e-07$
15	$2.0407e-04$	$4.9178e-06$	$1.0005e-02$	$7.2345e-07$

Table 4: Maximum absolute error, $\|e\|_\infty$, for the Example 3.3 **B**.

	$\varepsilon = 10^{-12}$		$\varepsilon = 10^{-14}$	
n	CM	LAM	CM	LAM
1	$2.2327e-02$	$2.2327e-02$	$2.2327e-02$	$2.2327e-02$
2	$3.4053e-04$	$1.7924e-03$	$1.7414e-03$	$1.7916e-03$
3	$9.9720e-05$	$6.3374e-04$	$5.8595e-03$	$6.3851e-04$
4	$4.2923e-05$	$2.6353e-05$	$4.7434e-03$	$2.7155e-05$
5	$4.7167e-05$	$5.4294e-06$	$8.7616e-03$	$6.4914e-06$
6	$6.4304e-05$	$9.3344e-07$	$3.7614e-03$	$6.4517e-06$
7	$8.9088e-05$	$7.5238e-07$	$1.2446e-02$	$6.4519e-06$
8	$3.7391e-04$	$7.7152e-07$	$1.6332e-02$	$6.4520e-06$
9	$1.5598e-04$	$7.7213e-07$	$1.8612e-02$	$6.4520e-06$
10	$3.3079e-04$	$7.7207e-07$	$7.6010e-02$	$6.4520e-06$
11	$1.0750e-04$	$7.7207e-07$	$1.8536e-02$	$6.4520e-06$
12	$6.5696e-05$	$7.7207e-07$	$2.0952e-02$	$6.4520e-06$
13	$1.4975e-04$	$7.7207e-07$	$3.3348e-03$	$6.4520e-06$
14	$1.3171e-04$	$7.7207e-07$	$6.0423e-03$	$6.4520e-06$
15	$5.1317e-05$	$7.7207e-07$	$1.0364e-02$	$6.4520e-06$

and $|\Delta|< 1$. *Then we have*

$$|r_n(s)|\leq M(\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\cdots).$$

$$for\ all\ s\in[a,b]$$

Let $s\in[a,b]$ *be arbitrary. Then there exists a positive integer* $0\leq l\leq m$ *such that*

$$|s-s_l|=\min_{0\leq i\leq m}|s-s_i|.$$

By using Taylor expansion of $r_n(s)$ *around* s_l, *we have*

$$r_n(s)=r_n(s_l)+\frac{(s-s_l)^1}{1!}r_n'(s_l)+\cdots+\frac{(s-s_l)^n}{n!}r_n^{(n)}(s_l)+\frac{(s-s_l)^{n+1}}{(n+1)!}r_n^{(n+1)}(s_l)+\cdots.$$

By the requirements (2.5) one gets

$$r_n(s)=\frac{(s-s_l)^{n+1}}{(n+1)!}r_n^{(n+1)}(s_l)+\frac{(s-s_l)^{n+2}}{(n+2)!}r_n^{(n+2)}(s_l)+\cdots.$$

Therefore, the assumptions

$$|r_n^{(k)}(s_l)|\leq M,\quad k=n+1,n+2,\cdots,$$

and

$$|s-s_l|^k\leq|\Delta|^k<1,\quad k=n+1,n+2,\cdots,$$

Table 5: Confirming the presented error bound and CPU time for the Example 3.4.

n	Maximum absolute errors of LAM	Presented error bound$\simeq \frac{1}{n!n}$	CPU time(in seconds)
1	$1.6389e-01$	$1.0000e+00$	0.44
2	$3.6323e-02$	$2.5000e-01$	0.57
3	$7.7801e-03$	$5.5556e-02$	0.67
4	$1.3005e-03$	$1.0417e-02$	0.79
5	$1.8838e-04$	$1.6667e-03$	0.92
6	$2.3754e-05$	$2.3148e-04$	1.06
7	$2.6591e-06$	$2.8345e-05$	1.21
8	$2.6754e-07$	$3.1002e-06$	1.38
9	$2.4447e-08$	$3.0619e-07$	1.60
10	$2.0462e-09$	$2.7557e-08$	1.73
11	$1.5800e-10$	$2.2775e-09$	1.92
12	$1.1323e-11$	$1.7397e-10$	2.12
13	$7.5684e-13$	$1.2353e-11$	2.46
14	$4.7296e-14$	$8.1934e-13$	2.63
15	$1.7986e-14$	$5.0981e-14$	2.81

complete the proof.

Lemma 2.2 *[25] For any positive integer n we have*

$$e-\sum_{k=0}^{n}\frac{1}{k!}<\frac{1}{n!\,n}.$$

Corollary 2.1 *By the assumptions of Lemma 2.1, for each positive integer n, we have*

$$|r_n(s)|\leq \frac{M}{n!\,n},\quad s\in[a,b].$$

Since

$$\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\cdots=e-\sum_{k=0}^{n}\frac{1}{k!},$$

then by Lemmas 2.1 and 2.2, we have

$$|r_n(s)|\leq \frac{M}{n!\,n},$$

that is,

$$|r_n(s)|= O(\frac{1}{n!\,n}). \tag{2.11}$$

It should be mentioned that, the error bound in Corollary 2.1 may be violated by ill-posed problems, since in this case the linear systems (2.7) usually have large condition numbers.

3 Test problems

In this section, we consider some stiff and ill-posed functional equations of the form (1.1) in order to compare accuracy of collocation method with our new local annihilation method.

Case 1:

In this case, we deal with the stiff initial-value problem [11].

$$(Ax)(s)=\varepsilon x'(s)+a(s)x(s),$$

$$x(0)=x_0,\quad s\in[0,T].$$

These equations are characterized by the presence of a small parameter as coefficient of the first-order derivative. These problems are stiff and ill-posed when $\varepsilon\to 0$ and have been treated numerically by using various approaches [13, 24].

Case 2:

Let

$$(Ax)(s)=\int_a^b k(s,t)x(t)dt.$$

In this case, we deal with a Fredholm integral equation of the first kind which is considered as an ill-posed problem. The solution of this problem may not exist, and if it exists it may not be unique [7, 10]. Many problems in engineering can be modelled by this kind of equations. For example, one and two-dimensional scattering from conducting bodies can be modelled by them [6]. In most numerical approaches for solving these equations, the attempt is to get accurate approximations by reducing their ill-posedness [3, 4, 5, 10]. One of the reliable treatments for these equations is the regularization method that

is presented by Tikhonov and Phillips [27, 23] independently. This method transforms the Fredholm integral equation of the first kind

$$\int_a^b k(s,t)x(t)dt = f(s), \quad s \in [a,b],$$

to the following Fredholm integral equation of the second kind(regularized equation)

$$\varepsilon x_\varepsilon(s) + \int_a^b k^*k(s,t)x_\varepsilon(t)dt = \tag{3.12}$$

$$(k^*f)(s), \quad s \in [a,b],$$

where [10]

$$k^*k(s,t) = \int_a^b k(u,s)k(u,t)du,$$

$$(k^*f)(s) = \int_a^b k(u,s)f(u)du, \quad s,t \in [a,b],$$

and ε is a small positive parameter called the regularization parameter. Also, Tikhonov and Phillips [16, 23] showed that

$$\lim_{\varepsilon \to 0} x_\varepsilon(s) = x(s)$$

By the above assumptions, we solve equation (3.12) by collocation and the new method. The presented numerical examples show that the new method is more accurate and stable than the collocation method.

Note

For computing the related integrals, we use a Gaussian quadrature rule of order 10 with 16 significant digits(by MATLAB R2013b) and maximum precision is used in solving stiff problems. In the following tables the collocation method and the local annihilation method are denoted by CM and LAM respectively. The maximum absolute error $\|e\|_\infty$ for the following examples is computed by $\|e\|_\infty = \max|x(s) - x_n(s_i)|$ in collocation points. The basis functions are chosen as $p_j(t) = t^j, j = 0, \cdots, n$.

Example 3.1 *Consider the initial value problem*

$$\varepsilon x'(s) + x(s) = g(s) + \varepsilon g'(s),$$

$$x(0) = 10, \quad s \in [0,T],$$

where $g(s) = 10 - (10+s)exp(-s)$ *and the exact solution is* $x_\varepsilon(s) = g(s) + 10exp(-\frac{s}{\varepsilon})$. *The problem explains an initial layer of thickness* $O(\varepsilon)$ *at* $x = 0$ *(see [13, 24] for more details). We solve this perturbed problem for* $\varepsilon = 10^{-3}$ *and* 10^{-5} *with* $T = 1$. *For these values of* ε, *the errors of dynamical systems method(DSM), proposed in [24], are about* 2.9×10^{-3} *while our method shows more accurate results (see Table 1).*

Example 3.2 *Consider*

$$\varepsilon x'(s) + \frac{1}{1+s}x(s) = e^{-\frac{s}{\sqrt{\varepsilon}}}(\frac{1}{1+s} - \sqrt{\varepsilon}),$$

$$x(0) = 1, \quad s \in [0,1],$$

with the exact solution $x_\varepsilon(s) = e^{-\frac{s}{\sqrt{\varepsilon}}}$. *The numerical results are given in Table 2 for* $\varepsilon = 10^{-3}$ *and* 10^{-5}.

Example 3.3 A:

$$\int_0^1 e^{st}x(t)dt = \frac{e^{s+1}-1}{s+1}, \quad s \in [0,1].$$

B:

$$\int_0^{\frac{\pi}{6}} cos(s-t)x(t)dt =$$

$$\frac{3sin(s) + (2\pi + 3\sqrt{3})cos(s)}{24}, \quad s \in [0, \frac{\pi}{6}].$$

The exact solutions are respectively e^s *and* $cos(s)$. *We solve the perturbed regularized forms of* **A** *and* **B** *for* $\varepsilon = 10^{-12}$ *and* 10^{-14} *with the collocation and the new method. The error of approximate solution for the equation* **A** *is of order* $O(10^{-5})$ *by augmented Galerkin method [4], while our new method is more accurate than the augmented Galerkin method (see Table 3 and Table 4).*

Finally, to confirm the presented error bound (2.11), we solve the following well-posed problem [18] by new method. The maximum absolute errors (Table 5) are compatible with the given error bound (2.11).

Example 3.4

$$x(s) - \frac{1}{2}\int_0^1 (1+s)e^{-st}x(t)dt =$$

$$e^{-s} + \frac{1}{2}(e^{-(1+s)} - 1), \quad s \in [0,1].$$

The exact solution is e^{-s}. *The numerical results and CPU time for local annihilation method(LAM) are shown in Table 5.*

4 Conclusion

The numerical results show that the local annihilation method is often more accurate and more stable than the collocation method for the numerical solution of the presented stiff and ill-posed problems. Moreover, this new method can be considered as a new approach in solving many stiff engineering problems. It is also shown by examples 1-3 that this method can be used as an efficient method for solving problems where the collocation method fails. Finally, the numerical results of example 4 confirm the presented error bound of the well-posed problems.

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