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Research Article

Algebraic Solving of Complex Interval Linear Systems by Limiting Factors

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Abstract

In this work, we propose a simple method for obtaining the algebraic solution of a complex interval linear system where coefficient matrix is a complex matrix and the right-hand-side vector is a complex interval vector. We first use a complex interval version of the Doolittle decomposition method and then we restrict the Doolittle's solution, by complex limiting factors, to achieve a complex interval vector that satisfies the mentioned system.

Keywords : Complex interval vector; Complex interval linear system; Complex interval Doolittle decomposition method.

1 Introduction

In many engineering and scientific problems, such as engineering analysis or design [8, 13] and control engineering [4], the linear systems of equations involving uncertain model parameters are very important. However, in some practical problems, for example the electrical circuits [13, 14], the model parameters are complex variables. Therefore, in such problems we have a linear system of equations with uncertain complex parameters. As we know, this uncertainty can be represented by various ways. In this paper, we focus on the use of intervals to represent the uncertain quantities in a system of complex linear equations [11].

Unfortunately, there are no many numerical procedures and mathematical models for algebraic solving an interval complex linear system. The interval complex linear systems were investigated in [11, 13, 14]. In 1999 , Rump [18, 19] implemented the complex interval arithmetic for Matlab in the package Intlab . In 2006, Djanybekov [7] have used interval Householder method for presenting an outer estimation of solution set of an interval linear algebraic system with complex interval parameters. In 2010, Hladik [11] have described the solution set for complex interval systems of equations (rectangular case) by a system of nonlinear inequalities and also, he showed how it can be used to obtain a very accurate approximation of the interval hull of the solution set. In the same year, Popova et al. [17] have reported on new software for solving linear systems where the coefficient matrix and the right-hand-side vector are affine linear func-

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tions of parameters varying within given complex intervals. Recently, Ghanbari [9] have introduced an algorithm for presenting an inner estimation of the solution set of a complex interval linear system. Also, he showed that under some certain conditions, the obtained inner estimation is an algebraic solution.

In this paper, we investigate the complex interval linear systems where the coefficient matrix is complex crisp-valued and the right-hand-side column is complex interval-valued. Also, we present a new approach for obtaining a complex interval vector, as an "algebraic solution", such that satisfies the complex interval linear system. To this end, we first use a complex interval version of the classic Doolittle decomposition method [6] to obtain a solution set, namely "Doolittle's solution". In the next step, we will limit the Doolittle's solution by some parameters, namely "complex limiting factors". Finally, we show that the obtained complex interval vector is an algebraic solution, that means it satisfies the complex interval linear system.

The outline of the paper is as follows. In Section 2 we state some basic definitions and theorems about the real and complex intervals. In Section 3, we present a complex interval version of the classic Doolittle decomposition method. In Section 4, we present a new approach for obtaining the algebraic solution of a complex interval linear system. In Section 5, we use the proposed method for solving two numerical examples. Conclusion is drawn in Section 6.

2 Preliminaries

In this section, we first briefly present the basic definitions of the real interval theory, which are used throughout this paper. Full aspects of all definitions can be found in [2, 3, 15, 11].

Definition 2.1 *A real interval $[x]$ is defined as the set of real numbers such that $[x] = [\underline{x}, \overline{x}] = \{x' \in \mathbb{R} : \underline{x} \leqslant x' \leqslant \overline{x}\}$ where $\underline{x} \leqslant \overline{x}$.*

In this paper, the set of all real intervals is denoted by $\mathbb{IR}$.

Definition 2.2 *We define the midpoint and width of the real interval $[x] = [\underline{x}, \overline{x}]$ respectively as follows:*

$$[x]^c = \frac{\underline{x} + \overline{x}}{2},$$

$$[x]^\Delta = \overline{x} - \underline{x}.$$

Definition 2.3 *For the arbitrary real intervals $[x] = [\underline{x}, \overline{x}]$ and $[y] = [\underline{y}, \overline{y}]$, we define addition and multiplication by a scalar λ as*

$$[x] + [y] = [\underline{x} + \underline{y}, \overline{y} + \overline{x}],$$

$$\lambda[x] = \begin{cases} [\lambda\underline{x}, \lambda\overline{x}], & \text{if } \lambda \geqslant 0, \\ [\lambda\overline{x}, \lambda\underline{x}], & \text{if } \lambda < 0. \end{cases}$$

Remark 2.1 [10] For the real intervals $[x_1]$, $[x_2]$, ..., $[x_n]$ and the real numbers a_1, a_2, ..., a_n we have

$$\left(\sum_{i=1}^{n} a_i\,[x_i]\right)^c = \sum_{i=1}^{n} a_i\ [x_i]^c,$$

$$\left(\sum_{i=1}^{n} a_i\,[x_i]\right)^\Delta = \sum_{i=1}^{n} |a_i|\ [x_i]^\Delta.$$

Definition 2.4 *We define the midpoint vector and the width vector of the real interval vector $[X] = ([x_1], [x_2], \ldots, [x_n])^T$ as follows:*

$$[X]^c = (\,[x_1]^c,\ [x_2]^c,\ \ldots,\ [x_n]^c\,)^T,$$

$$[X]^\Delta = (\,[x_1]^\Delta,\ [x_2]^\Delta,\ \ldots,\ [x_n]^\Delta\,)^T.$$

Now, we briefly remind some basic concepts of complex interval theory.

Definition 2.5 *[1, 5] A complex interval $[z]$ is defined as*

$$\begin{aligned} [z] &= [\underline{a}, \overline{a}] + i[\underline{b}, \overline{b}] \\ &:= \{a + ib \in \mathbb{C} \ \mid\ \underline{a} \leqslant a \leqslant \overline{a},\ \underline{b} \leqslant b \leqslant \overline{b}\}, \end{aligned}$$

where $[a] = [\underline{a}, \overline{a}]$ and $[b] = [\underline{b}, \overline{b}]$ are two arbitrary real intervals.

The set of all such complex intervals will be denoted by $\mathbb{IC}$. Based on Definition 2.5, we can denote

$$[z] = [\underline{z}, \overline{z}],$$

where

$$\underline{z} = \underline{a} + i\underline{b}, \qquad \overline{z} = \overline{a} + i\overline{b}.$$

Obviously, we can take $\mathbb{IR} \subset \mathbb{IC}$, because the real interval $[a]$ can be regarded as a complex interval $[a] = [a] + i[0, 0] \in \mathbb{IC}$.

Definition 2.6 *[1, 5] For two complex intervals* $[z_1] = [p_1] + i[q_1]$ *and* $[z_2] = [p_2] + i[q_2]$, *where* $[p_j] = [\underline{p_j}, \overline{p_j}]$ *and* $[q_j] = [\underline{q_j}, \overline{q_j}]$, $j = 1, 2$ *and the complex number* $c = a + ib$, *we will have*

$$\begin{aligned}[z_1] + [z_2] &= ([p_1] + [p_2]) + i\,([q_1] + [q_2]) \\ &= [\underline{p_1} + \underline{p_2}, \overline{p_1} + \overline{p_2}] \\ &+ \; i\,[\underline{q_1} + \underline{q_2}, \overline{q_1} + \overline{q_2}],\end{aligned}$$

and

$$\begin{aligned}c \cdot [z_1] &= (a + ib) \cdot ([p_1] + i\,[q_1]) \\ &= (a[p_1] - b[q_1]) + i\,(a[q_1] + b[p_1])\,.\end{aligned}$$

Definition 2.7 *[11, 12] We define the midpoint and width of the complex interval* $[z] = [\underline{z}, \overline{z}]$ *respectively as follows:*

$$[z]^c = \frac{\underline{z} + \overline{z}}{2},$$

$$[z]^\Delta = \overline{z} - \underline{z}.$$

Remark 2.2 Based on Definitions 2.2 and 2.7, for the complex interval $[z] = [p] + i[q]$, it can be easily shown that

$$[z]^c = [p]^c + i[q]^c, \qquad [z]^\Delta = [p]^\Delta + i[q]^\Delta.$$

In the following theorem, we obtain the midpoint and width of a linear combination of the complex intervals.

Theorem 2.1 *[9] For the complex intervals* $[z_j] = [p_j] + i[q_j]$, *and the complex numbers* $c_j = a_j + ib_j$, $j = 1, 2, ..., n$ *we have*

$$\left(\sum_{j=1}^{n} c_j\,[z_j]\right)^c = \sum_{j=1}^{n} (a_j\;[p_j]^c - b_j\;[q_j]^c)$$
$$+i\left(\sum_{j=1}^{n} (a_j\;[q_j]^c + b_j\;[p_j]^c)\right),$$
$$\left(\sum_{j=1}^{n} c_j[z_j]\right)^\Delta = \sum_{j=1}^{n} \left(|a_j|[p_j]^\Delta + |b_j|[q_j]^\Delta\right)$$
$$+i\left(\sum_{j=1}^{n} \left(|a_j|[q_j]^\Delta + |b_j|[p_j]^\Delta\right)\right).$$

Definition 2.8 *A vector* $[\boldsymbol{z}] = ([z_1], [z_2], \ldots, [z_n])^T$ *where* $[z_i]$, $i = 1, 2, \ldots, n$, *are the complex intervals, is called a complex interval vector.*

Definition 2.9 *We define the midpoint vector and the width vector of the complex interval vector* $[\boldsymbol{z}] = ([z_1], [z_2], \ldots, [z_n])^T$ *respectively as follows:*

$$[\boldsymbol{z}]^c = ([z_1]^c, [z_2]^c, \ldots, [z_n]^c)^T,$$

$$[\boldsymbol{z}]^\Delta = \left([z_1]^\Delta, [z_2]^\Delta, \ldots, [z_n]^\Delta\right)^T.$$

In the following, we provide a generalized definition of completely nonsingular matrices. Its classic definition can be found in [20, 16].

Definition 2.10 *Let* $\boldsymbol{C} = (c_{kj})_{n\times n}$ *be a crisp complex-valued matrix, i.e.* $c_{kj} = a_{kj} + i\,b_{kj}$ *and also* $\boldsymbol{A} = (a_{kj})_{n\times n}$ *and* $\boldsymbol{B} = (b_{kj})_{n\times n}$ *be the real and imaginary parts of the matrix* $\boldsymbol{C}$, *respectively. We say that the matrix* $\boldsymbol{C}$ *is completely nonsingular, if all matrices* $\boldsymbol{C}$, $|\boldsymbol{A}|+|\boldsymbol{B}|$ *and* $|\boldsymbol{A}|-|\boldsymbol{B}|$ *are nonsingular, where* $|\boldsymbol{A}|= (|a_{kj}|)_{n\times n}$ *and* $|\boldsymbol{B}|= (|b_{kj}|)_{n\times n}$ *are two nonnegative real matrices.*

Definition 2.11 *The* $n \times n$ *linear system*

$$\begin{cases} c_{11}\,[z_1] + c_{12}\,[z_2] + \cdots + c_{1n}\,[z_n] = [w_1], \\ c_{21}\,[z_1] + c_{22}\,[z_2] + \cdots + c_{2n}\,[z_n] = [w_2], \\ \qquad\vdots \\ c_{n1}\,[z_1] + c_{n2}\,[z_2] + \cdots + c_{nn}\,[z_n] = [w_n], \end{cases} \tag{2.1}$$

where the coefficient matrix $\boldsymbol{C} = (c_{kj})_{n\times n}$, $c_{kj} = a_{kj} + ib_{kj}$, *is an* $n\times n$ *crisp complex-valued matrix and* $[w_j] = [u_j] + i\,[v_j]$, $1 \leqslant j \leqslant n$ *are the complex intervals, is called a complex interval linear system.*

We denote the matrix form of Eq. (2.1) as

$$\mathbf{C}[\mathbf{z}] = [\mathbf{w}],$$

where

$$[\mathbf{z}] = ([z_1], [z_2], \ldots, [z_n])^T,$$

and

$$[\mathbf{w}] = ([w_1], [w_2], \ldots, [w_n])^T,$$

are two complex interval vectors. Also, if we set $[z_j] = [p_j] + i\,[q_j]$ and $[w_j] = [u_j] + i\,[vj]$, $1 \leqslant j \leqslant n$, then we can write

$$\mathbf{C} = \mathbf{A} + i\,\mathbf{B}, \qquad [\mathbf{z}] = [\mathbf{p}] + i\,[\mathbf{q}],$$

and

$$[\mathbf{w}] = [\mathbf{u}] + i\,[\mathbf{v}].$$

Definition 2.12 *[7, 17, 11] The solution set of the complex interval linear system (2.1) is defined traditionally as*

$$\Sigma = \left\{ \boldsymbol{z}' \in \mathbb{C}^n \mid (\exists \boldsymbol{w}' \in [\boldsymbol{w}])(\boldsymbol{C}\,\boldsymbol{z}' = \boldsymbol{w}') \right\}.$$

Theorem 2.2 *[9] Suppose that the coefficient matrix $\boldsymbol{C}$ is nonsingular, i.e. $det(\boldsymbol{C}) \neq 0 + i0$. then we have*

$$\Sigma = \boldsymbol{C}^{-1}\,[\boldsymbol{w}].$$

For the complex interval linear system (2.1), in addition to the solution set, there is another solution namely "algebraic solution", that is defined in [9, 21] as follows.

Definition 2.13 *A complex interval vector*

$$[\boldsymbol{z}] = ([z_1], [z_2], \ldots, [z_n])^T,$$

where $[z_j] = [\underline{z_j}, \overline{z_j}]$, is called an "algebraic solution" of the complex interval linear system (2.1) if

$$\sum_{j=1}^{n} c_{kj} \cdot [\underline{z_j}, \overline{z_j}] = [\underline{w_k}, \overline{w_k}], \quad k = 1, 2, \ldots, n.$$

In this paper, the algebraic solution of complex interval linear system (2.1) is denoted by means of

$$[\mathbf{z}_A] = ([z_{1A}], [z_{2A}], \ldots, [z_{nA}])^T,$$

where $[z_{jA}] = [\underline{z_{jA}}, \overline{z_{jA}}]$ is a complex interval for $j = 1, 2, \ldots, n$.

Remark 2.3 From Definition 2.13, it is clear that $\mathbf{C}\,[\mathbf{z}_A] = [\mathbf{w}]$ and consequently $\mathbf{C}\,[\mathbf{z}_A]^c = [\mathbf{w}]^c$. Therefore, by supposing that the matrix $\mathbf{C}$ be nonsingular, we have $[\mathbf{z}_A]^c = \mathbf{C}^{-1}\,[\mathbf{w}]^c$.

Theorem 2.3 *[9] Suppose that $[\boldsymbol{z}_A] = ([z_{1A}], [z_{2A}], \ldots, [z_{nA}])^T$, be an algebraic solution for the complex interval linear system (2.1). Then we have*

$$[\boldsymbol{z}_A] \subseteq \Sigma.$$

3 Complex interval Doolittle decomposition

In this section, we first present a complex interval version of Doolittle decomposition for solving the complex interval linear system (2.1). This method is obtained from its real version (see [6]), replacing the real numbers by the complex intervals and the real classic operations by the corresponding complex interval operations. Also, we will investigate the relation between the algebraic solution, the solution set and the obtained solution by complex interval Doolittle decomposition method.

Consider again the complex interval linear system

$$\mathbf{C}[\mathbf{z}] = [\mathbf{w}]. \tag{3.2}$$

Similar to the classic Doolittle method, we decompose the complex matrix $\mathbf{C} = (c_{kj})_{n\times n}$ into the product of the lower-triangular complex matrix $\mathbf{L} = (l_{kj})_{n\times n}$ and the upper-triangular complex matrix $\mathbf{U} = (u_{kj})_{n\times n}$, where the main diagonal of $\mathbf{L}$ consists of all $1 + 0i$s. In other words

$$\mathbf{C} = \mathbf{L}\mathbf{U}, \tag{3.3}$$

where

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ l'_{21} + il''_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l'_{n1} + il''_{n1} & l'_{n2} + il''_{n2} & \cdots & 1 \end{pmatrix},$$

and

$$\mathbf{U} = \begin{pmatrix} u'_{11} + iu''_{11} & \cdots & u'_{1n} + iu''_{1n} \\ 0 & \cdots & u'_{2n} + iu''_{2n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u'_{nn} + iu''_{nn} \end{pmatrix}.$$

Remark 3.1 [6] In the above decomposition, if all ones be on the main diagonal of $\mathbf{U}$, then the corresponding decomposition is called "*Crout method*".

From Eqs. (3.2) and (3.3), we have

$$\mathbf{L}\mathbf{U}[\mathbf{z}] = [\mathbf{w}], \tag{3.4}$$

if we set $[\mathbf{y}] = \mathbf{U}[\mathbf{z}]$, we have $\mathbf{L}[\mathbf{y}] = [\mathbf{w}]$. But $[\mathbf{w}]$ is given data, and $\mathbf{L}$ is a lower triangular matrix;

hence, we can directly solve for $[\mathbf{y}]$ by forward substitution, starting with the first component of $[\mathbf{y}]$, as follows

$$[y_1] = [w_1], \tag{3.5}$$

$$[y_k] = [w_k] - \sum_{j=1}^{k-1} l_{kj}[y_j], \qquad k = 2, 3, \dots, n. \tag{3.6}$$

In the next step, we can find the desired solution, $[\mathbf{z}]$, from $\mathbf{U}[\mathbf{z}] = [\mathbf{y}]$ via backward substitution, since $\mathbf{U}$ is upper triangular, as follows

$$[z_n] = \frac{1}{u_{nn}}[y_n], \tag{3.7}$$

$$[z_k] = \frac{1}{u_{kk}} \left([y_k] - \sum_{j=k+1}^{n} u_{kj}[z_j] \right), \tag{3.8}$$

where $k = n-1, n-2, \dots, 1$. It should be noted that the complex interval arithmetic rules, defined in Definition 2.6, have been used in the Eqs. (3.5)-(3.8).

In this paper, we denote the obtained solution via the complex interval Doolittle method by

$$[\mathbf{z}_D] = ([z_{1D}], [z_{2D}], \dots, [z_{nD}])^T,$$

where $[z_{jD}] = [p_j] + i[q_j]$, $j = 1, 2, \dots, n$.

In the following theorem, we investigate the relation between the complex interval vector $[\mathbf{z}_D]$ and the solution set Σ for the complex interval linear system (2.1).

Theorem 3.1 *Suppose that the coefficient matrix* $\mathbf{C}$ *of the system (2.1) be nonsingular, then we have*

$$\Sigma \subseteq [\mathbf{z}_D].$$

Proof. At first, based on Theorem 2.2, the solution set Σ is obtained as $\Sigma = \mathbf{C}^{-1}[\mathbf{w}]$. Also, let us consider $\mathbf{z}' \in \Sigma$. Then

$$\exists \mathbf{w}' \in [\mathbf{w}]; \qquad \mathbf{C}\,\mathbf{z}' = \mathbf{w}'.$$

We know that $\mathbf{C} = \mathbf{LU}$ (Doolittle decomposition), then $\mathbf{LU}\,\mathbf{z}' = \mathbf{w}'$. If we set $\mathbf{y}' = \mathbf{U}\,\mathbf{z}'$, then we have $\mathbf{L}\,\mathbf{y}' = \mathbf{w}'$. From the other point of view, since $\mathbf{L}$ is a lower triangular matrix, by forward substitution we conclude

$$y_1' = w_1',$$

$$y_k' = w_k' - \sum_{j=1}^{k-1} l_{kj} y_j', \qquad k = 2, 3, \dots, n.$$

Now, since $\mathbf{w}' = (w_1', w_2', \dots, w_n')^T \in [\mathbf{w}]$, then $\mathbf{y}' = (y_1', y_2', \dots, y_n')^T \in [\mathbf{y}]$, where $[\mathbf{y}]$ is defined in the Eqs. (3.4)-(3.6).

On the other hand, since $\mathbf{U}$ is a upper triangular matrix, for the system $\mathbf{y}' = \mathbf{U}\,\mathbf{z}'$, by backward substitution we have

$$z_n' = \frac{1}{u_{nn}} y_n',$$

$$z_k' = \frac{1}{u_{kk}} \left(y_k' - \sum_{j=k+1}^{n} u_{kj} z_j' \right),$$

where $k = n-1, n-2, \dots, 1$. Now, since $\mathbf{y}' = (y_1', y_2', \dots, y_n')^T \in [\mathbf{y}]$, then $\mathbf{z}' = (z_1', z_2', \dots, z_n')^T \in [\mathbf{z}_D]$, where $[\mathbf{z}_D]$ is obtained by the Eqs. (3.7) and (3.8). Consequently $\Sigma \subseteq [\mathbf{z}_D]$.

Remark 3.2 From Theorems 3.1 and 2.3, we conclude that if the system (2.1) have the algebraic solution $[\mathbf{z}_A]$ and also the coefficient matrix $\mathbf{C}$ be nonsingular, then

$$[\mathbf{z}_A] \subseteq \Sigma \subseteq [\mathbf{z}_D].$$

In the following theorem, we prove that the complex interval vector $[\mathbf{z}_D]$ satisfies the midpoint system of Eq. (2.1).

Theorem 3.2 *Let the coefficient matrix* $\mathbf{C}$ *be nonsingular. Then, we have*

$$\mathbf{C}\,[\mathbf{z}_D]^c = [\mathbf{w}]^c,$$

and consequently

$$\sum_{j=1}^{n} c_{kj} \left(\underline{z_{jD}} + \overline{z_{jD}} \right) = \left(\underline{w_k} + \overline{w_k} \right),$$

where $1 \le k \le n$.

Proof. From Remark 2.1 and Eqs. (3.5) and (3.6), we have

$$[y_1]^c = [w_1]^c,$$

$$[y_k]^c = [w_k]^c - \sum_{j=1}^{k-1} l_{kj}[y_j]^c, \quad k = 2, 3, \dots, n.$$

Since the midpoint of a complex interval number is a complex crisp number, therefore we can rewrite the above equations as follows

$$[\mathbf{y}]^c = \mathbf{L}^{-1}\,[\mathbf{w}]^c. \tag{3.9}$$

On the other hand, from Remark 2.1 and Eqs. (3.7) and (3.8), we have

$$[z_n]^c = \frac{1}{u_{nn}}[y_n]^c,$$

$$[z_k]^c = \frac{1}{u_{kk}}\left([y_k]^c - \sum_{j=k+1}^{n} u_{kj}[z_j]^c\right),$$

where $k = n-1, n-2, \ldots, 1$. As before, we can rewrite the above equations as follows

$$[\mathbf{z}_D]^c = \mathbf{U}^{-1}\,[\mathbf{y}]^c. \tag{3.10}$$

From Eqs. (3.9) and (3.10) we conclude

$$[\mathbf{z}_D]^c = \mathbf{U}^{-1}\,\mathbf{L}^{-1}\,[\mathbf{w}]^c = \mathbf{C}^{-1}[\mathbf{w}]^c,$$

and therefore

$$\mathbf{C}[\mathbf{z}_D]^c = [\mathbf{w}]^c.$$

Remark 3.3 Assuming that $[w_k] = [\underline{u_k}, \overline{u_k}] + i\,[\underline{v_k}, \overline{v_k}]$, and $[z_{kD}] = [\underline{p_k}, \overline{p_k}] + i\,[\underline{q_k}, \overline{q_k}]$, for $k = 1, 2, \ldots, n$, by virtue of Theorem 3.2 and the complex interval arithmetic, we conclude

$$\sum_{j=1}^{n} a_{kj}\left(\underline{p_j} + \overline{p_j}\right) - \sum_{j=1}^{n} b_{kj}\left(\underline{q_j} + \overline{q_j}\right) = \underline{u_k} + \overline{u_k},$$

$$\sum_{j=1}^{n} a_{kj}\left(\underline{q_j} + \overline{q_j}\right) + \sum_{j=1}^{n} b_{kj}\left(\underline{p_j} + \overline{p_j}\right) = \underline{v_k} + \overline{v_k},$$

for $k = 1, 2, \ldots, n$.

By Theorem 3.2, we can show a new relation between the Doolittle's solution $[\mathbf{z}_D]$, the algebraic solution $[\mathbf{z}_A]$ and the solution set Σ for the complex interval linear system (2.1) as follows.

Theorem 3.3 *Suppose that the complex interval linear system (2.1) has an algebraic solution and also the complex coefficient matrix* $\mathbf{C}$ *be nonsingular. Then, we have*

$$[\mathbf{z}_D]^c = [\mathbf{z}_A]^c = \Sigma^c.$$

Proof. By Remark 2.3 and Theorems 2.2 and 3.2, it is concluded that $[\mathbf{z}_D]^c = [\mathbf{z}_A]^c = \Sigma^c = \mathbf{C}^{-1}\,[\mathbf{w}]^c$.

For a numerical illustration of the above discussion, we present the following example. All final results are obtained by MATLAB software.

Example 3.1 *Consider the following* 4×4 *complex interval linear system*

$$\begin{cases} (2-i)[z_1] + (1-i)[z_2] + (-1+i)[z_3] + (i)[z_4] \\ \qquad = [-4,6] + i\,[-6,5], \\ (1-3i)[z_1] + (i)[z_2] + (2-2i)[z_3] + (1+i)[z_4] \\ \qquad = [-3,11] + i\,[-4,11], \\ (-1-i)[z_1] + (2-i)[z_2] + (i)[z_3] + (3-i)[z_4] \\ \qquad = [-9,10] + i\,[-9,7], \\ (i)[z_1] + (2-2i)[z_2] + (1+i)[z_3] + (1+2i)[z_4] \\ \qquad = [-9,7] + i\,[-5,12], \end{cases}$$

with the unique algebraic solution

$$[\mathbf{z}_A] = \begin{pmatrix} [0,1] + i\,[0,1] \\ [-1,1] + i\,[0,1] \\ [-1,0] + i\,[1,2] \\ [-1,2] + i\,[-1,1] \end{pmatrix}.$$

For the above system, by Theorem 2.2, the solution set Σ *can be obtained as follows*

$$\Sigma = \boldsymbol{C}^{-1}\,[\boldsymbol{w}]$$

$$= \begin{pmatrix} [-4.950, 5.950] + i\,[-4.840, 5.840] \\ [-7.297, 7.297] + i\,[-6.728, 7.728] \\ [-6.807, 5.807] + i\,[-5.043, 8.043] \\ [-6.402, 7.402] + i\,[-6.910, 6.910] \end{pmatrix}.$$

Also, by using the complex interval Doolittle decomposition method , we obtain

$$[\mathbf{z}_D] = \begin{pmatrix} [z_{1D}] \\ [z_{2D}] \\ [z_{3D}] \\ [z_{4D}] \end{pmatrix}$$

$$= \begin{pmatrix} [-162.344, 163.34] + i\,[-162.43, 163.43] \\ [-83.17, 83.17] + i\,[-82.34, 83.34] \\ [-39.16, 38.16] + i\,[-36.93, 39.93] \\ [-22.60, 23.60] + i\,[-23.15, 23.15] \end{pmatrix}.$$

From the above solutions, it is clear that

$$[\boldsymbol{z}_A] \subseteq \Sigma \subseteq [\boldsymbol{z}_D],$$

$$[\mathbf{z}_A]^c = \Sigma^c = [\mathbf{z}_D]^c = \begin{pmatrix} 0.5+0.5i \\ 0+0.5i \\ -0.5+1.5i \\ 0.5+0i \end{pmatrix},$$

and

$$\mathbf{C}\,[\mathbf{z}_D]^c = \mathbf{C}\,[\mathbf{z}_A]^c = \mathbf{C}\,\Sigma^c = [\mathbf{w}]^c$$
$$= \begin{pmatrix} 1-0.5i \\ 4+3.5i \\ 0.5-i \\ -1+3.5i \end{pmatrix}.$$

In the next section, we present a new way for obtaining the algebraic solution of a complex interval linear system.

4 The proposed method

In this section, we introduce a simple method for obtaining the algebraic solution of the complex interval linear system (2.1). In the proposed method, we first apply the complex interval Doolittle decomposition method and then restrict the obtained solution by using of some parameters, such that the final solution be a complex interval vector and also satisfies the complex interval linear system (2.1) . We propose our method as follows.

Let $[\mathbf{z}_D] = ([z_{1D}], [z_{2D}], \dots, [z_{nD}])^T$ be the obtained solution by the complex interval Doolittle decomposition method. Obviously, $[\mathbf{z}_D]$ is a complex interval vector. We define

$$[\mathbf{z}_A] = \begin{pmatrix} [z_{1A}] \\ [z_{2A}] \\ \vdots \\ [z_{nA}] \end{pmatrix} = \begin{pmatrix} [\underline{z_{1D}}+\theta_1, \overline{z_{1D}}-\theta_1] \\ [\underline{z_{2D}}+\theta_2, \overline{z_{2D}}-\theta_2] \\ \vdots \\ [\underline{z_{2D}}+\theta_n, \overline{z_{2D}}-\theta_n] \end{pmatrix}, \tag{4.11}$$

where θ_j, $j = 1, 2, \dots, n$ are called "complex limiting factors" and

$$0 \leqslant \theta_j \leqslant \frac{\overline{z_{jD}} - \underline{z_{jD}}}{2}, \qquad j = 1, 2, \dots, n, \tag{4.12}$$

this means that

$$0 \leqslant Real(\theta_j) \leqslant \frac{1}{2} Real\left([z_{jD}]^{\Delta}\right), \tag{4.13}$$

$$0 \leqslant Imag(\theta_j) \leqslant \frac{1}{2} Imag\left([z_{jD}]^{\Delta}\right), \tag{4.14}$$

where $j = 1, 2, \dots, n$. If we set $\theta_j = \alpha_j + i\beta_j$ and $[z_{jD}] = [p_j] + i[q_j]$ for $j = 1, 2, \dots, n$, then obviously $\underline{z_{jD}} = \underline{p_j} + i\underline{q_j}$ and $\overline{z_{jD}} = \overline{p_j} + i\overline{q_j}$. Also, we can rewrite the Eqs. (4.11)-(4.14) as follows:

$$[\mathbf{z}_A] = \begin{pmatrix} [\underline{p_1}+\alpha_1, \overline{p_1}-\alpha_1] + i\,[\underline{q_1}+\beta_1, \overline{q_1}-\beta_1] \\ [\underline{p_2}+\alpha_2, \overline{p_2}-\alpha_2] + i\,[\underline{q_2}+\beta_2, \overline{q_2}-\beta_2] \\ \vdots \\ [\underline{p_n}+\alpha_n, \overline{p_n}-\alpha_n] + i\,[\underline{q_n}+\beta_n, \overline{q_n}-\beta_n] \end{pmatrix}, \tag{4.15}$$

where the parameters α_j and β_j satisfy the following conditions

$$0 \leqslant \alpha_j \leqslant \frac{1}{2}[p_j]^{\Delta}, \quad j = 1, 2, \dots, n, \tag{4.16}$$

$$0 \leqslant \beta_j \leqslant \frac{1}{2}[q_j]^{\Delta}, \quad j = 1, 2, \dots, n. \tag{4.17}$$

It should be noted that the conditions (4.16) and (4.17) on α_j and β_j imply $[\mathbf{z}_D]$ be a complex interval vector.

Now, we are going to obtain the value of parameters α_j and β_j such that the obtained solution via Eq. (4.15) be an algebraic solution for the complex interval linear system (2.1). To this end, we must have

$$\sum_{j=1}^{n} c_{kj}[z_{jA}] = [w_k], \quad k = 1, 2, \dots, n,$$

or

$$\sum_{j=1}^{n} c_{kj}\left([\underline{p_j}+\alpha_j, \overline{p_j}-\alpha_j] + i\,[\underline{q_j}+\beta_1, \overline{q_j}-\beta_j]\right)$$
$$= [w_k],$$

for $k = 1, 2, \dots, n$. Due to $[w_k] = [\underline{u_k}, \overline{u_k}] + i\,[\underline{v_k}, \overline{v_k}]$ and $c_{kj} = a_{kj} + i\,b_{kj}$, by using of the complex interval arithmetic, we conclude

$$\begin{aligned} \underline{u_k} &= \sum_{a_{kj}\geqslant 0} a_{kj}\left(\underline{p_j}+\alpha_j\right) \\ &+ \sum_{a_{kj}<0} a_{kj}\left(\overline{p_j}-\alpha_j\right) \\ &- \sum_{b_{kj}<0} b_{kj}\left(\underline{q_j}+\beta_j\right) \\ &- \sum_{b_{kj}\geqslant 0} b_{kj}\left(\overline{q_j}-\beta_j\right), \end{aligned}$$

$$\begin{aligned}\overline{u_k} &= \sum_{a_{kj}\geqslant 0} a_{kj}\left(\overline{p_j}-\alpha_j\right)\\ &+ \sum_{a_{kj}<0} a_{kj}\left(\underline{p_j}+\alpha_j\right)\\ &- \sum_{b_{kj}<0} b_{kj}\left(\overline{q_j}-\beta_j\right)\\ &- \sum_{b_{kj}\geqslant 0} b_{kj}\left(\underline{q_j}+\beta_j\right),\end{aligned}$$

$$\begin{aligned}\underline{v_k} &= \sum_{a_{kj}\geqslant 0} a_{kj}\left(\underline{q_j}+\beta_j\right)\\ &+ \sum_{a_{kj}<0} a_{kj}\left(\overline{q_j}-\beta_j\right)\\ &+ \sum_{b_{kj}\geqslant 0} b_{kj}\left(\underline{p_j}+\alpha_j\right)\\ &+ \sum_{b_{kj}<0} b_{kj}\left(\overline{p_j}-\alpha_j\right),\end{aligned}$$

$$\begin{aligned}\overline{v_k} &= \sum_{a_{kj}\geqslant 0} a_{kj}\left(\overline{q_j}-\beta_j\right)\\ &+ \sum_{a_{kj}<0} a_{kj}\left(\underline{q_j}+\beta_j\right)\\ &+ \sum_{b_{kj}\geqslant 0} b_{kj}\left(\overline{p_j}-\alpha_j\right)\\ &+ \sum_{b_{kj}<0} b_{kj}\left(\underline{p_j}+\alpha_j\right).\end{aligned}$$

From the above equations, we have

$$\begin{aligned}\overline{u_k}-\underline{u_k} &= \sum_{j=1}^{n}|a_{kj}|(\overline{p_j}-\alpha_j)-\sum_{j=1}^{n}|a_{kj}|\left(\underline{p_j}+\alpha_j\right)\\ &+ \sum_{j=1}^{n}|b_{kj}|(\overline{q_j}-\beta_j)-\sum_{j=1}^{n}|b_{kj}|\left(\underline{q_j}+\beta_j\right),\end{aligned}$$

$$\begin{aligned}\overline{v_k}-\underline{v_k} &= \sum_{j=1}^{n}|a_{kj}|(\overline{q_j}-\beta_j)-\sum_{j=1}^{n}|a_{kj}|\left(\underline{q_j}+\beta_j\right)\\ &+ \sum_{j=1}^{n}|b_{kj}|(\overline{p_j}-\alpha_j)-\sum_{j=1}^{n}|b_{kj}|\left(\underline{p_j}+\alpha_j\right).\end{aligned}$$

In other words

$$\begin{aligned}[u_k]^{\Delta} &= \sum_{j=1}^{n}|a_{kj}|\left(\overline{p_j}-\underline{p_j}\right)-2\sum_{j=1}^{n}|a_{kj}|\alpha_j\\ &+ \sum_{j=1}^{n}|b_{kj}|\left(\overline{q_j}-\underline{q_j}\right)-2\sum_{j=1}^{n}|b_{kj}|\beta_j,\end{aligned}$$

$$\begin{aligned}[v_k]^{\Delta} &= \sum_{j=1}^{n}|a_{kj}|\left(\overline{q_j}-\underline{q_j}\right)-2\sum_{j=1}^{n}|a_{kj}|\beta_j\\ &+ \sum_{j=1}^{n}|b_{kj}|\left(\overline{p_j}-\underline{p_j}\right)-2\sum_{j=1}^{n}|b_{kj}|\alpha_j,\end{aligned}$$

for $k=1,2,\ldots,n$. Therefore, in the matrix form, we have

$$\begin{cases}[\mathbf{u}]^{\Delta}=|\mathbf{A}|\cdot[\mathbf{p}]^{\Delta}-2|\mathbf{A}|\cdot\boldsymbol{\alpha}\\ \quad+|\mathbf{B}|\cdot[\mathbf{q}]^{\Delta}-2|\mathbf{B}|\cdot\boldsymbol{\beta},\\ [\mathbf{v}]^{\Delta}=|\mathbf{A}|\cdot[\mathbf{q}]^{\Delta}-2|\mathbf{A}|\cdot\boldsymbol{\beta}\\ \quad+|\mathbf{B}|\cdot[\mathbf{p}]^{\Delta}-2|\mathbf{B}|\cdot\boldsymbol{\alpha},\end{cases} \tag{4.18}$$

where

$$|\mathbf{A}|=(|a_{kj}|)_{n\times n},\qquad |\mathbf{B}|=(|b_{kj}|)_{n\times n},$$

$$\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)^T,$$

and

$$\beta=(\beta_1,\beta_2,\ldots,\beta_n)^T.$$

From Eq. (4.18) we conclude

$$\begin{cases}|\mathbf{A}|\,\boldsymbol{\alpha}+|\mathbf{B}|\,\boldsymbol{\beta}\\ \quad=\frac{1}{2}\left(|\mathbf{A}|\,[\mathbf{p}]^{\Delta}+|\mathbf{B}|\,[\mathbf{q}]^{\Delta}-[\mathbf{u}]^{\Delta}\right),\\ |\mathbf{B}|\,\boldsymbol{\alpha}+|\mathbf{A}|\,\boldsymbol{\beta}\\ \quad=\frac{1}{2}\left(|\mathbf{A}|\,[\mathbf{q}]^{\Delta}+|\mathbf{B}|\,[\mathbf{p}]^{\Delta}-[\mathbf{v}]^{\Delta}\right).\end{cases} \tag{4.19}$$

Obviously, the above equation (4.19) is a $2n\times 2n$ real linear system and also its right hand side vector can be easily computed, because all parameters of right hand side are known. Finally, The real linear system (4.19) can be now uniquely solved for α and β, if and only if its coefficient matrix is nonsingular.

Theorem 4.1 *[9] The coefficient matrix of the real linear system (4.19) is nonsingular if and only if the matrices* $|\mathbf{A}|+|\mathbf{B}|$ *and* $|\mathbf{A}|-|\mathbf{B}|$ *are both nonsingular.*

Remark 4.1 [9] Supposing that

$$\mathbf{S} = \begin{pmatrix} |\mathbf{A}| & |\mathbf{B}| \\ |\mathbf{B}| & |\mathbf{A}| \end{pmatrix},$$

we conclude that if $\mathbf{S}^{-1}$ exists it must have the same structure as $\mathbf{S}$, i.e.

$$\mathbf{S}^{-1} = \begin{pmatrix} \mathbf{D} & \mathbf{E} \\ \mathbf{E} & \mathbf{D} \end{pmatrix},$$

where

$$\mathbf{D} = \frac{1}{2}\left[(|\mathbf{A}|+|\mathbf{B}|)^{-1} + (|\mathbf{A}|-|\mathbf{B}|)^{-1}\right],$$

$$\mathbf{E} = \frac{1}{2}\left[(|\mathbf{A}|+|\mathbf{B}|)^{-1} - (|\mathbf{A}|-|\mathbf{B}|)^{-1}\right].$$

Theorem 4.2 *[9] Suppose that in the real linear system (4.19) the matrices* $|\mathbf{A}|$ *and* $\mathbf{H} = |\mathbf{A}|-|\mathbf{B}|\cdot|\mathbf{A}|^{-1}\cdot|\mathbf{B}|$ *are both nonsingular. Then the system (4.19) has a unique solution as follows*

$$\beta = \mathbf{H}^{-1}\cdot\left(\mathbf{G} - |\mathbf{B}|\cdot|\mathbf{A}|^{-1}\cdot\mathbf{F}\right),$$
$$\alpha = |\mathbf{A}|^{-1}\cdot(\mathbf{F} - |\mathbf{B}|\cdot\beta),$$

where

$$\mathbf{F} = \frac{1}{2}\left(|\mathbf{A}|\cdot[\mathbf{p}]^{\Delta} + |\mathbf{B}|\cdot[\mathbf{q}]^{\Delta} - [\mathbf{u}]^{\Delta}\right),$$

and

$$\mathbf{G} = \frac{1}{2}\left(|\mathbf{A}|\cdot[\mathbf{q}]^{\Delta} + |\mathbf{B}|\cdot[\mathbf{p}]^{\Delta} - [\mathbf{v}]^{\Delta}\right).$$

In general, according to the defined notations in Remark 4.1 and Theorem 4.2 and also assuming that the coefficient matrix of real linear system (4.19) is nonsingular, we achieve

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \mathbf{S}^{-1}\begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix},$$

and consequently

$$\alpha = \mathbf{D}\cdot\mathbf{F} + \mathbf{E}\cdot\mathbf{G},$$

$$\beta = \mathbf{E}\cdot\mathbf{F} + \mathbf{D}\cdot\mathbf{G}.$$

Finally, we obtain

$$[\mathbf{z}_A] = [\underline{\mathbf{p}} + \alpha, \overline{\mathbf{p}} - \alpha] + i[\underline{\mathbf{q}} + \beta, \overline{\mathbf{q}} - \beta],$$

where

$$[\underline{\mathbf{p}}, \overline{\mathbf{p}}] = Real([\mathbf{z}_D]), \qquad [\underline{\mathbf{q}}, \overline{\mathbf{q}}] = Imag([\mathbf{z}_D]).$$

The above solution vector is thus unique but may still not be a complex interval vector, because it is possible that either $\underline{Real([\mathbf{z}_A])} > \overline{Real([\mathbf{z}_A])}$ or $\underline{Imag([\mathbf{z}_A])} > \overline{Imag([\mathbf{z}_A])}$. A necessary condition for the unique solution vector to be a complex interval vector is that the vectors α and β satisfy the conditions (4.16) and (4.17), respectively.

In the following theorem, we prove that the obtained solution via the above method is an "algebraic solution" for the complex interval linear system (2.1).

Theorem 4.3 *Let the coefficient matrix C of the complex interval linear system (2.1) be completely nonsingular (see Definition 2.10) and the solution of system (4.19) satisfies the conditions (4.16) and (4.17). Then the complex interval vector* $[\mathbf{z}_A]$ *obtained via Eq. (4.15) is a unique algebraic solution of the complex interval linear system (2.1).*

Since the coefficient matrix C is completely nonsingular, then according to Theorem 4.1, the system (4.19) has a unique solution. Also the conditions (4.16) and (4.17) guarantees that this unique solution is a complex interval vector. Now, to prove the theorem, we must show

$$\sum_{j=1}^{n} c_{kj}[z_{jA}] = [w_k], \quad k = 1, 2, \ldots, n,$$

or

$$\sum_{j=1}^{n} c_{kj}\left([\underline{p_j} + \alpha_j, \overline{p_j} - \alpha_j] + i\,[\underline{q_j} + \beta_1, \overline{q_j} - \beta_j]\right)$$
$$= [w_k],$$

for $k = 1, 2, \ldots, n$. Since $[w_k] = [\underline{u_k}, \overline{u_k}] + i\,[\underline{v_k}, \overline{v_k}]$ and $c_{kj} = a_{kj} + i\,b_{kj}$, by using the complex interval arithmetic, it is sufficient to prove that

$$\underline{u_k} = \sum_{a_{kj}\geqslant 0} a_{kj}\left(\underline{p_j} + \alpha_j\right) + \sum_{a_{kj}<0} a_{kj}\left(\overline{p_j} - \alpha_j\right)$$
$$- \sum_{b_{kj}<0} b_{kj}\left(\underline{q_j} + \beta_j\right) - \sum_{b_{kj}\geqslant 0} b_{kj}\left(\overline{q_j} - \beta_j\right), \quad (4.20)$$

$$\overline{u_k} = \sum_{a_{kj}\geqslant 0} a_{kj}\left(\overline{p_j} - \alpha_j\right) + \sum_{a_{kj}<0} a_{kj}\left(\underline{p_j} + \alpha_j\right)$$

$$-\sum_{b_{kj}<0} b_{kj}\left(\overline{q_j}-\beta_j\right)-\sum_{b_{kj}\geqslant 0} b_{kj}\left(\underline{q_j}+\beta_j\right), \quad (4.21)$$

$$\underline{v_k}=\sum_{a_{kj}\geqslant 0} a_{kj}\left(\underline{q_j}+\beta_j\right)+\sum_{a_{kj}<0} a_{kj}\left(\overline{q_j}-\beta_j\right)$$
$$+\sum_{b_{kj}\geqslant 0} b_{kj}\left(\underline{p_j}+\alpha_j\right)+\sum_{b_{kj}<0} b_{kj}\left(\overline{p_j}-\alpha_j\right), \quad (4.22)$$

$$\overline{v_k}=\sum_{a_{kj}\geqslant 0} a_{kj}\left(\overline{q_j}-\beta_j\right)+\sum_{a_{kj}<0} a_{kj}\left(\underline{q_j}+\beta_j\right)$$
$$+\sum_{b_{kj}\geqslant 0} b_{kj}\left(\overline{p_j}-\alpha_j\right)+\sum_{b_{kj}<0} b_{kj}\left(\underline{p_j}+\alpha_j\right). \quad (4.23)$$

Here, we prove only the Eq. (4.20). The Eqs. (4.21)-(4.23) can be proven similarly. For this end, from Eq. (4.19) and Remark 3.3, we have

$$\sum_{a_{kj}\geqslant 0} a_{kj}\left(\underline{p_j}+\alpha_j\right)+\sum_{a_{kj}<0} a_{kj}\left(\overline{p_j}-\alpha_j\right)$$
$$-\sum_{b_{kj}<0} b_{kj}\left(\underline{q_j}+\beta_j\right)-\sum_{b_{kj}\geqslant 0} b_{kj}\left(\overline{q_j}-\beta_j\right)$$

$$\begin{aligned}
&= \sum_{a_{kj}\geqslant 0} a_{kj}\underline{p_j}+\sum_{a_{kj}\geqslant 0} a_{kj}\alpha_j \\
&+ \sum_{a_{kj}<0} a_{kj}\overline{p_j}-\sum_{a_{kj}<0} a_{kj}\alpha_j \\
&- \sum_{b_{kj}<0} b_{kj}\underline{q_j}-\sum_{b_{kj}<0} b_{kj}\beta_j \\
&- \sum_{b_{kj}\geqslant 0} b_{kj}\overline{q_j}+\sum_{b_{kj}\geqslant 0} b_{kj}\beta_j \\
&= \sum_{a_{kj}\geqslant 0} a_{kj}\underline{p_j}+\sum_{a_{kj}<0} a_{kj}\overline{p_j} \\
&- \sum_{b_{kj}<0} b_{kj}\underline{q_j}-\sum_{b_{kj}\geqslant 0} b_{kj}\overline{q_j} \\
&+ \sum_{j=1}^{n}|a_{kj}|\alpha_j+\sum_{j=1}^{n}|b_{kj}|\beta_j \\
&= \sum_{a_{kj}\geqslant 0} a_{kj}\underline{p_j}+\sum_{a_{kj}<0} a_{kj}\overline{p_j} \\
&- \sum_{b_{kj}<0} b_{kj}\underline{q_j}-\sum_{b_{kj}\geqslant 0} b_{kj}\overline{q_j} \\
&+ \frac{1}{2}\sum_{j=1}^{n}|a_{kj}|\left(\overline{p_j}-\underline{p_j}\right)+\frac{1}{2}\sum_{j=1}^{n}|b_{kj}|\left(\overline{q_j}-\underline{q_j}\right) \\
&- \frac{1}{2}\left(\overline{u_k}-\underline{u_k}\right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{a_{kj}\geqslant 0} a_{kj}\underline{p_j}+\sum_{a_{kj}<0} a_{kj}\overline{p_j} \\
&- \sum_{b_{kj}<0} b_{kj}\underline{q_j}-\sum_{b_{kj}\geqslant 0} b_{kj}\overline{q_j} \\
&+ \frac{1}{2}\left(\sum_{a_{kj}\geqslant 0} a_{kj}\left(\overline{p_j}-\underline{p_j}\right)\right) \\
&- \left(\sum_{a_{kj}<0} a_{kj}\left(\overline{p_j}-\underline{p_j}\right)\right) \\
&+ \frac{1}{2}\left(\sum_{b_{kj}\geqslant 0} b_{kj}\left(\overline{q_j}-\underline{q_j}\right)\right) \\
&- \left(\sum_{b_{kj}<0} b_{kj}\left(\overline{q_j}-\underline{q_j}\right)\right)-\frac{1}{2}\left(\overline{u_k}-\underline{u_k}\right) \\
&= \frac{1}{2}\left(\sum_{a_{kj}\geqslant 0} a_{kj}\left(\overline{p_j}+\underline{p_j}\right)\right) \\
&+ \left(\sum_{a_{kj}<0} a_{kj}\left(\overline{p_j}+\underline{p_j}\right)\right) \\
&- \frac{1}{2}\left(\sum_{b_{kj}\geqslant 0} b_{kj}\left(\overline{q_j}+\underline{q_j}\right)\right) \\
&+ \left(\sum_{b_{kj}<0} b_{kj}\left(\overline{q_j}+\underline{q_j}\right)\right)-\frac{1}{2}\left(\overline{u_k}-\underline{u_k}\right) \\
&= \frac{1}{2}\left(\sum_{j=1}^{n} a_{kj}\left(\overline{p_j}+\underline{p_j}\right)\right) \\
&- \left(\sum_{j=1}^{n} b_{kj}\left(\overline{q_j}+\underline{q_j}\right)\right)-\frac{1}{2}\left(\overline{u_k}-\underline{u_k}\right) \\
&= \frac{1}{2}\left(\underline{u_k}+\overline{u_k}-\overline{u_k}+\underline{u_k}\right) \\
&= \underline{u_k}.
\end{aligned}$$

Therefore, the proof of the theorem is completed.

Based on the above mentioned discussion, we can propose an algorithm with three steps for obtaining the algebraic solution of the complex interval linear system (2.1) as follows. In this algorithm we suppose that the coefficient matrix **C** is completely nonsingular.

Algorithm model:

Step 1: Solve the complex interval linear system (2.1) by the complex interval Doolittle decomposition method and obtain the solution $[\mathbf{z}_D]$.

Step 2: Solve the real linear system (4.19) and obtain the real and imaginary parts (α_j and β_j) of the complex limiting factors θ_j, $j = 1, 2, \ldots, n$.

Step 3: If α_j and β_j satisfy the conditions (4.13) and (4.14) for all $j = 1, 2, \ldots, n$, respectively, then the presented $[\mathbf{z}_A]$ by Eq. (4.15) is an unique algebraic solution for the complex interval linear system (2.1). Otherwise the system (2.1) does not have any algebraic solution.

5 Numerical examples

In this section, we present two numerical examples to show ability and efficiency of our Algorithm. It should be noted that all numerical results are obtained by MATLAB software.

Example 5.1 *Consider the* 3×3 *complex interval linear system*

$$\begin{cases} (1+i)[z_1] + (2-3i)[z_2] + (-1-i)[z_3] \\ \qquad = [-8, 7] + i[-8, 8], \\ (1+2i)[z_1] + (-1-i)[z_2] + (2+i)[z_3] \\ \qquad = [-4, 12] + i[-4, 10], \\ (2-3i)[z_1] + (1-2i)[z_2] + (1+2i)[z_3] \\ \qquad = [-5, 16] + i[-6, 14], \end{cases} \tag{5.24}$$

It can be easily verified that $det(\mathbf{C}) = -18 - 58\,i$, $det(|\mathbf{A}|+|\mathbf{B}|) = 22$ and $det(|\mathbf{A}|-|\mathbf{B}|) = 2$, where $\mathbf{C}$ is the complex coefficient matrix of system (5.24) and $\mathbf{A}$ and $\mathbf{B}$ are the real and imaginary parts of the matrix $\mathbf{C}$, respectively. Therefore, according to Definition 2.10, we conclude that the the matrix $\mathbf{C}$ is completely nonsingular. Now, based on our algorithm, we first solve the complex interval linear system (5.24) by the complex interval Doolittle method. For this, we decompose the matrix $\mathbf{C}$ into the product of the lower-triangular complex matrix $\mathbf{L}$ and the upper-triangular complex matrix $\mathbf{U}$, where the main diagonal of $\mathbf{L}$ consists of all $1 + 0i$s, as follows

$$\mathbf{C} = \mathbf{L}\,\mathbf{U},$$

where

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 1.5+0.5i & 1 & 0 \\ -0.5-2.5i & -1.33-0.88i & 1 \end{pmatrix},$$

and

$$\mathbf{U} = \begin{pmatrix} 1+i & 2-3i & -1-i \\ 0 & -5.5+2.5i & 3+3i \\ 0 & 0 & 4.36+5.62i \end{pmatrix}.$$

Since $\mathbf{L}$ is a lower-triangular matrix , we can easily solve the system $\mathbf{L}\,[\mathbf{y}] = [\mathbf{w}]$ by forward substitution. Therefore, from Eqs. (3.5) and (3.6) we obtain

$$[\mathbf{y}] = \begin{pmatrix} [y_1] \\ [y_2] \\ [y_3] \end{pmatrix}$$

$$= \begin{pmatrix} [-8, 7] + i\,[-8, 8] \\ [-18.5, 28] + i\,[-19.5, 26] \\ [-76.38, 93.80] + i\,[-72.13, 94.60] \end{pmatrix}.$$

Now, by solving $\mathbf{L}\,[\mathbf{z}] = [\mathbf{y}]$ via backward substitution, we can find the Doolittle's solution $[\mathbf{z}_D]$. Therefore, from Eqs. (3.6) and (3.7) we obtain

$$[\mathbf{z}_D] = \begin{pmatrix} [z_{1D}] \\ [z_{2D}] \\ [z_{3D}] \end{pmatrix}$$

$$= \begin{pmatrix} [-175.03, 176.03] + i\,[-175.03, 176.03] \\ [-26.93, 26.93] + i\,[-26.39, 27.39] \\ [-14.60, 18.60] + i\,[-16.65, 16.65] \end{pmatrix}.$$

In the next step, by solving the real linear system (4.19), we obtain the real and imaginary parts of the complex limiting factors $\theta_j = \alpha_j + i\,\beta_j$, $j = 1, 2, \ldots, n$, respectively as follows

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 175.03 \\ 25.93 \\ 15.60 \end{pmatrix},$$

and

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 174.03 \\ 26.39 \\ 15.65 \end{pmatrix}.$$

It can be easily investigated that the above obtained vectors α and β satisfy the conditions (4.13) and (4.14), respectively. Consequently, based on the proposed algorithm, by using the Eqs. (4.11) or (4.15) we obtain the unique algebraic solution of the complex interval linear system (5.24) as follows

$$[\mathbf{z}_A] = \begin{pmatrix} [z_{1A}] \\ [z_{2A}] \\ [z_{3A}] \end{pmatrix} = \begin{pmatrix} [0, 1] + i\,[-1, 2] \\ [-1, 1] + i\,[0, 1] \\ [1, 3] + i\,[-1, 1] \end{pmatrix}.$$

Example 5.2 *Consider the* 5×5 *complex interval linear system*

$$\begin{cases} (1+2i)[z_1]+(-1-i)[z_2]+(1+3i)[z_3] \\ +(-1+i)[z_4]+(-2+i)[z_5] \\ =[-7,6]+i[9,20], \\ \\ (3-2i)[z_1]+(-2+i)[z_2]+(-3-i)[z_3] \\ +(-2-2i)[z_4]+(2+2i)[z_5] \\ =[3,21]+i[-21,-3], \\ \\ (-1-i)[z_1]+(2-i)[z_2]+(3+i)[z_3] \\ +(4-3i)[z_4]+(5-i)[z_5] \\ =[1,24]+i[-24,-8], \\ \\ (-1-5i)[z_1]+(1-2i)[z_2] \\ +(1+3i)[z_3] \\ +(2-4i)[z_4]+(3-2i)[z_5] \\ =[-12,12]+i[-30,-10], \\ \\ (2+i)[z_1]+(-1+i)[z_2]+(1+i)[z_3] \\ +(2-3i)[z_4]+(4-5i)[z_5] \\ =[-2,12]+i[-35,-20], \end{cases} \tag{5.25}$$

It can be easily verified that $det(\mathbf{C})=2511-2247\,i$, $det(|\mathbf{A}|+|\mathbf{B}|)=192$ and $det(|\mathbf{A}|-|\mathbf{B}|)=30$. Therefore, we conclude that the matrix $\mathbf{C}$ is completely nonsingular. Now, based on our algorithm, we first solve the complex interval linear system (5.25) by the complex interval Doolittle method. For this, we decompose the matrix $\mathbf{C}$ into

$$\mathbf{C}=\mathbf{L}\,\mathbf{U},$$

where

$$\mathbf{L}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -0.2-1.6i & 1 & 0 & 0 & 0 \\ -0.6+0.2i & 0.4+1.8i & 1 & 0 & 0 \\ -2.2-0.6i & 4.2+2.4i & 2.10-1.04i & 1 & 0 \\ 0.8-0.6i & -1.2-0.4i & -0.42+0.46i & -0.77-0.48i & 1 \end{pmatrix},$$

and

$$\mathbf{U}=\begin{pmatrix} 1+2i & -1-i & 1+3i & -1+i & -2+i \\ 0 & -0.6-0.8i & -7.6+1.2i & -3.8-3.4i & -i \\ 0 & 0 & 9.4+15.8i & -1+6i & 2.2+0.4i \\ 0 & 0 & 0 & 2.85+7.36i & -9.44+4.65i \\ 0 & 0 & 0 & 0 & -2.94-9.97i \end{pmatrix}.$$

By solving the system $\mathbf{L}\,[\mathbf{y}]=[\mathbf{w}]$ by forward substitution and also Eqs. (3.5) and (3.6), we obtain

$$[\mathbf{y}]=\begin{pmatrix} [y_1] \\ [y_2] \\ [y_3] \\ [y_4] \\ [y_5] \end{pmatrix}$$

$$=\begin{pmatrix} [-7,6]+i[9,20] \\ [-30.4,7.8]+i[-30.4,10.6] \\ [-59.24,62.84]+i[-38.08,72.28] \\ [-352.40,337.02]+i[-291.16,383.67] \\ [-555.17,490.64]+i[-539.96,504.84] \end{pmatrix}.$$

Now, by solving $\mathbf{L}\,[\mathbf{z}]=[\mathbf{y}]$ via backward substitution and Eqs. (3.6) and (3.7), we can find the Doolittle's solution $[\mathbf{z}_D]$ as follows

$$[\mathbf{z}_D]=\begin{pmatrix} [z_{1D}] \\ [z_{2D}] \\ [z_{3D}] \\ [z_{4D}] \\ [z_{5D}] \end{pmatrix}$$

$$=\begin{pmatrix} [-4995.49,4998.49]+i[-4997.74,4996.74] \\ [-3626.97,3623.97]+i[-3624.45,3625.45] \\ [-120.42,121.42]+i[-120.58,121.58] \\ [-198.57,201.57]+i[-200.59,200.59] \\ [-59.97,64.97]+i[-65.00,60.00] \end{pmatrix}.$$

In the next step, by solving the real linear system (4.19), we obtain the real parts α_j and the imaginary parts β_j of the complex limiting factors $\theta_j=\alpha_j+i\,\beta_j$, $j=1,2,\ldots,n$, as follows

$$\alpha=\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix}=\begin{pmatrix} 4996.49 \\ 3624.97 \\ 120.42 \\ 199.57 \\ 61.97 \end{pmatrix},$$

and

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix} = \begin{pmatrix} 4996.74 \\ 3624.45 \\ 120.58 \\ 199.59 \\ 63.00 \end{pmatrix}.$$

It can be easily investigated that the above obtained vectors α and β satisfy the conditions (4.13) and (4.14), respectively. Consequently, based on the proposed algorithm, by using the Eqs. (4.11) or (4.15), we obtain the unique algebraic solution of the complex interval linear system (5.25) as follows

$$[\mathbf{z}_A] = \begin{pmatrix} [z_{1A}] \\ [z_{2A}] \\ [z_{3A}] \\ [z_{4A}] \\ [z_{5A}] \end{pmatrix} = \begin{pmatrix} [1,2] + i\,[-10] \\ [-2,-1] + i\,[0,1] \\ [0,1] + i\,[0,1] \\ [1,2] + i\,[-1,1] \\ [2,3] + i\,[-2,-3] \end{pmatrix}.$$

6 Conclusion

In this paper, we have presented a simple method for solving the complex interval linear system $\mathbf{C}[\mathbf{z}] = [\mathbf{w}]$, where $\mathbf{C}$ is a complex $n \times n$ matrix, the unknown vector $[\mathbf{z}]$ and the right-hand side vector $[\mathbf{w}]$ are all vectors consisting of n complex interval numbers. In the proposed method, by restricting the solutions of complex interval Doolittle method, we achieved a complex interval vector such that satisfies the system $\mathbf{C}[\mathbf{z}] = [\mathbf{w}]$. For future work, we try to extend our method to solve complex fuzzy linear systems when, at least one of the elements of right-hand side is a complex fuzzy number.

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