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Research Article

Solving Fully Fuzzy Dual Matrix System With Optimization Problem

F. Babakordi *†, M. Adabitabar Firozja ‡

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Abstract

In this paper, the fuzzy dual matrix system as $\widetilde{A}\widetilde{X}+\widetilde{B}=\widetilde{C}\widetilde{X}+\widetilde{D}$ in which $\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D},\widetilde{X}$ are LR fuzzy matrices is studied. At first we solve 1-cut system in order to find the core of LR fuzzy solution; then to obtain the spreads of the LR fuzzy solution, we discuss in several cases. The spreads are obtained by using multiplication, quasi norm and minimization problem with a special objective function. We prove some theorems and we suggest conditions in which the fuzzy dual matrix system of $\widetilde{A}\widetilde{X}+\widetilde{B}=\widetilde{C}\widetilde{X}+\widetilde{D}$ and the fuzzy matrix system of $(\widetilde{A}\ominus_H \widetilde{C})\widetilde{X}=(\widetilde{D}\ominus_H \widetilde{B})$ have the same LR fuzzy solution and we discuss about the conditions where LR fuzzy dual matrix system has crisp solution. Finally numerical examples are solved to illustrate the ability, accuracy and capability of the proposed method.

Keywords : Fuzzy number; Optimization problem; Fuzzy dual matrix system; Linear programming; Hukuhara difference.

1 Introduction

Linear systems have important applications in many branches of science and engineering. In many applications, at least some of the parameters of the system are represented by fuzzy rather than crisp numbers. So, it is immensely important to develop a numerical procedure that would appropriately treat general fuzzy linear systems and solve them [2]. Friedman et al. [4] introduced a general model for solving a fuzzy $n\times n$ linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. They used the parametric form of fuzzy numbers and replaced the original fuzzy $n\times n$ linear system by a crisp $2n\times 2n$ linear system and studied duality in fuzzy linear systems $Ax=Bx+y$ where A, B are real $n\times n$ matrices, the unknown vector x is vector $n\times 1$ of fuzzy numbers and the constant y is vector $n\times 1$ of arbitrary fuzzy numbers, in [5]. Buckley and Qu in the continuous of their works in [5, 6], proposed different solutions for fully fuzzy linear system(FFLS) in [8]. Based on their work, Muzzioli and Reynaerts have studied FFLS in a dual form [9]. Their approach to solving a $n\times n$ FFLS leads to solving a $2^{n(n+1)}$ crisp system. Clearly, for a large n, obtaining such a solution is not easy work and such an approach has a big error. Then numerical method applied to solve fuzzy linear systems [10, 11, 12, 13, 14, 15]. Recently fuzzy systems have been studied by many authors [16, 17, 18].

*Corresponding author. babakordif@yahoo.com, Tel:+98(911)1788478.
†Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.
‡Department of Mathematics, Science and Research Branch, Islamic Azad University, Ghaemshahr, Iran.

In [19, 20], the solution of fuzzy linear system (FLS) was investigated based on a 1-level expansion. In this paper, we are finding the solution of a fully fuzzy dual matrix system of the form $\widetilde{A}\widetilde{X}+\widetilde{B}=\widetilde{C}\widetilde{X}+\widetilde{D}$, using multiplication LR fuzzy matrix, quasi norm, minimization problem. The paper is organized as follows, In Section 2, we recall some fundamental results on fuzzy numbers. The proposed model for solving the system $\widetilde{A}\widetilde{X}+\widetilde{B}=\widetilde{C}\widetilde{X}+\widetilde{D}$ are discussed in Section 3. Section 4 introduces three examples for illustration of method and conclusions are drawn in Section 5.

2 Preliminaries

In this section, some basic definitions which are required in the following are defined. These notations can be found in [1, 3, 23, 25].

Definition 2.1 *[21] A fuzzy number is a fuzzy set like $\widetilde{u}(x):[a,d]\longrightarrow[0,1]$ which satisfies*

1. *$\widetilde{u}$ is upper semi-continuous,*

2. *$\widetilde{u}(x)=0$ outside some interval $[a,d]$,*

3. *There are real numbers b,c such that $a\le b\le c\le d$ and,*

$\widetilde{u}(x)$ *is monotonic increasing on* $[a,b]$,

$\widetilde{u}(x)$ *is monotonic decreasing on* $[c,d]$,

$\widetilde{u}(x)=1, b\le x\le c.$

The membership function $\widetilde{u}(x)$ can be expressed as:

$$\widetilde{u}(x)=\begin{cases} f^L_{\widetilde{u}}(x) & a\le x\le b,\\ 1 & b\le x\le c,\\ f^R_{\widetilde{u}}(x) & c\le x\le d,\\ 0 & otherwise,\end{cases}$$

where $f^L_{\widetilde{u}}(x):[a,b]\to[0,1]$ and $f^R_{\widetilde{u}}(x):[c,d]\to[0,1]$ are left and right membership functions of fuzzy number $\widetilde{u}$ respectively . (see [22])

Definition 2.2 *[26] A fuzzy number $\widetilde{u}$ is said to be a LR fuzzy number if*

$$\widetilde{u}(x)=\begin{cases} L(\frac{a-x}{\alpha}), & x\le a,\alpha>0\\ R(\frac{x-a}{\beta}), & x\ge a,\beta>0\end{cases}$$

where a is the core of $\widetilde{u}$, α and β are left and right spreads, respectively, and the function $L(.)$, which is called left shape function, satisfies:

$L(x)=L(-x)$

$L(0)=1$ *and* $L(1)=0$

$L(x)$ *is non-increasing on* $[0,\infty)$

The definition of right shape function $R(.)$ is similar to that of $L(.)$. The core, left and right spreads, and the shape function of LR fuzzy number $\widetilde{u}$ are symbolically shown as: $\widetilde{u}=(a,\alpha,\beta)$. We say that $\widetilde{u}=(a,\alpha,\beta)$ is positive (negative)if $a-\alpha\ge 0(a+\beta\le 0)$.

Definition 2.3 *[6] Let $\widetilde{u}=(a,\alpha,\beta)$ and $\widetilde{v}=(b,\gamma,\delta)$, we have:*

Addition:

$$\widetilde{u}+\widetilde{v}=(a,\alpha,\beta)+(b,\gamma,\delta)=(a+b,\alpha+\gamma,\beta+\delta)$$

Multiplication:

$$\widetilde{u}.\widetilde{v}\approx\begin{cases}(a.b,a.\gamma+b.\alpha,a.\delta+b.\beta), & \widetilde{u}>0,\widetilde{v}>0\\ (a.b,b.\alpha-a.\delta,b.\beta-a.\gamma), & \widetilde{u}<0,\widetilde{v}>0\\ (a.b,a.\gamma-b.\beta,a.\delta-b.\alpha), & \widetilde{u}>0,\widetilde{v}<0\\ (a.b,-a.\delta-b.\beta,-a.\gamma-b.\alpha), & \widetilde{u}<0,\widetilde{v}<0\\ (a.b,a.\gamma-b.\beta,a.\delta+b.\beta), & \widetilde{u}>0,\ 0\in\widetilde{v}\\ (a.b,b.\alpha-a.\delta,-a.\gamma-b.\alpha), & \widetilde{u}<0,\ 0\in\widetilde{v}\\ (a.b,b.\alpha-a.\delta,a.\delta+b.\beta), & 0\in\widetilde{u},\ \widetilde{v}>0\\ (a.b,a.\gamma-b.\beta,-a.\gamma-b.\alpha), & 0\in\widetilde{u},\ \widetilde{v}<0\\ (a.b,a.\gamma-b.\beta,a.\delta+b.\beta), & 0\in\widetilde{u},0\in\widetilde{v}\end{cases}$$

Scalar multiplication:

$$\lambda\bigotimes\widetilde{u}=\lambda\bigotimes(a,\alpha,\beta)=\begin{cases}(\lambda a,\lambda\alpha,\lambda\beta), & \lambda>0\\ (\lambda a,-\lambda\beta,-\lambda\alpha), & \lambda<0\end{cases}$$

Definition 2.4 *Let $\widetilde{A} = [(a_{ij}, \alpha_{ij}, \beta_{ij})]$ and $\widetilde{B} = [(b_{ij}, \delta_{ij}, \gamma_{ij})]$ where $1 \leq i, j \leq n$, then Hukuhara difference between $\widetilde{A}$ and $\widetilde{B}$ define as follows:*

$$\widetilde{A} \ominus_H \widetilde{B} = [(a_{ij} - b_{ij}, \alpha_{ij} - \delta_{ij}, \beta_{ij} - \gamma_{ij})]$$

Definition 2.5 *The fuzzy linear system*

$$\begin{bmatrix} (a_{11}, \alpha_{a_{11}}, \beta_{a_{11}}) & \dots & (a_{1n}, \alpha_{a_{1n}}, \beta_{b_{1n}}) \\ \vdots & \vdots & \vdots \\ (a_{n1}, \alpha_{a_{n1}}, \beta_{a_{n1}}) & \dots & (a_{nn}, \alpha_{a_{nn}}, \beta_{a_{nn}}) \end{bmatrix} \begin{bmatrix} (x_{11}, \alpha_{11}, \beta_{11}) & \dots & (x_{1n}, \alpha_{1n}, \beta_{1n}) \\ \vdots & \vdots & \vdots \\ (x_{n1}, \alpha_{n1}, \beta_{n1}) & \dots & (x_{nn}, \alpha_{nn}, \beta_{nn}) \end{bmatrix} + \begin{bmatrix} (b_{11}, \alpha_{b_{11}}, \beta_{b_{11}}) & \dots & (b_{1n}, \alpha_{b_{1n}}, \beta_{b_{1n}}) \\ \vdots & \vdots & \vdots \\ (b_{n1}, \alpha_{b_{n1}}, \beta_{b_{n1}}) & \dots & (b_{nn}, \alpha_{b_{nn}}, \beta_{b_{nn}}) \end{bmatrix}$$

$$= \begin{bmatrix} (c_{11}, \alpha_{c_{11}}, \beta_{c_{11}}) & \dots & (c_{1n}, \alpha_{c_{1n}}, \beta_{c_{1n}}) \\ \vdots & \vdots & \vdots \\ (c_{n1}, \alpha_{c_{n1}}, \beta_{c_{n1}}) & \dots & (c_{nn}, \alpha_{c_{nn}}, \beta_{c_{nn}}) \end{bmatrix} \begin{bmatrix} (x_{11}, \alpha_{11}, \beta_{11}) & \dots & (x_{1n}, \alpha_{1n}, \beta_{1n}) \\ \vdots & \vdots & \vdots \\ (x_{n1}, \alpha_{n1}, \beta_{n1}) & \dots & (x_{nn}, \alpha_{nn}, \beta_{nn}) \end{bmatrix} + \begin{bmatrix} (d_{11}, \alpha_{d_{11}}, \beta_{d_{11}}) & \dots & (d_{1n}, \alpha_{d_{1n}}, \beta_{d_{1n}}) \\ \vdots & \vdots & \vdots \\ (d_{n1}, \alpha_{d_{n1}}, \beta_{d_{n1}}) & \dots & (d_{nn}, \alpha_{d_{nn}}, \beta_{d_{nn}}) \end{bmatrix} \quad (2.1)$$

is called a fuzzy dual matrix system in which $\widetilde{A} = [(a_{ij}, \alpha_{a_{ij}}, \beta_{a_{ij}})]$ and $\widetilde{X} = [(x_{ij}, \alpha_{ij}, \beta_{ij})]$, $\widetilde{B} = [(b_{ij}, \alpha_{b_{ij}}, \beta_{b_{ij}})]$, $\widetilde{C} = [(c_{ij}, \alpha_{c_{ij}}, \beta_{c_{ij}})]$, $\widetilde{D} = [(d_{ij}, \alpha_{d_{ij}}, \beta_{d_{ij}})]$, are LR fuzzy matrices and denoted by $\widetilde{A}\widetilde{X}+\widetilde{B} = \widetilde{C}\widetilde{X}+\widetilde{D}$, if $A = [a_{i,j}]_{n\times n}, C = [c_{i,j}]_{n\times n}$, we suppose that $A - C$ is nonsingular.

3 The solution of Fuzzy Dual Matrix System

Definition 3.1 *If $A = [a_{ij}]_{n\times n}$, $i, j = 1, 2, ..., n$ be a crisp matrix then we define quasi norm A as :*

$$\|A\| = max_{i,j}\{| a_{ij} |\}$$

Definition 3.2 *Matrix $\widetilde{A}$ is called a LR fuzzy matrix if each element of $\widetilde{A}$ is a LR fuzzy number and showing as $\widetilde{A} = [(a_{ij}, \alpha_{ij}, \beta_{ij})]$.*

Definition 3.3 *We say that $\widetilde{u} = (a, \alpha, \beta)$ is fuzzy zero if $a = 0, a - \alpha \leq 0, a + \beta \geq 0$.*

Now we solve the system of fuzzy Dual matrix (2.1). First, by solving the 1-cut system, we have:

$$\sum_{k=1}^{n} a_{ik}x_{kj} + b_{ij} = \sum_{k=1}^{n} c_{1k}x_{kj} + d_{ij} \quad (3.2)$$

$$i, j = 1, \dots, n.$$

from 3.2, we obtain x_{kj}, $k, j = 1, \dots, n$. Now we consider three situations to find the spreads. If $x_{kj} > 0$, then we consider $\tilde{x}_{kj} = (x_{kj}, \alpha_{kj}, \beta_{kj}) > 0$ hence, $x_{kj} - \alpha_{kj} \geq 0$. If $x_{kj} < 0$, then we consider $\tilde{x}_{kj} = (x_{kj}, \alpha_{kj}, \beta_{kj}) < 0$ hence, $x_{kj} + \beta_{kj} \leq 0$. If $x_{kj} = 0$, then we consider $\tilde{x}_{kj} = (x_{kj}, \alpha_{kj}, \beta_{kj}) \approx 0$ hence, $x_{kj} - \alpha_{kj} \leq 0$ and $x_{kj} + \beta_{kj} \geq 0$. Now, with above conditions we solve the following system for $i, j = 1, \dots, n$.

$$\begin{gathered} \textstyle\sum_{\tilde{a}_{ik}>0, x_{kj}>0} \tilde{a}_{ik}\tilde{x}_{kj} + \sum_{\tilde{a}_{ik}<0, x_{kj}<0} \tilde{a}_{ik}\tilde{x}_{kj} \\ + \textstyle\sum_{\tilde{a}_{ik}>0, x_{kj}<0} \tilde{a}_{ik}\tilde{x}_{kj} + \sum_{\tilde{a}_{ik}<0, x_{kj}>0} \tilde{a}_{ik}\tilde{x}_{kj} \\ + \textstyle\sum_{\tilde{a}_{ik}>0, x_{kj}=0} \tilde{a}_{ik}\tilde{x}_{kj} + \sum_{\tilde{a}_{ik}<0, x_{kj}=0} \tilde{a}_{ik}\tilde{x}_{kj} \\ + \textstyle\sum_{0\in\tilde{a}_{ik}, x_{kj}>0} \tilde{a}_{ik}\tilde{x}_{kj} + \sum_{0\in\tilde{a}_{ik}, x_{kj}<0} \tilde{a}_{ik}\tilde{x}_{kj} \\ + \textstyle\sum_{0\in\tilde{a}_{ik}, x_{kj}=0} \tilde{a}_{ik}\tilde{x}_{kj} + \tilde{b}_{ij} \\ = \\ \textstyle\sum_{\tilde{c}_{1k}>0, x_{kj}>0} \tilde{c}_{1k}\tilde{x}_{kj} + \sum_{\tilde{c}_{1k}<0, x_{kj}<0} \tilde{c}_{1k}\tilde{x}_{kj} \\ + \textstyle\sum_{\tilde{c}_{1k}>0, x_{kj}<0} \tilde{c}_{1k}\tilde{x}_{kj} + \sum_{\tilde{c}_{1k}<0, x_{kj}>0} \tilde{c}_{1k}\tilde{x}_{kj} \\ + \textstyle\sum_{\tilde{c}_{1k}>0, x_{kj}=0} \tilde{c}_{1k}\tilde{x}_{kj} + \sum_{\tilde{c}_{1k}<0, x_{kj}=0} \tilde{c}_{1k}\tilde{x}_{kj} \\ + \textstyle\sum_{0\in\tilde{c}_{1k}, x_{kj}>0} \tilde{c}_{1k}\tilde{x}_{kj} + \sum_{0\in\tilde{c}_{1k}, x_{kj}<0} \tilde{c}_{1k}\tilde{x}_{kj} \\ + \textstyle\sum_{0\in\tilde{c}_{1k}, x_{kj}=0} \tilde{c}_{1k}\tilde{x}_{kj} + \tilde{d}_{ij} \end{gathered} \quad (3.3)$$

And rewrite above system regarding to Definition 2.3 as follows.

If by solving system (3.2) for all $k, j = 1, \dots, n$, $x_{kj} > 0$, we find positive fuzzy solution. By using Definition 2.3 in system (3.3), we have :

$$\begin{gathered} \sum_{\tilde{a}_{ik}>0} (a_{ik}\alpha_{kj} + x_{kj}\alpha_{a_{ik}}) + \\ \sum_{\tilde{a}_{ik}<0 or 0\in\tilde{a}_{ik}} (\alpha_{a_{ik}}x_{kj} - a_{ik}\beta_{kj}) + \alpha_{b_{ij}} = \\ \sum_{\tilde{c}_{ik}>0} (c_{ik}\alpha_{kj} + x_{kj}\alpha_{c_{ik}}) + \\ \sum_{\tilde{c}_{ik}<0 or 0\in\tilde{c}_{ik}} (\alpha_{c_{ik}}x_{kj} - c_{ik}\beta_{kj}) \\ + \alpha_{d_{ij}}, \end{gathered} \quad (3.4)$$

$$\sum_{0\in\tilde{a}or\tilde{a}_{ik}>0}(a_{ik}\beta_{kj}+x_{kj}\beta_{a_{ik}})+ \sum_{\tilde{a}_{ik}<0}(\beta_{a_{ik}}x_{kj}-a_{ik}\alpha_{kj}) +\beta_{b_{ij}} = \sum_{\tilde{c}_{ik}>0or0\in\tilde{c}_{ik}}(c_{ik}\beta_{kj}+x_{kj}\beta_{c_{ik}})+ \sum_{\tilde{c}_{ik}<0}(\beta_{c_{ik}}x_{kj}-c_{ik}\alpha_{kj}) +\beta_{d_{ij}} \quad i,j=1,\ldots,n. \tag{3.5}$$

By using quasi norm, we solve the following minimization problem to find spreads:

$$Min\{\| \sum_{\tilde{a}_{ik}>0}(a_{ik}\alpha_{kj}+x_{kj}\alpha_{a_{ik}})+ \sum_{\tilde{a}_{ik}<0or0\in\tilde{a}_{ik}}(\alpha_{a_{ik}}x_{kj}-a_{ik}\beta_{kj})+\alpha_{b_{ij}} - \sum_{\tilde{c}_{ik}>0}(c_{ik}\alpha_{kj}+x_{kj}\alpha_{c_{ik}}) - \sum_{\tilde{c}_{ik}<0or0\in\tilde{c}_{ik}}(\alpha_{c_{ik}}x_{kj}-c_{ik}\beta_{kj})-\alpha_{d_{ij}} \| + \| \sum_{0\in\tilde{a}or\tilde{a}_{ik}>0}(a_{ik}\beta_{kj}+x_{kj}\beta_{a_{ik}}) + \sum_{\tilde{a}_{ik}<0}(\beta_{a_{ik}}x_{kj}-a_{ik}\alpha_{kj}) + \beta_{b_{ij}} - \sum_{\tilde{c}_{ik}>0or0\in\tilde{c}_{ik}}(c_{ik}\beta_{kj}+x_{kj}\beta_{c_{ik}}) - \sum_{\tilde{c}_{ik}<0}(\beta_{c_{ik}}x_{kj}-c_{ik}\alpha_{kj})-\beta_{d_{ij}} \|\}$$

$$s.t.\quad 0\le\alpha_{kj}\le x_{kj},\beta_{kj}\ge 0, \quad i,j,k=1,\ldots,n. \tag{3.6}$$

where by using definition 3.1, it can be written as follows:

$$Min\{Max_{i,j} \mid \sum_{\tilde{a}_{ik}>0}(a_{ik}\alpha_{kj}+x_{kj}\alpha_{a_{ik}})+ \sum_{\tilde{a}_{ik}<0or0\in\tilde{a}_{ik}}(\alpha_{a_{ik}}x_{kj}-a_{ik}\beta_{kj})+\alpha_{b_{ij}} - \sum_{\tilde{c}_{ik}>0}(c_{ik}\alpha_{kj}+x_{kj}\alpha_{c_{ik}}) - \sum_{\tilde{c}_{ik}<0or0\in\tilde{c}_{ik}}(\alpha_{a_{ik}}x_{kj}-c_{ik}\beta_{kj})-\alpha_{d_{ij}} \mid + Max_{i,j} \mid \sum_{0\in\tilde{a}or\tilde{a}_{ik}>0}(a_{ik}\beta_{kj}+x_{kj}\beta_{a_{ik}}) + \sum_{\tilde{a}_{ik}<0}(\beta_{a_{ik}}x_{kj}-a_{ik}\alpha_{kj})+\beta_{b_{ij}} - \sum_{\tilde{c}_{ik}>0or0\in\tilde{c}_{ik}}(c_{ik}\beta_{kj}+x_{kj}\beta_{c_{ik}}) - \sum_{\tilde{c}_{ik}<0}(\beta_{c_{ik}}x_{kj}-c_{ik}\alpha_{kj})-\beta_{d_{ij}} \mid\}$$

$$s.t.\quad 0\le\alpha_{kj}\le x_{kj},\beta_{kj}\ge 0, \quad i,j,k=1,\ldots,n. \tag{3.7}$$

By using

$$z_1 = Max_{i,j} \mid \sum_{\tilde{a}_{ik}>0}(a_{ik}\alpha_{kj}+x_{kj}\alpha_{a_{ik}})+ \sum_{\tilde{a}_{ik}<0or0\in\tilde{a}_{ik}}(\alpha_{a_{ik}}x_{kj}-a_{ik}\beta_{kj})+\alpha_{b_{ij}} - \sum_{\tilde{c}_{ik}>0}(c_{ik}\alpha_{kj}+x_{kj}\alpha_{c_{ik}}) - \sum_{\tilde{c}_{ik}<0or0\in\tilde{c}_{ik}}(\alpha_{c_{ik}}x_{kj}-c_{ik}\beta_{kj})-\alpha_{d_{ij}} \mid$$

$$z_2 = Max_{i,j} \mid \sum_{0\in\tilde{a}or\tilde{a}_{ik}>0}(a_{ik}\beta_{kj}+x_{kj}\beta_{a_{ik}}) + \sum_{\tilde{a}_{ik}<0}(\beta_{a_{ik}}x_{kj}-a_{ik}\alpha_{kj})+\beta_{b_{ij}} - \sum_{\tilde{c}_{ik}>0or0\in\tilde{c}_{ik}}(c_{ik}\beta_{kj}+x_{kj}\beta_{c_{ik}}) - \sum_{\tilde{c}_{ik}<0}(\beta_{c_{ik}}x_{kj}-c_{ik}\alpha_{kj})-\beta_{d_{ij}} \mid$$

we will have

$$Min \quad z_1+z_2$$

$$s.t.\quad -z_1 \le \sum_{\tilde{a}_{ik}>0}(a_{ik}\alpha_{kj}+x_{kj}\alpha_{a_{ik}})+ \sum_{\tilde{a}_{ik}<0or0\in\tilde{a}_{ik}}(\alpha_{a_{ik}}x_{kj}-a_{ik}\beta_{kj})+\alpha_{b_{ij}} - \sum_{\tilde{c}_{ik}>0}(c_{ik}\alpha_{kj}+x_{kj}\alpha_{c_{ik}})$$

$$- \sum_{\tilde{c}_{ik}<0 or 0\in\tilde{c}_{ik}} (\alpha_{c_{ik}} x_{kj} - c_{ik}\beta_{kj}) - \alpha_{d_{ij}} \le z_1$$

$$-z_2 \le \sum_{0\in\tilde{a} or \tilde{a}_{ik}>0} (a_{ik}\beta_{kj} + x_{kj}\beta_{a_{ik}})$$
$$+ \sum_{\tilde{a}_{ik}<0} (\beta_{a_{ik}} x_{kj} - a_{ik}\alpha_{kj}) + \beta_{b_{ij}}$$
$$- \sum_{\tilde{c}_{ik}>0 or 0\in\tilde{c}_{ik}} (c_{ik}\beta_{kj} + x_{kj}\beta_{c_{ik}})$$
$$- \sum_{\tilde{c}_{ik}<0} (\beta_{c_{ik}} x_{kj} - c_{ik}\alpha_{kj}) - \beta_{d_{ij}} \le z_2$$

$$0 \le \alpha_{kj} \le x_{kj}, \beta_{kj} \ge 0, \quad k, j, i = 1, \ldots, n \quad (3.8)$$

If by solving system (3.2) for all $k, j = 1, \ldots, n$, $x_{kj} < 0$, we find negative fuzzy solution. To find the spreads of negative fuzzy solution, we solve following minimization problem:

$$Min \quad z_1 + z_2$$

$$s.t. \quad -z_1 \le \sum_{\tilde{a}_{ik}>0 or 0\in\tilde{a}_{ik}} (a_{ik}\alpha_{kj} - x_{kj}\beta_{a_{ik}})$$
$$+ \sum_{\tilde{a}_{ik}<0} (-a_{ik}\beta_{kj} - \beta_{a_{ik}} x_{kj}) + \alpha_{b_{ij}}$$
$$- \sum_{\tilde{c}_{ik}>0 or 0\in\tilde{c}_{ik}} (c_{ik}\alpha_{kj} - x_{kj}\beta_{c_{ik}})$$
$$- \sum_{\tilde{c}_{ik}<0} (-c_{ik}\beta_{kj} - \beta_{c_{ik}} x_{kj}) - \alpha_{d_{ij}} \le z_1$$
$$-z_2 \le \sum_{\tilde{a}_{ik}>0} (a_{ik}\beta_{kj} - \alpha_{a_{ik}x_{kj}})$$
$$+ \sum_{\tilde{a}_{ik}<0 or 0\in\tilde{a}_{ik}} (-a_{ik}\alpha_{kj} - x_{kj}\alpha_{a_{ik}}) + \beta_{b_{ij}}$$
$$- \sum_{\tilde{c}_{ik}>0} (c_{ik}\beta_{kj} - \alpha_{c_{ik}x_{kj}})$$
$$- \sum_{\tilde{c}_{ik}<0 or 0\in\tilde{c}_{ik}} (-c_{ik}\alpha_{kj} - x_{kj}\alpha_{c_{ik}}) - \beta_{d_{ij}} \le z_2$$

$$\alpha_{kj} \ge 0, 0 \le \beta_{kj} \le -x_{kj}, \quad k, j, i = 1, \ldots, n \quad (3.9)$$

in which

$$z_1 = Max_{i,j} \mid \sum_{\tilde{a}_{ik}>0 or 0\in\tilde{a}_{ik}} (a_{ik}\alpha_{kj} - x_{kj}\beta_{a_{ik}})$$
$$+ \sum_{\tilde{a}_{ik}<0} (-a_{ik}\beta_{kj} - \beta_{a_{ik}} x_{kj}) + \alpha_{b_{ij}}$$
$$- \sum_{\tilde{c}_{ik}>0 or 0\in\tilde{c}_{ik}} (c_{ik}\alpha_{kj} - x_{kj}\beta_{c_{ik}})$$
$$- \sum_{\tilde{c}_{ik}<0} (-c_{ik}\beta_{kj} - \beta_{c_{ik}} x_{kj}) - \alpha_{d_{ij}} \mid$$

$$z_2 = Max_{i,j} \mid \le \sum_{\tilde{a}_{ik}>0} (a_{ik}\beta_{kj} - \alpha_{a_{ik}x_{kj}})$$
$$+ \sum_{\tilde{a}_{ik}<0 or 0\in\tilde{a}_{ik}} (-a_{ik}\alpha_{kj} - x_{kj}\alpha_{a_{ik}}) + \beta_{b_{ij}}$$
$$- \sum_{\tilde{c}_{ik}>0} (c_{ik}\beta_{kj} - \alpha_{c_{ik}x_{kj}})$$
$$- \sum_{\tilde{c}_{ik}<0 or 0\in\tilde{c}_{ik}} (-c_{ik}\alpha_{kj} - x_{kj}\alpha_{c_{ik}}) - \beta_{d_{ij}} \mid$$

If by solving system (3.2) for all $k, j = 1, \ldots, n$, $x_{kj} = 0$, we find quasi zero fuzzy solution. To find the spreads of quasi zero fuzzy solution, we solve the following minimization problem:

$$Min \quad z_1 + z_2$$

$$s.t. \quad -z_1 \le \sum_{\tilde{a}_{ik}>0 or 0\in\tilde{a}_{ik}} (a_{ik}\alpha_{kj})$$
$$+ \sum_{\tilde{a}_{ik}<0} (-a_{ik}\beta_{kj}) + \alpha_{b_{ij}}$$
$$- \sum_{\tilde{c}_{ik}>0 or 0\in\tilde{c}_{ik}} (c_{ik}\alpha_{kj})$$
$$- \sum_{\tilde{c}_{ik}<0} (-c_{ik}\beta_{kj}) - \alpha_{d_{ij}} \le z_1$$
$$-z_2 \le \sum_{\tilde{a}_{ik}>0 or 0\in\tilde{a}_{ik}} (a_{ik}\beta_{kj})$$
$$+ \sum_{\tilde{a}_{ik}<0} (-a_{ik}\alpha_{kj}) + \beta_{b_{ij}}$$
$$- \sum_{\tilde{c}_{ik}>0 or 0\in\tilde{c}_{ik}} (c_{ik}\beta_{kj})$$
$$- \sum_{\tilde{c}_{ik}<0} (-c_{ik}\alpha_{kj}) - \beta_{d_{ij}} \le z_2$$

$$\alpha_{kj} \ge 0, \beta_{kj} \ge 0, \quad k, j, i = 1, \ldots, n \quad (3.10)$$

in which

$$z_1 = Max_{i,j} \mid \sum_{\widetilde{a}_{ik}>0 or 0\in\widetilde{a}_{ik}} (a_{ik}\alpha_{kj}) + \sum_{\widetilde{a}_{ik}<0} (-a_{ik}\beta_{kj}) + \alpha_{b_{ij}} - \sum_{\widetilde{c}_{ik}>0 or 0\in\widetilde{c}_{ik}} (c_{ik}\alpha_{kj}) - \sum_{\widetilde{c}_{ik}<0} (-c_{ik}\beta_{kj}) - \alpha_{d_{ij}} \mid$$

$$z_2 = Max_{i,j} \mid \sum_{\widetilde{a}_{ik}>0 or 0\in\widetilde{a}_{ik}} (a_{ik}\beta_{kj}) + sum_{\widetilde{a}_{ik}<0}(-a_{ik}\alpha_{kj}) + \beta_{b_{ij}} - \sum_{\widetilde{c}_{ik}>0 or 0\in\widetilde{c}_{ik}} (c_{ik}\beta_{kj}) - \sum_{\widetilde{c}_{ik}<0} (-c_{ik}\alpha_{kj}) - \beta_{d_{ij}} \mid$$

Total form

If by solving system (3.2), some of the variables are positive, some of the variables are negative, some of the variables are zero, to find spreads we solve the following minimization problem

$$Min \quad z_1 + z_2$$

$$\begin{aligned} s.t. \quad -z_1 \leq & \sum_{\widetilde{a}_{ik}>0, x_{kj}>0} (a_{ik}\alpha_{kj} + x_{kj}\alpha_{a_{ik}}) + \sum_{\widetilde{a}_{ik}<0 or 0\in\widetilde{a}_{ik}, x_{kj}>0} (\alpha_{a_{ik}}x_{kj} - a_{ik}\beta_{kj}) \\ & + \sum_{\widetilde{a}_{ik}>0 or 0\in\widetilde{a}_{ik}, x_{kj}<0} (a_{ik}\alpha_{kj} - x_{kj}\beta_{a_{ik}}) + \sum_{\widetilde{a}_{ik}<0, x_{kj}<0} (-a_{ik}\beta_{kj} - \beta_{a_{ik}}x_{kj}) + \\ & \sum_{\widetilde{a}_{ik}>0 or 0\in\widetilde{a}_{ik}, x_{kj}=0} (a_{ik}\alpha_{kj}) + \sum_{\widetilde{a}_{ik}<0, x_{kj}=0} (-a_{ik}\beta_{kj}) + \alpha_{b_{ij}} \\ & - \sum_{\widetilde{c}_{ik}>0, x_{kj}>0} (c_{ik}\alpha_{kj} + x_{kj}\alpha_{c_{ik}}) - \sum_{\widetilde{c}_{ik}<0 or 0\in\widetilde{c}_{ik}, x_{kj}>0} (\alpha_{c_{ik}}x_{kj} - c_{ik}\beta_{kj}) \\ & - \sum_{\widetilde{c}_{ik}>0 or 0\in\widetilde{c}_{ik}, x_{kj}<0} (c_{ik}\alpha_{kj} - x_{kj}\beta_{c_{ik}}) - \sum_{\widetilde{c}_{ik}<0, x_{kj}<0} (-c_{ik}\beta_{kj} - \beta_{c_{ik}}x_{kj}) \\ & - \sum_{\widetilde{c}_{ik}>0 or 0\in\widetilde{c}_{ik}, x_{kj}=0} (c_{ik}\alpha_{kj}) - \sum_{\widetilde{c}_{ik}<0, x_{kj}=0} (-c_{ik}\beta_{kj}) - \alpha_{d_{ij}} \leq z_1 \end{aligned}$$

$$\begin{aligned} -z_2 \leq & \sum_{0\in\widetilde{a} or \widetilde{a}_{ik}>0, x_{kj}>0} (a_{ik}\beta_{kj} + x_{kj}\beta_{a_{ik}}) + \sum_{\widetilde{a}_{ik}<0, x_{kj}>0} (\beta_{a_{ik}}x_{kj} - a_{ik}\alpha_{kj}) + \\ & \sum_{\widetilde{a}_{ik}>0, x_{kj}<0} (a_{ik}\beta_{kj} - \alpha_{a_{ik}x_{kj}}) + \sum_{\widetilde{a}_{ik}<0 or 0\in\widetilde{a}_{ik}, x_{kj}<0} (-a_{ik}\alpha_{kj} - x_{kj}\alpha_{a_{ik}}) + \\ & \sum_{\widetilde{a}_{ik}>0 or 0\in\widetilde{a}_{ik}, x_{kj}=0} (a_{ik}\beta_{kj}) + \sum_{\widetilde{a}_{ik}<0, x_{kj}=0} (-a_{ik}\alpha_{kj}) + \beta_{b_{ij}} \\ & - \sum_{\widetilde{c}_{ik}>0 or 0\in\widetilde{c}_{ik}, x_{kj}>0} (c_{ik}\beta_{kj} + x_{kj}\beta_{c_{ik}}) - \sum_{\widetilde{c}_{ik}<0, x_{kj}>0} (\beta_{c_{ik}x_{kj}} - c_{ik}\alpha_{kj}) - \\ & - \sum_{\widetilde{c}_{ik}>0, x_{kj}<0} (c_{ik}\beta_{kj} - \alpha_{c_{ik}x_{kj}}) - \sum_{\widetilde{c}_{ik}<0 or 0\in\widetilde{c}_{ik}, x_{kj}<0} (-c_{ik}\alpha_{kj} - x_{kj}\alpha_{c_{ik}}) \\ & - \sum_{\widetilde{c}_{ik}>0 or 0\in\widetilde{c}_{ik}, x_{kj}=0} (c_{ik}\beta_{kj}) - \sum_{\widetilde{c}_{ik}<0, x_{kj}=0} (-c_{ik}\alpha_{kj})\beta_{d_{ij}} \leq z_2 \end{aligned}$$

$$0 \leq \alpha_{kj} \leq x_{kj}, \beta_{kj} \geq 0, \quad if \quad x_{kj} > 0$$

$$\alpha_{kj} \geq 0, 0 \leq \beta_{kj} \leq -x_{kj}, \quad if \quad x_{kj} < 0$$

$$\alpha_{kj} \geq 0, \beta_{kj} \geq 0, \tag{3.11}$$

$$if \quad x_{kj} = 0 \tag{3.12}$$

in which

$$z_1 = Max_{i,j} \mid \sum_{\widetilde{a}_{ik}>0, x_{kj}>0} (a_{ik}\alpha_{kj} + x_{kj}\alpha_{a_{ik}}) + \sum_{\widetilde{a}_{ik}<0 or 0\in\widetilde{a}_{ik}, x_{kj}>0} (\alpha_{a_{ik}} x_{kj} - a_{ik}\beta_{kj}) + \sum_{\widetilde{a}_{ik}>0 or 0\in\widetilde{a}_{ik}, x_{kj}<0} (a_{ik}\alpha_{kj} - x_{kj}\beta_{a_{ik}}) + \sum_{\widetilde{a}_{ik}<0, x_{kj}<0} (-a_{ik}\beta_{kj} - \beta_{a_{ik}} x_{kj}) + \sum_{\widetilde{a}_{ik}>0 or 0\in\widetilde{a}_{ik}, x_{kj}=0} (a_{ik}\alpha_{kj}) + \sum_{\widetilde{a}_{ik}<0, x_{kj}=0} (-a_{ik}\beta_{kj}) + \alpha_{b_{ij}} - \sum_{\widetilde{c}_{ik}>0, x_{kj}>0} (c_{ik}\alpha_{kj} + x_{kj}\alpha_{c_{ik}}) - \sum_{\widetilde{c}_{ik}<0 or 0\in\widetilde{c}_{ik}, x_{kj}>0} (\alpha_{a_{ik}} x_{kj} - c_{ik}\beta_{kj}) - \sum_{\widetilde{c}_{ik}>0 or 0\in\widetilde{c}_{ik}, x_{kj}<0} (c_{ik}\alpha_{kj} - x_{kj}\beta_{c_{ik}}) - \sum_{\widetilde{c}_{ik}<0, x_{kj}<0} (-c_{ik}\beta_{kj} - \beta_{c_{ik}} x_{kj}) - \sum_{\widetilde{c}_{ik}>0 or 0\in\widetilde{c}_{ik}, x_{kj}=0} (c_{ik}\alpha_{kj}) - \sum_{\widetilde{c}_{ik}<0, x_{kj}=0} (-c_{ik}\beta_{kj}) - \alpha_{d_{ij}} \mid$$

$$z_2 = Max_{i,j} \mid \sum_{0\in\widetilde{a} or \widetilde{a}_{ik}>0, x_{kj}>0} (a_{ik}\beta_{kj} + x_{kj}\beta_{a_{ik}}) + \sum_{\widetilde{a}_{ik}<0, x_{kj}>0} (\beta_{a_{ik}} x_{kj} - a_{ik}\alpha_{kj}) + \sum_{\widetilde{a}_{ik}>0, x_{kj}<0} (a_{ik}\beta_{kj} - \alpha_{a_{ik} x_{kj}}) + \sum_{\widetilde{a}_{ik}<0 or 0\in\widetilde{a}_{ik}, x_{kj}<0} (-a_{ik}\alpha_{kj} - x_{kj}\alpha_{a_{ik}}) + - \sum_{\widetilde{a}_{ik}>0 or 0\in\widetilde{a}_{ik}, x_{kj}=0} (a_{ik}\beta_{kj}) + \sum_{\widetilde{a}_{ik}<0, x_{kj}=0} (-a_{ik}\alpha_{kj}) + \beta_{b_{ij}} - \sum_{\widetilde{c}_{ik}>0 or 0\in\widetilde{c}_{ik}, x_{kj}>0} (c_{ik}\beta_{kj} + x_{kj}\beta_{c_{ik}}) - \sum_{\widetilde{c}_{ik}<0, x_{kj}>0} (\beta_{c_{ik}} x_{kj} - c_{ik}\alpha_{kj}) - \sum_{\widetilde{c}_{ik}>0, x_{kj}<0} (c_{ik}\beta_{kj} - \alpha_{c_{ik} x_{kj}}) - \sum_{\widetilde{c}_{ik}<0 or 0\in\widetilde{c}_{ik}, x_{kj}<0} (-c_{ik}\alpha_{kj} - x_{kj}\alpha_{c_{ik}}) - - \sum_{\widetilde{c}_{ik}>0 or 0\in\widetilde{c}_{ik}, x_{kj}=0} (c_{ik}\beta_{kj}) - \sum_{\widetilde{c}_{ik}<0, x_{kj}=0} (-c_{ik}\alpha_{kj}) \beta_{d_{ij}} \mid$$

Theorem 3.1 *If models (3.12) have more than two optimal solutions being fuzzy solutions for system $\widetilde{A}\widetilde{X} + \widetilde{B} = \widetilde{C}\widetilde{X} + \widetilde{D}$, then it has infinite number of fuzzy solutions.*

It is similar to the proof of theorem 3.1 in [20].

Theorem 3.2 *Let $\widetilde{X}$ is the algebraic fuzzy solution of $\widetilde{A}\widetilde{X} + \widetilde{B} = \widetilde{C}\widetilde{X} + \widetilde{D}$. Then the optimal value of minimization problem is zero.*

Let $\widetilde{X} = (x_{kj}, \alpha_{kj}, \beta_{kj})$, $i = 1, \ldots, n, j = 1, \ldots, t, k = 1, \ldots, m$ is an algebraic solution, then:

$$\sum_{\widetilde{a}_{ik}>0} (a_{ik}\alpha_{kj} + x_{kj}\alpha_{a_{ik}}) + \sum_{\widetilde{a}_{ik}<0 or 0\in\widetilde{a}_{ik}} (\alpha_{a_{ik}} x_{kj} - a_{ik}\beta_{kj}) + \alpha_{b_{ij}} = \sum_{\widetilde{c}_{ik}>0} (c_{ik}\alpha_{kj} + x_{kj}\alpha_{c_{ik}}) + \sum_{\widetilde{c}_{ik}<0 or 0\in\widetilde{c}_{ik}} (\alpha_{c_{ik}} x_{kj} - c_{ik}\beta_{kj}) + \alpha_{d_{ij}},$$

$$\sum_{0\in\widetilde{a}_{ik} or \widetilde{a}_{ik}>0} (a_{ik}\beta_{kj} + x_{kj}\beta_{a_{ik}}) + \sum_{\widetilde{a}_{ik}<0} (\beta_{a_{ik}} x_{kj} - a_{ik}\alpha_{kj}) + \beta_{b_{ij}} = \sum_{\widetilde{c}_{ik}>0 or 0\in\widetilde{c}_{ik}} (c_{ik}\beta_{kj} + x_{kj}\beta_{c_{ik}}) + \sum_{\widetilde{c}_{ik}<0} (\beta_{c_{ik}} x_{kj} - c_{ik}\alpha_{kj}) + \beta_{d_{ij}}$$

by using the constraints of model (3.13), $z_1 = z_2 = 0$ and $z^* = 0$, so $\widetilde{X}$ is an optimal solution.

Theorem 3.3 *If $(\widetilde{A} \ominus_H \widetilde{C})$ and $(\widetilde{D} \ominus_H \widetilde{B})$ exist and $(\widetilde{a}_{ij} \ominus_H \widetilde{c}_{ij}) > 0$ for $\widetilde{a}_{ij} > 0,\ \widetilde{c}_{ij} > 0$, $(\widetilde{a}_{ij} \ominus_H \widetilde{c}_{ij}) < 0$ for $\widetilde{a}_{ij} < 0,\ \widetilde{c}_{ij} < 0$ and $0 \in (\widetilde{a}_{ij} \ominus_H \widetilde{c}_{ij})$ for $0 \in \widetilde{a}_{ij},\ 0 \in \widetilde{c}_{ij}$, then the systems $\widetilde{A}\widetilde{x} + \widetilde{B} = \widetilde{C}\widetilde{x} + \widetilde{D}$ and $(\widetilde{A} \ominus_H \widetilde{C})\widetilde{x} = (\widetilde{D} \ominus_H \widetilde{B})$ have the same fuzzy solution.*

For 1-cut system we have:

$$AX + B = CX + D \to (A - C)X = (D - B)$$

Since $(A - C)$ is nonsingular, the theorem is hold for 1-cut system. On other hand, because $(\widetilde{A} \ominus_H \widetilde{C})$ and $(\widetilde{D} \ominus_H \widetilde{B})$ exist and $(\widetilde{a}_{ij} \ominus_H \widetilde{c}_{ij}) > 0$ for $\widetilde{a}_{ij} > 0,\ \widetilde{c}_{ij} > 0$, $(\widetilde{a}_{ij} \ominus_H \widetilde{c}_{ij}) < 0$ for $\widetilde{a}_{ij} < 0,\ \widetilde{c}_{ij} < 0$ and $0 \in (\widetilde{a}_{ij} \ominus_H \widetilde{c}_{ij})$ for $0 \in \widetilde{a}_{ij},\ 0 \in \widetilde{c}_{ij}$, minimization problem for the systems of $\widetilde{A}\widetilde{x} + \widetilde{B} = \widetilde{C}\widetilde{x} + \widetilde{D}$ and $(\widetilde{A} \ominus_H \widetilde{C})\widetilde{x} = (\widetilde{D} \ominus_H \widetilde{B})$ is similar. Therefore, the spreads of fuzzy solution of the systems are equal too. Proof is complete.

Theorem 3.4 *If $(\widetilde{B} \ominus_H \widetilde{D})$ and $(\widetilde{A} \ominus_H \widetilde{C})$ exist, $\widetilde{A}, \widetilde{C}, (\widetilde{A} \ominus_H \widetilde{C}) > 0$ or $\widetilde{A}, \widetilde{C}, (\widetilde{A} \ominus_H \widetilde{C}) < 0$, system (2.1) has a crisp solution.*

First we suppose that $(\widetilde{B} \ominus_H \widetilde{D})$ and $(\widetilde{A} \ominus_H \widetilde{C})$ exist, $\widetilde{A}, \widetilde{C}, (\widetilde{A} \ominus_H \widetilde{C}) > 0$ and $x_{kj} > 0$, so from (3.8) we have:

$$Min \quad z_1 + z_2$$

$$\begin{aligned} s.t.\quad -z_1 \leq & \sum_{\widetilde{a}_{ik}>0} (a_{ik}\alpha_{kj} + x_{kj}\alpha_{a_{ik}}) + \alpha_{b_{ij}} \\ & - \sum_{\widetilde{c}_{ik}>0} (c_{ik}\alpha_{kj} + x_{kj}\alpha_{c_{ik}}) - \alpha_{d_{ij}} \leq z_1 \\ -z_2 \leq & \sum_{\widetilde{a}_{ik}>0} (a_{ik}\beta_{kj} + x_{kj}\beta_{a_{ik}}) + \beta_{b_{ij}} \\ & - \sum_{\widetilde{c}_{ik}>0} (c_{ik}\beta_{kj} + x_{kj}\beta_{c_{ik}}) - \beta_{d_{ij}} \leq z_2 \end{aligned}$$

$$0 \leq \alpha_{kj} \leq x_{kj}, \beta_{kj} \geq 0, k, j = 1, \ldots, n \quad (3.13)$$

On other hand

$$\begin{aligned} & \sum_{\widetilde{a}_{ik}>0} (a_{ik}\alpha_{kj} + x_{kj}\alpha_{a_{ik}}) + \alpha_{b_{ij}} \\ & - \sum_{\widetilde{c}_{ik}>0} (c_{ik}\alpha_{kj} + x_{kj}\alpha_{c_{ik}}) - \alpha_{d_{ij}} = \\ & \sum_{\widetilde{a}_{ik}>0, \widetilde{c}_{ik}>0} ((a_{ik} - c_{ik})\alpha_{kj} + x_{kj}(\alpha_{a_{ik}} - \alpha_{c_{ik}}) \\ & + (\alpha_{b_{ij}} - \alpha_{d_{ij}})) \sum_{\widetilde{a}_{ik}>0} (a_{ik}\beta_{kj} + x_{kj}\beta_{a_{ik}}) + \beta_{b_{ij}} \\ & - \sum_{\widetilde{c}_{ik}>0} (c_{ik}\beta_{kj} + x_{kj}\beta_{c_{ik}}) - \beta_{d_{ij}} = \\ & \sum_{\widetilde{a}_{ik}>0, \widetilde{c}_{ik}>0} ((a_{ik} - c_{ik})\beta_{kj} + x_{kj}(\beta_{a_{ik}} - \beta_{c_{ik}}) + \\ & (\beta_{b_{ij}} - \beta_{d_{ij}})) \end{aligned} \quad (3.14)$$

and $(\widetilde{B} \ominus_H \widetilde{D})$ exists i.e, $((\alpha_{b_{ij}} - \alpha_{d_{ij}}), (\beta_{b_{ij}} - \beta_{d_{ij}})) > 0$, $\widetilde{A}, \widetilde{C}$, and $(\widetilde{A} \ominus_H \widetilde{C}) > 0$ i.e, $((a_{ik} - c_{ik}), (\alpha_{a_{ik}} - \alpha_{c_{ik}}), (\beta_{a_{ik}} - \beta_{c_{ik}})) > 0$, so the constraint of (3.8) conclude that:

$$\alpha_{kj} = \beta_{kj} = 0$$

Therefore, fuzzy dual matrix system has a crisp solution.
The proof of other cases is similar and omitted.

4 Numerical Examples

Example 4.1 *Consider the following full fuzzy dual system:*

$$\begin{cases} (0.5, 0.2, 0.3)\widetilde{x}_1 + (0.4, 0.2, 0.1)\widetilde{x}_2 + (0.3, 0.2, 0.2) = \\ \quad (0.4, 0.1, 0.1)\widetilde{x}_1 + (0.3, 0.2, 0.1)\widetilde{x}_2 + (1, 0.4, 0.4) \\ \\ (0.3, 0.2, 0.2)\widetilde{x}_1 + (0.7, 0.1, 0.2)\widetilde{x}_2 + (0.5, 0.2, 0.1) = \\ \quad (0.2, 0.1, 0.1)\widetilde{x}_1 + (0.3, 0.1, 0.1)\widetilde{x}_2 + (1.5, 0.3, 0.6) \end{cases}$$

By solving 1-cut system, we obtain:

$$x_1 = 6, = x_2 = 1.$$

Since $x_1, x_2 > 0$, we use model (3.8):

$$min \quad z_1 + z_2$$

$$-z_1 \leq 0.1\alpha_1 + 0.1\alpha_2 + 0.4 \leq z_1$$

$$-z_1 \leq 0.1\alpha_1 + 0.4\alpha_2 + 0.5 \leq z_1$$

$$-z_2 \leq 0.1\beta_1 + 0.1\beta_2 + 1 \leq z_2$$

$$-z_2 \leq 0.1\beta_1 + 0.4\beta_2 + 0.2 \leq z_2$$

$$0 \leq \alpha_1 \leq 6, 0 \leq \alpha_2 \leq 1, \beta_1, \beta_2, z_1, z_2 \geq 0.$$

By solving above LP, we obtain:

$$z_1 = 0.5, z_2 = 1, \ \alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0.$$

So the system has a solution as:

$$\tilde{X} = \begin{bmatrix} (6,0,0) \\ (1,0,0) \end{bmatrix},$$

On other hand, we have $(\widetilde{a}_{ij} \ominus_H \widetilde{c}_{ij}), \widetilde{a}_{ij}, \widetilde{c}_{ij} > 0$, *so theorem 3.3 concludes that system* $((\widetilde{A} \ominus_H \widetilde{C})\widetilde{X} = (\widetilde{D} \ominus_H \widetilde{B}))$:

$$\begin{cases} (0.1,0.1,0.2)\widetilde{x}_1 + (0.1,0,0)\widetilde{x}_2 = (0.7,0.2,0.2) \\ (0.1,0.1,0.1)\widetilde{x}_1 + (0.4,0,0.1)\widetilde{x}_2 = (1,0.1,0.5) \end{cases}$$

has the same solution as dual system:

$$\tilde{X} = \begin{bmatrix} (6,0,0) \\ (1,0,0) \end{bmatrix}.$$

Example 4.2 *Consider the following fully fuzzy matrix dual system:*

$$\begin{bmatrix} (-1,1,1) & (1,1,1) \\ (1,1,0) & (2,0,0) \end{bmatrix} \begin{bmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} \\ \widetilde{x}_{21} & \widetilde{x}_{22} \end{bmatrix} + \begin{bmatrix} (3,1,0) & (2,1,1) \\ (0,1,1) & (1,1,1) \end{bmatrix} = \begin{bmatrix} (1,1,1) & (-1,0,0) \\ (3,1,0) & (2,0,0) \end{bmatrix} \begin{bmatrix} \widetilde{x}_{11} & \widetilde{x}_{12} \\ \widetilde{x}_{21} & \widetilde{x}_{22} \end{bmatrix} + \begin{bmatrix} (3,2,1) & (2,0,3) \\ (0,2,0) & (1,1,1) \end{bmatrix}$$

The solution of 1-cut system is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. *By using model (3.13), we have*

$$min \quad z_1 + z_2$$

$$-z_1 \leq \beta_{11} + \alpha_{21} - \alpha_{11} - \beta_{21} - 1 \leq z_1$$

$$-z_1 \leq \beta_{12} + \alpha_{22} - \alpha_{12} - \beta_{22} + 1 \leq z_1$$

$$-z_1 \leq -2\alpha_{11} - 1 \leq z_1$$

$$-z_1 \leq -2\alpha_{12} \leq z_1$$

$$-z_2 \leq \alpha_{11} + \beta_{21} - \beta_{11} - \alpha_{21} \leq z_2$$

$$-z_2 \leq \alpha_{12} + \beta_{22} - \beta_{12} - \alpha_{22} - 2 \leq z_2$$

$$-z_2 \leq -2\beta_{11} + 1 \leq z_2$$

$$-z_2 \leq -2\beta_{12} \leq z_2$$

$$\alpha_{i,j}, \beta_{i,j}, z_1, z_2 \geq 0, \quad i,j = 1,2.$$

The solutions are $\begin{bmatrix} (\alpha_{11},\beta_{11}) & (\alpha_{12},\beta_{12}) \\ (\alpha_{21},\beta_{21}) & (\alpha_{22},\beta_{22}) \end{bmatrix} = \begin{bmatrix} (0,0.5) & (0,0) \\ (0,0.5) & (0,2) \end{bmatrix}$, $z_1 = 1, z_2 = 0$, *then the fuzzy solution is* $\widetilde{X} = \begin{bmatrix} (0,0.0.5) & (0,0,0) \\ (0,0,0.5) & (0,0,2) \end{bmatrix}$. *We observe that* $(\widetilde{D} \ominus_H \widetilde{B})$ *and* $(\widetilde{B} \ominus_H \widetilde{D})$ *do not exist, so the conditions of theorem 3.3 and theorem 3.4 are not hold.*

Example 4.3 *Consider the following full fuzzy dual system:*

$$\begin{cases} (-5,1,3)\widetilde{x}_1 + (-2,2,1)\widetilde{x}_2 + (4,2,1) = \\ \quad (-3,1,1)\widetilde{x}_1 + (-1,1,0)\widetilde{x}_2 + (1,1,0) \\ (-2,1,0)\widetilde{x}_1 + (-3,1,1)\widetilde{x}_2 + (3,1,2) = \\ \quad (-1,1,0)\widetilde{x}_1 + (-2,0,1)\widetilde{x}_2 + (1,0,0) \end{cases}$$

By solving 1-cut system, we obtain:

$$x_1 = x_2 = 1.$$

Since $x_1, x_2 > 0$, *we use model (3.8):*

$$min \quad z_1 + z_2$$

$$-z_1 \leq 2\beta_1 + \beta_2 + 3 \leq z_1$$

$$-z_1 \leq \beta_1 + \beta_2 + 3 \leq z_1$$

$$-z_2 \leq 2\alpha_1 + \alpha_2 + 4 \leq z_2$$

$$-z_2 \leq \alpha_1 + \alpha_2 + 3 \leq z_2$$

$$0 \leq \alpha_1, \alpha_2 \leq 1, \beta_1, \beta_2, z_1, z_2 \geq 0.$$

By solving above LP, we obtain:

$$z_1 = 3, z_2 = 4, \ \alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0$$

so the system has a solution as:

$$\tilde{X} = \begin{bmatrix} (1,0,0) \\ (1,0,0) \end{bmatrix},$$

On other hand, we have $(\widetilde{B} \ominus_H \widetilde{D})$ *exists and* $(\widetilde{A}, \widetilde{C}, (\widetilde{A} \ominus_H \widetilde{C})) < 0$, *so we observe that theorem 3.4 is hold.*

5 Conclusion

In this paper, we suggested a method for solving the LR fuzzy dual matrix system of $\widetilde{A}\widetilde{X} + \widetilde{B} = \widetilde{C}\widetilde{X} + \widetilde{D}$ in which $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}, \widetilde{X}$ are LR fuzzy matrices. First, we solved 1-cut system for finding the core and obtained spreads by using multiplication, quasi norm and optimization problem. Then we explained the conditions in which the systems $\widetilde{A}\widetilde{X} + \widetilde{B} = \widetilde{C}\widetilde{X} + \widetilde{D}$ and $(\widetilde{A} \ominus_H \widetilde{C})\widetilde{X} = (\widetilde{D} \ominus \widetilde{B})$ have the same solution, and discussed about the conditions where LR fuzzy dual matrix system has crisp solution. For future research, we will compare the obtained solution of our method with the suggested solution in [9].

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Fatemeh Babakordi is an assistant Prof. in Department of Mathematics and statistics, Gonbad Kavous University, Gonbad Kavous, in Iran. She completed her Ph. D degree in Islamic Azad University in Tehran,Her interests research include fuzzy numerical analysis, fuzzy linear systems, fuzzy differential equations, fuzzy integral differential equations. Babakordi 's papers appear in journals such as Applied mathematics and computation, Journal of Applied mathematics, International journal of advanced manufacturing technology, International journal of Production Economics, Optimization, International Journal of Mathematics, Journal of Global Optimization and etc.

Mohamadsadegh Adabitabarfirozja is Associate Professor in Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr in Iran. His interests research include fuzzy linear system, fuzzy differential equation, fuzzy partial differential equation, fuzzy integral differential equation, fuzzy ranking.