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Research Article

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A Note On Dual Models Of Interval DEA and Its Extension To Interval Data

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Abstract

In this article, we investigate the measurement of performance in DMUs in which input and/or output values are given as imprecise data. By imprecise data, we mean that in some cases, we only know that the actual values are inside certain intervals, and in other cases, data are specified only as ordinal preference information. In this article, we present two distinct perspectives for determining the upper and lower bounds of the efficiency the DMU under evaluation can have with imprecise data: (1) The optimistic perspective, which uses DEA-efficient production frontier, and seeks the best score among various values of the efficiency score; the measured efficiency in this perspective is called the best relative efficiency or the optimistic efficiency. (2) The pessimistic perspective, which uses inefficiency frontier, also called input frontier, and seeks the lowest score among various values of the efficiency score; the measured efficiency in this perspective is called the worst relative efficiency or the pessimistic efficiency. For this reason and contrary to some DEA-related studies, we do not restrict our attention only to precise data. We will investigate a more general case of dealing with imprecise data, providing a method for obtaining the upper and lower bounds of efficiency. Two numerical examples will be presented to illustrate the application of the proposed DEA approach.

Keywords : Data envelopment analysis; Imprecise data; Optimistic efficiency interval; Pessimistic efficiency interval; Overall efficiency interval; Ranking.

1 Introduction

Data envelopment analysis (DEA) is extensively used to evaluate and estimate the efficiency of decision-making units (DMUs). DEA was first proposed by Charnes et al. [1]. It has been widely used for evaluating the relative efficiency of many decision-making entities in the public and private sectors. In recent years, numerous studies have been conducted on the application of DEA in educational and industrial centers [2, 3, 4].

DEA calculates an efficiency score for each DMU relative to a set of DMUs. DEA efficiency score (in input-oriented mode) defines the maximum possible proportional reduction in input usage with constant output level for each DMU. This increases the efficiency of a DMU up to the most efficient DMUs in the DMUs set. In other

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words, DEA chooses a set of the most favorable weights for each DMU under evaluation. Accordingly, the method proposed by Charnes et al. [1] measures the efficiency of DMUs from the optimistic viewpoint. The measured efficiency of this method is called the *best relative efficiency* or the *optimistic efficiency* where its value is less than or equal to one. If the optimistic efficiency of a DMU is equal to one, it is DEA-efficient or optimistic efficient; otherwise, it is DEAnon-efficient or optimistic non-efficient. It is believed that the performance of optimistic efficient DMUs is higher than that of optimistic non-efficient DMUs.

On the other hand, the approach proposed by Parkan and Wang [5] measures the efficiency of DMUs from the pessimistic viewpoint [6, 7]. In this approach, a set of the most unfavorable weights is selected for each DMU under evaluation. The measured efficiency of the pessimistic perspective is called the *worst relative efficiency* or the *pessimistic efficiency* where its value is greater than or equal to one. If the pessimistic efficiency of a DMU is equal to one, that DMU is called pessimistic inefficient or DEA-inefficient; otherwise it is called pessimistic non-inefficient or DEAnon-inefficient. It is commonly believed that the performance of the pessimistic inefficient DMUs is worse than the pessimistic non-inefficient DMUs.

Optimistic and pessimistic efficiencies measure two extremes of the performance of each DMU. Any method that considers only one of the perspectives is bias. To determine the overall performance of DMUs, both optimistic and pessimistic viewpoints should be considered simultaneously.

Entani et al. [8] proposed a paired DEA model with interval efficiencies measured from both optimistic and pessimistic perspectives. The paired DEA model was initially developed for crisp data and later was extended to interval and fuzzy data. Theoretically, their models were able to render both interval and fuzzy data. However, there were some problems with the models. The models only use one input and one output data to determine the lower bound of the efficiency interval for each DMU regardless of the number of inputs and outputs in the model. Consequently, their model leads to data loss concerning input and output data of the DMU under evaluation. In addition, the paired DEA model uses variable production frontiers to measure the efficiency intervals of different DMUs with interval data. Wang and Yang [9] proposed a pair of bounded DEA model for crisp data. The pair of bounded DEA model uses all possible inputs and outputs. It measures the best and the worst relative efficiencies of each DMU using a virtual DMU called anti-ideal DMU. The anti-ideal DMU employs the maximum input value to produce the minimum output. The efficiency of an anti-ideal DMU is zero when all output values are zero. As a result, their pair of bounded DEA model fails when determining the interval efficiency for each DMU. Recently, Azizi and Wang [10] developed improved bounded DEA models which are able to measure the efficiencies of DMUs in all situations. Wang et al. [11] proposed a pair of interval DEA model for precise data. The interval DEA models use the pessimistic efficiency of a virtual DMU called ideal DMU- which employs the minimum input to produce the maximum output- for determining the efficiency interval for each DMU. Accordingly, the optimistic and pessimistic efficiencies of each DMU are measured. Accordingly, Azizi and Jahed [12] noted that the interval DEA models of Wang et al. [11] are unable to determine the lower bound of the efficiency interval for each DMU when the input value is zero. To resolve this problem, Azizi and Jahed [12] developed an improved interval DEA model to measure the overall performances of DMUs in all conditions. Azizi and Fathi Ajirlu [13] used the optimistic efficiency of the ideal DMU and the pessimistic efficiency of the anti-ideal DMU to determine the lower bound of the efficiency interval for crisp data. Their DEA models were unable to determine the lower bound of the efficiency interval when there were zeros in each input and output. Foroughi and Aouni [14] proposed a mixed integer linear programming model to determine the lower bound of the efficiency interval for each DMU. The proposed model is unable to identify all pessimistic inefficient DMUs.

Wang and Luo [15] combined the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS), which is one of the topics of multi-attribute decision making, with DEA. To calculate the best and the worst relative efficiencies, they defined two virtual DMUs called ideal DMU and anti-ideal DMU to develop two DEA models. They combined these two distinct efficiencies and obtained a relative closeness index as a basis for ranking DMUs. Their proposed DEA models have two major disadvantages: (1) In most cases, their DEA models use constant weights for all DMUs, and (2) when there are zeros in each input and output, their DEA models are infeasible. Wang et al. [16] proposed a geometric average efficiency measure to assess the overall performance of each DMU. The geometric average efficiency combines both optimistic and pessimistic efficiencies of each DMU. Therefore, it is more comprehensive than either of the measures. Recently, Wang and Lan [17] and Chin et al. [18] extended this approach. Wang and Chin [19] proposed a new overall performance measure for ranking DMUs. The proposed DEA approach considers both optimistic and pessimistic efficiencies of the DMUs simultaneously. The overall performance measure not only considers the magnitude of the two different efficiencies, but also considers their directions. Therefore, it is assumed to be more comprehensive than the geometric average efficiency measure proposed by Wang et al. [16]. Amirteimoori [20] introduced an efficiency measure using the ideal and anti-ideal indicators formed on the basis of the efficient and inefficient frontiers of the DEA. These indicators maximize of the weighted L_1 distance from a particular DMU relative to the inefficient and efficient frontiers of the DEA. Amirteimoori et al. [21] also improved the cost efficiency interval of a DMU by adjusting the observed inputs and outputs. Based on returns to scale terms, Wang and Lan [22] examined the most productive scale size of a DMU from both optimistic and pessimistic perspectives.

According to literature, much effort is needed to measure the overall performances of DMUs, because it should be measured in a more general case in the presence of imprecise data. It is noteworthy that the papers concerning the simultaneous application of both optimistic and pessimistic viewpoints were reviewed. Entani et al. [8] examined the DEA structure in the presence of interval data from both optimistic and pessimistic perspectives. The drawbacks of DEA models are described in Section 3. The main objective of the present paper is to measure the overall performances of DMUs using DEA and simultaneous application of crisp, ordinal and interval data. The upper bound of the overall efficiency interval is obtained from the optimistic viewpoint based on the best position of each DMU using a set of the most favorable weights. The lower bound is obtained from the pessimistic viewpoint based on the most unfavorable position of each DMU using a set of the most unfavorable weights. The overall efficiency interval shows all possible evaluations through different perspectives. Accordingly, the decision maker is provided with the efficiency interval of all possible values of efficiencies reflecting the different views. Two numerical examples illustrate the application of the proposed method. Since the ultimate efficiency score for each DMU is characterized by an interval, a simple but practical ranking approach is needed to rank and compare DMUs efficiencies. Previously, several approaches have been developed to rank interval numbers. But all of them have some disadvantages. In particular, when the interval numbers have equal centers but different widths, all of them are incapable of distinguishing between these numbers. We use the minimax regret approach, developed by Wang et al. [23], for comparing and ranking the efficiency intervals of the DMUs.

This paper is organized as follows. Section 2 introduces the basic DEA models used for determining the best and the worst relative efficiencies of DMUs. Section 3 analyzes the DEA models proposed by Entani et al. [8], then the adjusted pessimistic efficiency interval is reviewed. Section 4 illustrates an empirical example of scoring the performances of a set of 20 branches of a commercial bank in Iran. The conclusion is presented in Section 5.

2 Interval DEA models for measurement of the best and the worst relative efficiencies

2.1 Interval DEA models for measurement of the best relative efficiencies of DMUs

In DEA analysis, it is usually assumed that there are n production units that consume m different inputs and produce s different outputs. Specifically, the jth production unit consumes x_{ij} units of input i $(i = 1, \ldots, m)$ and produces y_{rj} units of output r $(r = 1, \ldots, s)$. In interval DEA, it is assumed that some exact values of input x_{ij} and output y_{rj} are not known. It is only known that they are in the range of the upper and lower bounds specified by intervals $[x^L_{ij}, x^U_{ij}]$ and $[y^L_{rj}, y^U_{rj}]$, and each DMU has a positive lower bound input and a positive lower bound output.

To deal with such an uncertain situation, Wang et al. [23] presented the following pair of linear programming (LP) models that measure the best relative efficiencies of DMUs:

$$
\begin{aligned}
\max\ \theta^U_o &= \sum_{r=1}^{s} u_r y^U_{ro} \\
\text{s.t.}\ & \sum_{r=1}^{s} u_r y^U_{rj} - \sum_{i=1}^{m} v_i x^L_{ij} \le 0, \quad j = 1, \ldots, n, \\
& \sum_{i=1}^{m} v_i x^L_{io} = 1, \\
& u_r, v_i \ge \varepsilon, r = 1, \ldots, s; i = 1, \ldots, m.
\end{aligned} \tag{2.1}
$$

$$
\begin{aligned}
\max\ \theta^L_o &= \sum_{r=1}^{s} u_r y^L_{ro} \\
\text{s.t.}\ & \sum_{r=1}^{s} u_r y^U_{rj} - \sum_{i=1}^{m} v_i x^L_{ij} \le 0, \quad j = 1, \ldots, n, \\
& \sum_{i=1}^{m} v_i x^U_{io} = 1, \\
& u_r, v_i \ge \varepsilon, r = 1, \ldots, s; i = 1, \ldots, m.
\end{aligned} \tag{2.2}
$$

where DMU_o is the DMU under evaluation, v_i $(i = 1, \ldots, m)$ and u_r $(r = 1, \ldots, s)$ are decision variables, and ε is the non-Archimedean infinitesimal [24]. θ^U_o is the best relative efficiency under the most favorable conditions and θ^L_o is the best relative efficiency under the most unfavorable conditions for DMU_o. They form the optimistic efficiency interval $[\theta^L_o, \theta^U_o]$. If there is a set of positive weights u^*_r $(r = 1, \ldots, s)$ and v^*_i $(i = 1, \ldots, m)$ that make $\theta^{U*}_o = 1$, then DMU_o is called DEA-efficient or optimistic efficient; otherwise, it is called DEA-non-efficient or optimistic non-efficient. All DEA-efficient DMUs collectively form an efficiency frontier.

2.2 Interval DEA models for measurement of the worst relative efficiencies of DMUs

The *input-oriented* framework, which is based on the set of input requirement and its inefficiency frontier, tries to increase input values as much as possible, while keeping the output at most at its current level. This emphasizes the fact that output is kept constant and input values are increased proportionally, until the inefficient production frontier is obtained. DEA estimator for inefficient production possibility set is called the pessimistic efficiency or the worst relative efficiency. For a particular DMU, such as DMU_o, relative efficiencies can be calculated form the following pessimistic DEA models [25]:

$$
\begin{aligned}
\min\ \phi^L_o &= \sum_{r=1}^{s} u_r y^L_{ro} \\
\text{s.t.}\ & \sum_{r=1}^{s} u_r y^L_{rj} - \sum_{i=1}^{m} v_i x^U_{ij} \ge 0, \quad j = 1, \ldots, n, \\
& \sum_{i=1}^{m} v_i x^U_{io} = 1, \\
& u_r, v_i \ge \varepsilon, r = 1, \ldots, s; i = 1, \ldots, m.
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
\min\ \phi^U_o &= \sum_{r=1}^{s} u_r y^U_{ro} \\
\text{s.t.}\ & \sum_{r=1}^{s} u_r y^L_{rj} - \sum_{i=1}^{m} v_i x^U_{ij} \ge 0, \quad j = 1, \ldots, n, \\
& \sum_{i=1}^{m} v_i x^L_{io} = 1, \\
& u_r, v_i \ge \varepsilon, r = 1, \ldots, s; i = 1, \ldots, m.
\end{aligned} \tag{2.4}
$$

In models (2.3) and (2.4), ϕ^L_o is the worst relative efficiency under the most unfavorable condi-

Table 1: Data for five DMUs with one input and one output.

DMU	Input	Output
A	[1, 3]	[2, 3]
B	[4, 5]	[6, 7]
C	[9, 11]	[7, 9]
D	[6, 8]	[1, 3]
E	[5, 7]	[4, 5]

tions and ϕ_o^U is the worst relative efficiency under the most favorable conditions for DMU$_o$. They give the pessimistic efficiency interval $[\phi_o^L, \phi_o^U]$ for DMU$_o$. When there is a set of positive weights u_r^* $(r = 1, \ldots, s)$ and v_i^* $(i = 1, \ldots, m)$ that satisfy $\phi_o^{L*} = 1$, we say that DMU$_o$ is DEA-inefficient or pessimistic inefficient; otherwise, we say that DMU$_o$ is DEA-non-inefficient or pessimistic non-inefficient. All DEA-inefficient DMUs collectively form an inefficiency frontier.
In order to illustrate the difference between optimistic efficient, optimistic non-efficient, pessimistic inefficient, and pessimistic non-inefficient DMUs (i.e., the difference between the efficiency frontier and the inefficiency frontier), we use an example of a dataset with one input and one output, as shown in Table 1. The efficiency and inefficiency frontiers for this example are shown in Figure 1. As it is clear from the figure, three DMUs are on the efficient frontier, which we call DEA-efficient or optimistic efficient DMUs, and the rest of the DMUs are called DEA-non-efficient or optimistic non-efficient in relation to the efficient frontier. Also, there are two DMUs on the inefficient frontier, which we call DEA-inefficient or pessimistic inefficient, while we call the rest of the DMUs DEA-non-inefficient or pessimistic non-inefficient in relation to the inefficient frontier. Here, there is also some overlap, or common units, between optimistic efficient and pessimistic inefficient units.

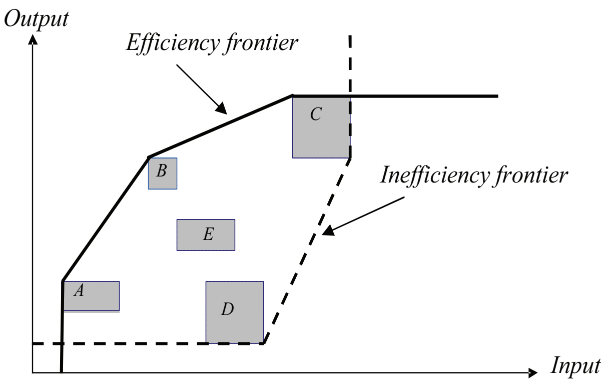

Figure 1: Efficiency and inefficiency frontiers for the five DMUs under variable return to scale.

3 The overall efficiency interval-Integration of the optimistic and pessimistic efficiencies

3.1 A review of Entani et al.'s [8] DEA models

To develop an overall efficiency interval for each DMU, Entani et al. [8] proposed the following mathematical programming model to determine the upper bound of the overall efficiency interval of DMU$_o$:

$$\max\ \Theta_o^U = \max_{y_{ij},x_{ij}} \frac{\frac{\sum_{r=1}^s u_r y_{ro}}{\sum_{i=1}^m v_i x_{io}}}{\underset{j}{max}\left\{\frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}}\right\}} \tag{3.5}$$
$$\text{s.t. } u_r, v_i \geq 0, r = 1, \ldots, s; i = 1, \ldots, m.$$

where $x_{ij} \in [x_{ij}^L, x_{ij}^U]$ and $y_{rj} \in [y_{rj}^L, y_{rj}^U]$. To obtain the optimum value of model (3.5), it was simplified to model (3.6) by Entani et al. [8]:

$$\begin{aligned} \max\ \Theta_o^U &= \frac{\sum_{r=1}^s u_r y_{ro}^U}{\sum_{i=1}^m v_i x_{io}^L} \\ \text{s.t. } \max&\left\{ \max_{j\neq o}\left\{\frac{\sum_{r=1}^s u_r y_{rj}^L}{\sum_{i=1}^m v_i x_{ij}^U}\right\}, \right. \\ &\left. \frac{\sum_{r=1}^s u_r y_{ro}^U}{\sum_{i=1}^m v_i x_{io}^L} \right\} = 1, \\ &u_r, v_i \geq 0, r = 1, \ldots, s; i = 1, \ldots, m. \end{aligned} \tag{3.6}$$

The upper bound of the overall efficiency interval for DMU$_o$ can be achieved using the following LP problem:

$$\begin{aligned} \max\ \Theta_o^U &= \sum_{r=1}^{s} u_r y_{ro}^U \\ \text{s.t.}\ & \sum_{r=1}^{s} u_r y_{rj}^L - \sum_{i=1}^{m} v_i x_{ij}^U \le 0,\ j=1,\dots,n; j\neq o, \\ & \sum_{r=1}^{s} u_r y_{ro}^U - \sum_{i=1}^{m} v_i x_{io}^L \le 0, \\ & \sum_{i=1}^{m} v_i x_{io}^L = 1, \\ & u_r, v_i \ge \varepsilon, r=1,\dots,s; i=1,\dots,m. \end{aligned} \tag{3.7}$$

In models (3.6) and (3.7), the lower bounds of input intervals x_{io}^L and the upper bounds of output intervals y_{ro}^U are used for DMU$_o$, and upper bounds of input intervals x_{ij}^U and the lower bounds of output intervals y_{rj}^L are used for other DMUs. Model (3.7) is equivalent to the upper bound DEA model of Despotis and Smirlis [26], and reports many DMUs that are not DEA-efficient as DEA-efficient. One of the drawbacks of model (3.7) is that it uses different constraint sets for evaluating the efficiencies of different DMUs. The main drawback of using different sets of constraints for efficiencies measurement of DMUs is the lack of possibility of comparison between efficiencies, since different production frontiers have been used in the process of efficiency measurement. We use LP model (2.1) for obtaining the upper bound of the overall efficiency interval for each DMU. This model is different from the existing DEA models for interval data in that the LP model (2.1) uses a fixed and unified production frontier for measuring the efficiency of each DMU. To obtain the lower bound of the overall efficiency interval for DMU$_o$, Entani et al. [8] proposed the following mathematical programming model for DMU$_o$:

$$\begin{aligned} \min\ \varphi_o^L &= \min_{y_{ij}, x_{ij}} \frac{\frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}}}{max_j \{\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}}\}} \\ \text{s.t.}\ & u_r, v_i \ge 0, r=1,\dots,s; i=1,\dots,m. \end{aligned} \tag{3.8}$$

To obtain the optimum value of model (3.8), it was simplified to model (3.9) by Entani et al. [8]:

$$\begin{aligned} \min\ \varphi_o^L &= \frac{\sum_{r=1}^{s} u_r y_{ro}^L}{\sum_{i=1}^{m} v_i x_{io}^U} \\ \text{s.t.}\ & \max\left\{ \max_{j\neq o} \left\{ \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i x_{ij}^L} \right\}, \frac{\sum_{r=1}^{s} u_r y_{ro}^L}{\sum_{i=1}^{m} v_i x_{io}^U} \right\} = 1, \\ & u_r, v_i \ge 0, r=1,\dots,s; i=1,\dots,m. \end{aligned} \tag{3.9}$$

The upper bounds of the input intervals, x_{io}^U and the lower bounds of the output intervals, y_{ro}^L in model (3.9) are used for DMU$_o$. The lower bounds of the input intervals, x_{ij}^L and the upper bounds of the output intervals, y_{rj}^U are used for other DMUs. Model (3.9) cannot be transformed into an equivalent LP model. To achieve the optimal value of model (3.9), assuming $\sum_{r=1}^{s} u_r y_{rj}^U / \sum_{i=1}^{m} v_i x_{ij}^L = 1$ for each DEA-efficient unit, Entani et al. [8] divided. Model (3.9) into e_1 sub-optimization problems $j = J_1, \dots, J_{e_1}$, in which e_1 is the number of DEA-efficient units, and $J_1, \dots, J_{e_1}$ are DEA-efficient units:

$$\begin{aligned} \min\ \varphi_{oj}^L &= \frac{\sum_{r=1}^{s} u_r y_{ro}^L}{\sum_{i=1}^{m} v_i x_{io}^U} \\ \text{s.t.}\ & \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i x_{ij}^L} = 1, \\ & u_r, v_i \ge 0, r=1,\dots,s; i=1,\dots,m. \end{aligned} \tag{3.10}$$

The sub-optimization problem (3.10) can be converted to e_1 LP models as follows:

$$\begin{aligned} \min\ \varphi_{oj}^L &= \sum_{r=1}^{s} u_r y_{ro}^L \\ \text{s.t.}\ & \sum_{r=1}^{s} u_r y_{rj}^U - \sum_{i=1}^{m} v_i x_{ij}^L = 0, \\ & \sum_{i=1}^{m} v_i x_{io}^U = 1 \\ & u_r, v_i \ge 0, r=1,\dots,s; i=1,\dots,m. \end{aligned} \tag{3.11}$$

Entani et al [8] claim that the minimum value out of the optimal values of (3.11) is the optimal value of model (3.9). Assuming that $\varphi_{oJ_1}^{L*}, \dots, \varphi_{oJ_{e_1}}^{L*}$ are the optimal values of the objectives functions in LPs of the sub-optimization problem (3.11), we can write the lower bound of the overall efficiency interval of DMU_o mathematically as follows:

$$\varphi_o^{L*} = \min_{j=J_1,\ldots,J_{e_1}} \{\varphi_{oj}^{L*}\} \tag{3.12}$$

Accordingly, $[\varphi_o^{L*}, \Theta_o^{U*}]$ is the overall efficiency interval of DMU_o, where Θ_o^{U*} is the optimal value of model (3.7).

In the sub-optimization problem (3.11), each LP model has only two linear constraints. Therefore, regardless of the number of inputs and outputs in the problem under consideration, only two decision variables can be non-zero, one for the input weight and the other for the output weight. As such, Entani et al.'s [8] DEA models measure the pessimistic efficiency of each DMU by taking into account only one input and one output. Compared with the sub-optimization problem (3.11), model (2.3) includes $(n + 1)$ linear constraints and, consequently, can make the most use of input and output information. Additionally, model (2.3) is able to accurately identify the pessimistic inefficient units and the inefficiency frontier. To clarify this point, consider the following numerical example.

Example 3.1 *Consider the example discussed by Cooper et al. [27]. We have five DMUs that use two inputs, one crisp and the other interval, and produce two outputs, one crisp and the other ordinal. The data set is shown in Table 2.*

For conversion of ordinal preference information into interval data, we used the approach proposed by Wang et al. [23]. For this example, the preference intensity parameter and the ratio parameter about the strong ordinal preference information were determined (or estimated) as $\chi_2 = 1.2$ and $\sigma_2 = 0.2$, respectively [28, 29]. Using the technique described in Wang et al. [23], we can obtain an interval estimate for the second output of each DMU, which is shown in the last column of Table 2.

First we obtain the optimistic and pessimistic efficiencies of the five DMUs using interval DEA models (2.1)-(2.4). These are shown in Table 3. From Table 3, it is clear that only one DMU, i.e. DMU_1, is DEA-efficient according to the optimistic DEA model (2.1). This DEA-efficient unit determines the efficiency frontier. It is usually believed that this DEA-efficient unit should have a better performance than the other four units that are identified as DEA-non-efficient. From the pessimistic efficiency perspective, two DMUs, i.e. DMU_4 and DMU_5, are identified as DEA-inefficient. Collectively, they define an inefficiency frontier. It is believed that these two DEA-non-efficient units have a poorer performance than the three units that are identified as DEA-non-inefficient. The evaluations above have been performed from different perspectives and, as such, may have different results. Any conclusion based on only one of these two perspectives will undoubtedly be unrealistic and unconvincing. In order to provide an overall assessment of the performance of each DMU, Entani et al. [8] considered both optimistic and pessimistic perspectives simultaneously. The results of Entani et al.'s [8] interval DEA models are shown in the last column of Table 3. As it can be seen from Table 3, due to the use of different production frontiers for measuring the efficiencies of different DMUs, model (3.7), which is used by Entani et al. [8] for obtaining the upper bound of the overall efficiency interval, evaluates all five DMUs as DEA-efficient. Also, the sub-optimization problem (3.11), which is used by Entani et al. [8] for obtaining the lower bound of the overall efficiency interval, identifies only DMU_4, which has the smallest lower-bound efficiency among the five DMUs, as a DEA-inefficient unit. But it cannot identify DMU_5 which is DEA-inefficient. Consequently, the efficiency and inefficiency frontiers cannot be determined using Entani et al.'s [8] interval DEA models.

For example, since five DMUs were identified as DEA-efficient in the approach proposed by Entani et al. [8], they used 5 LP models for determining the lower bound of the overall efficiency interval for each DMU (In sum, 25 LP models must be solved for determination of the lower bound of the overall efficiency interval for five DMUs.. Consider DMU_3 as an instance. To determine the value of the lower bound of the overall efficiency interval for this DMU, the following five

Table 2: Imprecise data and ordinal data converted for five DMUs.

DMU	Inputs		Outputs		Converted ordinal data
	x_{1j} (exact)	x_{2j} (interval)	y_{1j} (exact)	y_{2j} (ordinal*)	
1	100	[0.6, 0.7]	200	4	[0.3456, 0.8333]
2	150	[0.8, 0.9]	1000	2	[0.2400, 0.5787]
3	150	[1, 1]	1200	5	[0.4147, 1.0000]
4	200	[0.7, 0.8]	900	1	[0.2000, 0.4823]
5	200	[1, 1]	600	3	[0.2880, 0.6944]

* Ranking, such that 5 ≡ highest rank, ..., 1 ≡ lowest rank ($y_{23} \succ y_{21} \succ \cdots \succ y_{24}$)

Table 3: Imprecise data and ordinal data converted for five DMUs.

DMU	Optimistic efficiency interval $[\theta_j^{L*}, \theta_j^{U*}]$	Pessimistic efficiency interval $[\phi_j^{L*}, \phi_j^{U*}]$	Overall efficiency interval according to Entani et al.'s [8] DEA models (models (3.7) and (3.11)
1	[1.0000, 1.0000]	[1.9749, 4.9184]	[0.3555, 1.0000]
2	[0.3333, 0.5209]	[1.0473, 2.0833]	[0.1920, 1.0000]
3	[0.4000, 0.8000]	[1.5133, 2.0000]	[0.2986, 1.0000]
4	[0.3375, 0.4961]	[1.0000, 1.5000]	[0.1200, 1.0000]
5	[0.2074, 0.5000]	[1.0000, 1.0000]	[0.1500, 1.0000]

LP models must be solved:

$$\text{(LP1)}: \ \varphi_{31}^{L*} = \min\ 1200u_1 + 0.4147u_2$$

$$\text{s.t.} \begin{cases} 150v_1 + v_2 = 1, \\ 2000u_1 + 0.8333u_2 - 100v_1 - 0.6v_2 = 0, \\ u_1, u_2, v_1, v_2 \geq 0. \end{cases}$$

$$\text{(LP2)}: \ \varphi_{32}^{L*} = \min\ 1200u_1 + 0.4147u_2$$

$$\text{s.t.} \begin{cases} 150v_1 + v_2 = 1, \\ 1000u_1 + 0.5787u_2 - 150v_1 - 0.8v_2 = 0, \\ u_1, u_2, v_1, v_2 \geq 0. \end{cases}$$

$$\text{(LP3)}: \ \varphi_{33}^{L*} = \min\ 1200u_1 + 0.4147u_2$$

$$\text{s.t.} \begin{cases} 150v_1 + v_2 = 1, \\ 1200u_1 + u_2 - 150v_1 - v_2 = 0, \\ u_1, u_2, v_1, v_2 \geq 0. \end{cases}$$

$$\text{(LP4)}: \ \varphi_{34}^{L*} = \min\ 1200u_1 + 0.4147u_2$$

$$\text{s.t.} \begin{cases} 150v_1 + v_2 = 1, \\ 900u_1 + 0.4823u_2 - 200v_1 - 0.7v_2 = 0, \\ u_1, u_2, v_1, v_2 \geq 0. \end{cases}$$

$$\text{(LP5)}: \ \varphi_{35}^{L*} = \min\ 1200u_1 + 0.4147u_2$$

$$\text{s.t.} \begin{cases} 150v_1 + v_2 = 1, \\ 600u_1 + 0.6944u_2 - 200v_1 - v_2 = 0, \\ u_1, u_2, v_1, v_2 \geq 0. \end{cases}$$

The solutions of these five LP models are as follows:
$\varphi_{31}^{L*} = 0.2986, u_1^* = 0, u_2^* = 0.7200, v_1^* = 0$ and $v_2^* = 1$,
$\varphi_{32}^{L*} = 0.5733, u_1^* = 0, u_2^* = 1.3824, v_1^* = 0$ and $v_2^* = 1$,
$\varphi_{33}^{L*} = 0.4147, u_1^* = 0, u_2^* = 1, v_1^* = 0$ and $v_2^* = 1$,
$\varphi_{34}^{L*} = 0.6019, u_1^* = 0, u_2^* = 1.4514, v_1^* = 0$ and $v_2^* = 1$,
$\varphi_{35}^{L*} = 0.5972, u_1^* = 0, u_2^* = 1.4401, v_1^* = 0$ and $v_2^* = 1$.
Finally, the lower bound of the overall efficiency interval of DMU_3 is obtained as follows:

$$\varphi_3^{L*} = \min\{0.2986, 0.5733, 0.4147, 0.6019, 0.5972\} = 0.2986$$

The lower bounds of the overall efficiency intervals for the other four DMUs are calculated similarly. Also, based on the five sets of input and output weights obtained above, it is evident that only one input (either input 1 or input 2) and one output (either output 1 or output 2) are involved in the calculation of the lower bound of the overall efficiency interval. For instance, consider the fourth set of weights, i.e.

$u_1^* = 0, u_2^* = 1.4514, v_1^* = 0, v_2^* = 1$. We substituted this set of weights in the constraints of model (9) and obtained the following efficiencies for the DMUs:
$\varphi_1^{L*} = 2.0158, \varphi_2^{L*} = 1.0499, \varphi_3^{L*} = 1.4514, \varphi_4^{L*} = 0.3629, \varphi_5^{L*} = 1.0079$.
Except for the efficiency of DMU_4, all other efficiency values are greater than one. Such results evidently violate the condition $max\left\{\underset{j\neq o}{max}\{\sum_{r=1}^{s} u_r y_{rj}^L / \sum_{i=1}^{m} v_i x_{ij}^U\}, \sum_{r=1}^{s} u_r y_{ro}^U / \sum_{i=1}^{m} v_i x_{io}^L\right\} = 1$. Therefore, the approach proposed by Entani et al. [8] for obtaining the lower bound of the overall efficiency interval is illogical and unacceptable.

In the next section, we will develop novel DEA models for determination of the lower bound of the overall efficiency interval in order to overcome these drawbacks.

The pessimistic efficiency score is the opposite of the optimistic efficiency score. It is a score that each DMU obtains in its most unfavorable situation (or the most favorable situation) using a set of *the most unfavorable weights*. Theoretically, the best and the worst relative efficiencies should be calculated in a common range and should form an interval for each DMU. For example, they can be measured in the interval $[\beta, 1]$, where $\beta > 0$ is a parameter. In the next section, we will find a suitable value for β.

3.2 Adjusting the worst relative efficiencies

Theoretically, the best and the worst relative efficiencies should form an interval. For this purpose, the worst relative efficiencies determined by models (2.3) and (2.4) should be adjusted [30, 31]. Suppose that β $(0 < \beta \leq 1)$ is the adjustment coefficient, then the adjusted worst relative efficiencies can be written as $\beta\phi_j^* = \beta[\phi_j^{L*}, \phi_j^{U*}] = \hat{\phi}_j^* = [\hat{\phi}_j^{L*}, \hat{\phi}_j^{U*}]$ $(j = 1, \ldots, n)$ satisfying $\hat{\phi}_j^* = \beta\phi_j^* = \beta[\phi_j^{L*}, \phi_j^{U*}] \leq \theta_j^* = [\theta_j^{L*}, \theta_j^{U*}]$ $(j = 1, \ldots, n)$ or $\beta \leq \underset{j=1,\ldots,n}{min}\{\theta_j^{L*}/\phi_j^{U*}\}$. Assuming $\phi_{max}^{U*} = \underset{j=1,\ldots,n}{max}\{\phi_j^{U*}\}$ and $\theta_{min}^{L*} = \underset{j=1,\ldots,n}{min}\{\theta_j^{L*}\}$, then $\underset{j=1,\ldots,n}{min}\{\theta_j^{L*}/\phi_j^{U*}\} \geq \underset{j=1,\ldots,n}{min}\{\theta_j^{L*}\}/\underset{j=1,\ldots,n}{max}\{\phi_j^{U*}\}$. Substituting $\beta = \theta_{min}^{L*}/\phi_{max}^{U*}$, it is ensured that $\beta \leq \underset{j=1,\ldots,n}{min}\{\theta_j^{L*}/\phi_j^{U*}\}$. Since β is not zero, the worst performance of DMUs in the interval $[\beta, 1]$ can be measured by the following models:

$$\begin{aligned} \min\ \Psi_o^L &= \frac{\sum_{r=1}^{s} u_r y_{ro}^L}{\sum_{i=1}^{m} v_i x_{io}^U} \\ \text{s.t.}\ & \frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i x_{ij}^U} \geq \beta, \quad j = 1, \ldots, n, \\ & u_r, v_i \geq \varepsilon, r = 1, \ldots, s; i = 1, \ldots, m. \end{aligned} \tag{3.13}$$

$$\begin{aligned} \min\ \Psi_o^U &= \frac{\sum_{r=1}^{s} u_r y_{ro}^U}{\sum_{i=1}^{m} v_i x_{io}^L} \\ \text{s.t.}\ & \frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i x_{ij}^U} \geq \beta, \quad j = 1, \ldots, n, \\ & u_r, v_i \geq \varepsilon, r = 1, \ldots, s; i = 1, \ldots, m. \end{aligned} \tag{3.14}$$

Models (3.13) and (3.14) can be transformed into the following two LP models:

$$\begin{aligned} \min\ \Psi_o^L &= \sum_{r=1}^{s} u_r y_{ro}^L \\ \text{s.t.}\ & \sum_{r=1}^{s} u_r y_{rj}^L - \sum_{i=1}^{m} v_i(\beta x_{ij}^U) \geq 0,\ j = 1, \ldots, n, \\ & \sum_{i=1}^{m} v_i x_{io}^U = 1 \\ & u_r, v_i \geq \varepsilon, r = 1, \ldots, s; i = 1, \ldots, m. \end{aligned} \tag{3.15}$$

$$\begin{aligned} \min\ \Psi_o^U &= \sum_{r=1}^{s} u_r y_{ro}^U \\ \text{s.t.}\ & \sum_{r=1}^{s} u_r y_{rj}^L - \sum_{i=1}^{m} v_i(\beta x_{ij}^U) \geq 0,\ j = 1, \ldots, n, \\ & \sum_{i=1}^{m} v_i x_{io}^L = 1 \\ & u_r, v_i \geq \varepsilon, r = 1, \ldots, s; i = 1, \ldots, m. \end{aligned} \tag{3.16}$$

Suppose that Ψ_o^{L*} and Ψ_o^{U*} are the optimal values of models (3.15) and (3.16), respectively. Then they form a pessimistic efficiency interval, $[\Psi_o^{L*}, \Psi_o^{U*}]$. The worst performances of the n DMUs is obtained by repeatedly solving models (3.15) and (3.16) for each DMU. The pessimistic efficiency interval is shown by $[\Psi_j^{L*}, \Psi_j^{U*}]$ $(j = 1, \ldots, n)$. To obtain more reliable results, both optimistic and pessimistic perspectives should be applied simultaneously to score a problem. For this reason, the optimistic and pessimistic efficiency intervals of DMUs are merged to obtain a new interval called the *overall efficiency interval*. Both upper and lower bounds (extreme values) of the overall efficiency interval are considered from two different viewpoints. As a result, an overall efficiency interval of $[\Psi_j^{L*}, \Psi_j^{U*}]$ $(j = 1, \ldots, n)$ is defined for each DMU_j. The upper bound of the overall efficiency interval is obtained from the optimistic viewpoint based on the best position of each DMU using a set of the most favorable weights. The lower bound is obtained from the pessimistic viewpoint based on the most unfavorable position of each DMU using a set of the most unfavorable weights. The overall efficiency interval shows all possible evaluations of different perspectives. Accordingly, the decision maker is provided with the overall efficiency interval of all possible efficiencies reflecting different perspectives.

The following definitions are concerned for the overall efficiency interval, $[\Psi_o^{L*}, \theta_o^{U*}]$.

Definition 3.1 *DMU_o is called DEA-efficient or optimistic efficient, if $\theta_o^{U*} = 1$, otherwise it is called DEA-non-efficient.*

Definition 3.2 *DMU_o is called DEA-inefficient or pessimistic inefficient, if $\Psi_o^{L*} = \beta$, otherwise it is called DEA-non-inefficient.*

Definition 3.3 *DMU_o is called DEA-unspecified if and only if it is neither DEA-efficient nor DEA-inefficient.*

Regarding DEA-unspecified units, we could say that they are always circumscribed between the efficient and inefficient frontiers (see DMU_E in Figure 1) [8].

For a comparison of our proposed overall efficiency interval with the overall efficiency interval obtained from Entani et al.'s [8] DEA models, consider the numerical example presented in Section *3.1*. First we determine the value of β using Table 3 and we obtain $\beta = \theta_{min}^{L*}/\phi_{max}^{U*} = 0.2074/4.9184 = 0.0422$.

Then we run interval DEA models (3.15) and (3.16) for each DMU to obtain the adjusted pessimistic efficiency interval for the five DMUs. By integrating the optimistic efficiency interval and the adjusted pessimistic efficiency interval for the five DMUs, we obtain the overall performance score, i.e. the overall efficiency interval, of each DMU. Noting the obtained overall efficiency intervals in Table 4, it can be clearly seen that the proposed DEA approach identifies both pessimistic inefficient DMUs accurately (DMU_4 and DMU_5 are identified as pessimistic inefficient DMUs). These evaluation results are completely compatible with the results of the pessimistic DEA model (2.3). In this example, DMU_2 and DMU_3 are identified as DEA-unspecified units. Furthermore, for comparison and ranking of the overall efficiency intervals of the five DMUs, we used the minimax regret approach for calculating the maximum loss of efficiency for each DMU. The last column of Table 4 shows the ranking of the five DMUs according to the overall efficiency interval, from which it can be seen that DMU_1 has the best overall performance.

It should be noted that Entani et al. [8] have developed an approach for finding overall efficiency intervals for crisp data, interval data, and fuzzy data. However, they have not described the method of computation of the overall efficiency interval for ordinal data. Besides, they have not considered the overall efficiency interval for a mixture of crisp data, interval data, and ordinal data.

4 An empirical example

In this section, the DEA approach with both efficient and inefficient frontiers addressed in this paper is used to evaluate the performances of the commercial bank branches. In this example, the value of the non-Archimedean infinitesimal is assumed to be $\varepsilon = 10^{-10}$.

Consider, the performance measurement of 20 branches of a set of commercial bank (DMUs) in Iran. Each branch is examined in terms of three inputs including *payable interest*, *personnel*, and *non-performing loans* and five outputs including *the total sum of four main deposits*, *other deposits*, *loans granted*, *received interest*, and *fee*. The data set used in this analysis was adopted from Jahanshahloo et al. [32]. Tables 5 and 6 show the interval inputs and outputs for DMUs. Table 7 shows the overall efficiency intervals scores, optimistic efficiency intervals, pessimistic efficiency intervals, and adjusted pes-

Table 4: Imprecise data and ordinal data converted for five DMUs.

DMU	Adjusted pessimistic efficiency interval $[\Psi_j^{L*}, \Psi_j^{U*}]$	Overall efficiency interval $[\Psi_j^{L*}, \theta_j^{U*}]$	Rank
1	[0.0833, 0.2076]	[0.0833, 1.0000]	1
2	[0.0442, 0.0879]	[0.0442, 0.5209]	3
3	[0.0639, 0.0844]	[0.0639, 0.8000]	2
4	[0.0422, 0.0633]	[0.0422, 0.4961]	5
5	[0.0422, 0.0422]	[0.0422, 0.5000]	4

Table 5: Input data for 20 bank branches.

DMU$_j$	x_{1j}^L	x_{1j}^U	x_{2j}^L	x_{2j}^U	x_{3j}^L	x_{3j}^U
1	5007.37	9613.37	36.29	36.86	87243	87243
2	2926.81	5961.55	18.8	20.16	9945	12120
3	8732.7	17752.5	25.74	27.17	47575	50013
4	945.93	1966.39	20.81	22.54	19292	19753
5	8487.07	17521.66	14.16	14.8	3428	3911
6	13759.35	27359.36	19.46	19.46	13929	15657
7	587.69	1205.47	27.29	27.48	27827	29005
8	4646.39	9559.61	24.52	25.07	9070	9983
9	1554.29	3427.89	20.47	21.59	412036	413902
10	17528.31	36297.54	14.84	15.05	8638	10229
11	2444.34	4955.78	20.42	20.54	500	937
12	7303.27	14178.11	22.87	23.19	16148	21353
13	9852.15	19742.89	18.47	21.83	17163	17290
14	4540.75	9312.24	22.83	23.96	17918	17964
15	3039.58	6304.01	39.32	39.86	51582	55136
16	6585.81	13453.58	25.57	26.52	20975	23992
17	4209.18	8603.79	27.59	27.95	41960	43103
18	1015.52	2037.82	13.63	13.93	18641	19354
19	5800.38	11875.39	27.12	27.26	19500	19569
20	1445.68	2922.15	28.96	28.96	31700	32061

simistic efficiency intervals for these DMUs based on models (2.1)-(2.4), (3.15) and (3.16). The optimistic assessment of the bank branches revealed that 11 DMUs were in the best position and obtained the maximum efficiency score of 100%. These 11 DMUs are classified as optimistic efficient DMUs with the best performance (best productive DMUs are DEA-efficient, otherwise they are DEA-non-efficient). However, the pessimistic assessment of the bank branches showed that 8 DMUs were in the worst position and obtained the smallest efficiency score. Accordingly, these 8 DMUs are classified as pessimistic inefficient DMUs with the worst performance (the worst productive DMUs are DEA-inefficient, otherwise they are DEA-non-inefficient). These 8 DMUs are candidates for bankruptcy. Investment risk assessment is considered to be an important issue for financial institutions or investors investing in businesses or bank branches. Thus, the financial institutions or individual investors should certainly assess the performance of bank branches in the banking industry before investment. Now, using the set of the upper bounds of the pessimistic efficiency interval and the lower bounds of the optimistic interval, the lower bound of the overall efficiency interval of DMUs can be determined. The calculated lower bound is $\beta = \theta_{min}^{L*}/\phi_{max}^{U*} = 0.1841/3.9815 = 0.0462$. The last column of Table 7 shows DMUs ranks in terms of the overall efficiency interval. It is noteworthy that the DMUs 7, 9, and 10 are optimistic efficient under the optimistic assessment used in model (2.1). However, they are pessimistic inefficient under the pessimistic assessment used in model (2.3). Therefore, they are not classified as

Table 6: Output data for 20 bank branches.

DMU	y_{1j}^L	y_{1j}^U	y_{2j}^L	y_{2j}^U	y_{3j}^L	y_{3j}^U	y_{4j}^L	y_{4j}^U	y_{5j}^L	y_{5j}^U
1	2696995	3126798	263643	382545	1675519	1853365	108634.76	125740.28	965.97	6957.33
2	340377	440355	95978	117659	377309	390203	32396.65	37836.56	304.67	749.4
3	1027546	1061260	37911	503089	1233548	1822028	96842.33	108080.01	2285.03	3174
4	1145235	1213541	229646	268460	468520	542101	32362.8	39273.37	207.98	510.93
5	390902	395241	4924	12136	129751	142873	12662.71	14165.44	63.32	92.3
6	988115	1087392	74133	111324	507502	574355	53591.3	72257.28	480.16	869.52
7	144906	165818	180530	180617	288513	323721	40507.97	45847.48	176.58	370.81
8	408163	416416	405396	486431	1044221	1071812	56260.09	73948.09	4654.71	5882.53
9	335070	410427	337971	449336	1584722	1802942	176436.81	189006.12	560.26	2506.67
10	700842	768593	14378	15192	2290745	2573512	662725.21	791463.08	58.89	86.86
11	641680	696338	114183	241081	1579961	2285079	17527.58	20773.91	1070.81	2283.08
12	453170	481943	27196	29553	245726	275717	35757.83	42790.14	375.07	559.85
13	553167	574989	21298	23043	425886	431815	45652.24	50255.75	438.43	836.82
14	309670	342598	20168	26172	124188	126930	8143.79	11948.04	936.62	1468.45
15	286149	317186	149183	270708	787959	810088	106798.63	111962.3	1203.79	4335.24
16	321435	347848	66169	80453	360880	379488	89971.47	165524.22	200.36	399.8
17	618105	835839	244250	404579	9136507	9136507	33036.79	41826.51	2781.24	4555.42
18	248125	320974	3063	6330	26687	29173	9525.6	10877.78	240.04	274.7
19	640890	679916	490508	684372	2946797	3985900	66097.16	95329.87	961.56	1914.25
20	119948	120208	14943	17495	297674	308012	21991.53	27934.19	282.73	471.22

Table 7: Interval efficiencies of the 20 bank branches.

DMU	Optimistic efficiency int. $[\theta_j^{L*}, \theta_j^{U*}]$	Pessimistic efficiency int. $[\phi_j^{L*}, \phi_j^{U*}]$	Pessimistic efficiency int. $[\Psi_j^{L*}, \Psi_j^{U*}]$	Overall efficiency int. $[\Psi_j^{L*}, \theta_j^{U*}]$	Overall efficiency interval according to [8] DEA models	Rank
1	[0.8561, 1.0000]	[2.2649, 3.3214]	[0.1049, 0.1537]	[0.1049, 1.0000]	[0.0024, 1.0000]	1
2	[0.3603, 0.5105]	[1.8516, 2.7034]	[0.0855, 0.1249]	[0.0855, 0.5105]	[0.0055, 1.0000]	3
3	[0.5226, 0.9802]	[1.7464, 3.9815]	[0.0807, 0.1840]	[0.0807, 0.9802]	[0.0016, 1.0000]	6
4	[0.8891, 1.0000]	[1.4753, 2.4900]	[0.0682, 0.1152]	[0.0682, 1.0000]	[0.0023, 1.0000]	9
5	[0.6331, 0.6806]	[1.0000, 1.1809]	[0.0462, 0.0546]	[0.0462, 0.6806]	[0.0009, 0.7690]	16
6	[0.8835, 1.0000]	[1.6075, 3.2642]	[0.0743, 0.1508]	[0.0743, 1.0000]	[0.0067, 1.0000]	8
7	[0.6576, 1.0000]	[1.0000, 1.2160]	[0.0462, 0.0562]	[0.0462, 1.0000]	[0.0013, 1.0000]	13
8	[0.8320, 1.0000]	[1.6572, 2.8677]	[0.0767, 0.1326]	[0.0767, 1.0000]	[0.0229, 1.0000]	7
9	[0.7396, 1.0000]	[1.0000, 1.0761]	[0.0462, 0.0497]	[0.0462, 1.0000]	[0.0003, 1.0000]	14
10	[0.8839, 1.0000]	[1.0000, 1.3898]	[0.0462, 0.0643]	[0.0462, 1.0000]	[0.0010, 1.0000]	15
11	[0.8984, 1.0000]	[1.7788, 2.7646]	[0.0823, 0.1279]	[0.0823, 1.0000]	[0.0160, 1.0000]	4
12	[0.3288, 0.3991]	[1.2019, 1.6901]	[0.0555, 0.0781]	[0.0555, 0.3991]	[0.0025, 0.4999]	12
13	[0.4438, 0.5366]	[1.2225, 2.2980]	[0.0565, 0.1062]	[0.0565, 0.5366]	[0.0026, 0.7058]	11
14	[0.2690, 0.3586]	[1.0000, 1.1341]	[0.0462, 0.0524]	[0.0462, 0.3586]	[0.0015, 0.7285]	19
15	[0.4067, 1.0000]	[1.3402, 1.8771]	[0.0619, 0.0867]	[0.0619, 1.0000]	[0.0031, 1.0000]	10
16	[0.2227, 0.5530]	[1.0000, 1.5968]	[0.0462, 0.0738]	[0.0462, 0.5530]	[0.0018, 1.0000]	17
17	[0.9871, 1.0000]	[1.7289, 2.2712]	[0.0807, 0.1058]	[0.0807, 1.0000]	[0.0084, 1.0000]	5
18	[0.2651, 0.3534]	[1.0000, 1.1172]	[0.0462, 0.0516]	[0.0462, 0.3534]	[0.0003, 0.9548]	20
19	[0.7934, 1.0000]	[2.1424, 3.8900]	[0.0992, 0.1801]	[0.0992, 1.0000]	[0.0108, 1.0000]	2
20	[0.1841, 0.4003]	[1.0000, 1.0022]	[0.0462, 0.0463]	[0.0462, 0.4003]	[0.0010, 0.9789]	18

the best DMUs. The ranking results confirm that they are not among the first 11 DMUs. Thus, the results of the present study provide more useful information for the managers of bank branches. This is why the application of DEA in the presence of imprecise data is necessary.

It should be noted that only 80 LPs should be solved to obtain the overall efficiency interval of 20 branches using the approach proposed in this paper. To determine β, we need to be solved 20 LPs to calculate the lower bound of the optimistic efficiency intervals of the 20 branches (using model (2.2)). On the other hand, 20 LPs should be solved to calculate the upper bound of the pessimistic efficiency intervals of the 20 branches (using model (2.4)). Of the other 40 LPs, 20 LPs are used to calculate the upper bound of the optimistic efficiency intervals of the 20 bank branches using model (2.1) (constituting the upper bound of the overall efficiency intervals). To calculate the lower bound of the adjusted pessimistic efficiency intervals of the 20 bank branches (constituting the lower bound of the overall efficiency intervals), 20 LPs are solved using model (3.15). However, the DEA models proposed by Entani et al. [8] need to solve 300 LPs to obtain the overall efficiency interval of 20 branches (See Table 7). Of the 300 LPs, 20 LPs are used to calculate the upper bound of the overall efficiency intervals of the 20 branches using model (3.7). Of this, 14 DMUs are evaluated as DEA-efficient and 20×14 LPs are used to calculate the lower bound of the overall efficiency intervals of the 20 branches using the sub-optimization problem (3.11). More importantly, the lower-bound DEA model of Entani et al. [8] identifies only one DMU, i.e. DMU_9, as the pessimistic inefficient DMU with the minimum lower bound of the overall efficiency interval among the 20 DMUs. However, it is unable to detect the other seven pessimistic inefficient DMUs. The rounding error of DMU_{18} is identified as a pessimistic inefficient DMU while the pessimistic efficiencies of DMU_9 and DMU_{18} are 0.000296442840994248 and 0.00030171562671364, respectively. This example confirms the applicability and discriminating power of the approach proposed in this paper.

In addition to the above advantages, the other advantages of the proposed approach are as follows compared with Entani et al.'s [8] approach:

- In our proposed approach, the upper bounds of the overall efficiency intervals are measured according to the same constraints for different DMUs. However, the method proposed by Entani et al. [8] measures the upper bounds of the overall efficiency intervals under different constraints leading to incomparable efficiencies for different DMUs.
- One of the important features of measuring the pessimistic efficiency of DMUs is identification of pessimistic inefficient DMUs which, from the pessimistic point of view, have the role of the worst units among other DMUs and delineate the inefficiency frontier. Thus, the evaluators may know which DMU is pessimistic inefficient and which is not. The lower-bound DEA model of Entani et al. [8] is an exception. Their model identifies only one DMU with the minimum lower bound of the efficiency interval. Accordingly, it is unable to correctly identify all pessimistic inefficient DMUs. Basically, the lower-bound DEA model of Entani et al. [8] is unable to determine the inefficiency frontier. Therefore, much evaluation information is lost. On the other hand, each LP model in the sub-optimization problem (3.11) is subject to only two linear constraints. There is only one non-zero input and output weight and the weights of other inputs and outputs are zero. In other words, only one input and one output of DMU_o are used to calculate the lower bound of the overall efficiency interval and the other data are ignored. Obviously, this is irrational and unacceptable.
- The calculations in our approach are much less than those of Entani et al.'s [8] approach, in particular, when the number of DMUs under evaluation is high. The proposed approach reduces the computational effort. In our proposed approach, only $4n$ LPs need to be solved to calculate the overall efficiency intervals of n DMUs. However, the approach proposed by Entani et al. [8] needs to solve $(e_1 + 1)n$ LPs, where e_1 represents the number of optimistic efficient DMUs. Of $(e_1 + 1)n$ LPs, n LPs are solved to calculate the upper bound of the overall efficiency intervals (using model (3.7)) while $e_1 n$ LPs are solved to obtain the lower bound of the overall efficiency intervals (using the sub-optimization problem (3.11)).

5 Conclusions

The assessment of the efficiency of DMUs is a complex but important decision-making issue that involves multiple quantitative and qualitative selection criteria. The present article proposed a new approach to deal with interval data, ordinal preference data, and their mixtures in DEA. The proposed method allows the most efficient use of the conventional DEA with imprecise data. The proposed approach measures the efficiency of each DMU from both optimistic and pessimistic perspectives leading to upper and lower bounds for efficiency called the overall efficiency interval. The overall efficiency interval calculates the imprecise efficiency interval for each DMU. Using the overall efficiency interval, we can further prioritize DMUs performances. In comparison with the overall efficiency interval formed by Entani et al. [8], the overall efficiency interval formed by our approach employs fixed and unified production frontiers (i.e., the efficient and inefficient frontiers) as a benchmark for measuring the efficiency of all DMUs. This leads to a more rational, reliable, and applicable overall efficiency interval. The overall efficiency interval not only describes the true situation in more detail, but reduces the pressure on all evaluated DMUs and evaluators psychologically. Two numerical examples were examined to demonstrate the simplicity and utility of the proposed approach in measuring the efficiency of DMUs.

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