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Research Article

Numerical Solution of Two-Dimensional Hyperbolic Equations with Nonlocal Integral Conditions Using Radial Basis Functions

E. Shivanian *†, M. Aslefallah ‡

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Abstract

This paper proposes a numerical method to deal with the two-dimensional hyperbolic equations with nonlocal integral conditions. The nonlocal integral equation usually is of major challenge in the frame work of the numerical solutions of partial differential equations. The method benefits from collocation radial basis function method, the generalized thin plate splines (GTPS) radial basis functions are used. Therefore, it does not require any struggle to determine shape parameter (In other RBFs, it is time-consuming step). The present technique is one of the truly meshless methods in where it does not require any background integration cells over local or global domains and it is in contrast to weak form methods in where all integrations are carried out locally or globally over quadrature domains of regular shapes, such as lines in one dimensions, circles or squares in two dimensions and spheres or cubes in three dimensions. The obtained results for some numerical examples reveal that the proposed technique is very effective, convenient and quite accurate to such considered problems.

Keywords : Radial basis function; Hyperbolic equations with purely integral conditions; Kansa method; Finite differences $\theta-$ method.

1 Introduction

This paper is devoted to the numerical computation of the two-dimensional equation. In the domain $D = \{(x, y, t) : 0 < x < l\,,\ 0 < y < l\,,\ 0 < t < T\}$, we consider a third order two dimensional hyperbolic equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y, t), \tag{1.1}$$

with initial conditions

$$u(x, y, 0) = g(x, y)\,, \quad \frac{\partial u}{\partial t}(x, y, 0) = h(x, y)\,, \tag{1.2}$$

and nonlocal (integral) boundary conditions

$$\int_0^l u(x, y, t)\, dx = 0\,, \quad \int_0^l x\, u(x, y, t)\, dx = 0, \tag{1.3}$$

$$\int_0^l u(x, y, t)\, dy = 0\,, \quad \int_0^l y\, u(x, y, t)\, dy = 0, \tag{1.4}$$

where f, g and h are given functions.

Many problems in science and engineering modelled as differential equations. Solving equations by traditional numerical methods such as finite

*Corresponding author. shivanian@sci.ikiu.ac.ir, Tel:+98(912)6825371.
†Department of Applied Mathematics, Imam Khomeini International University, Qazvin, Iran.
‡Department of Applied Mathematics, Imam Khomeini International University, Qazvin, Iran.

difference (FDM), finite element (FEM) needs generation of a regular mesh in the domain of the problem which is computationally expensive [1, 2, 3, 4, 5]. During the last decade, meshless methods have received much attention. Due to the difficulty of the mesh generation problem, meshless methods for simulation of the numerical problems are employed. Radial basis functions (RBFs) interpolation is a technique for representing a function starting with data on scattered points [6, 7, 8, 9, 10, 11]. The RBFs can be of various types, such as: polynomials of a given degree; linear, quadratic, cubic, etc; thin plate spline (TPS), multiquadrics (MQ), inverse multiquadrics (IMQ), Gaussian forms (GA), etc. Most differential equations do not have exact analytic solutions, so approximation and numerical techniques must be used.

Development of constructive methods for the numerical solution of mathematical problems is a main branch of mathematics. Meshless methods have attracted much attention in the both mathematics and engineering community, recently. Extensive developments have been made in several varieties of meshless techniques and applied to many applications in science and engineering. These methods exist under different names, such as: the diffuse element method (DEM) [12], the hp-cloud method [13], Meshless Local Petrov-Galerkin (MLPG) method [14, 15, 16], the meshless local boundary integral equation (LBIE) method [17], the partition of unity method (PUM) [18], the meshless collocation method based on radial basis functions (RBFs) [19, 20, 21, 22], the smooth particle hydrodynamics (SPH)[23], the reproducing kernel particle method (RKPM) [24], the radial point interpolation method [25], meshless local radial point interpolation method (MLRPI) [26, 27], and so on. Recently, increasing attention has been paid to the development, analysis, and implementation of stable methods for the numerical solutions of hyperbolic equations. There have been many numerical methods for hyperbolic equations, such as the finite difference, the finite element, and the collocation methods, etc. [19, 27]. Mohanty et al. [31, 32] developed some alternating direction implicit schemes for the two- and three-dimensional linear hyperbolic equations. Most of these schemes are second-order accurate in both space and time.

The first investigation of this type of problems in one-dimensional case, goes back to [33] in 1996, in which the author proved the existence, uniqueness, and continuous dependence of the solution upon the data of certain hyperbolic problems with only integral boundary conditions. Later, similar problems have been studied in [34, 35] by using the energetic method and the Rothe time-discretization method. We refer the reader to [28, 29, 31, 33, 34, 36, 37, 39, 41, 42] for hyperbolic equations with Neumann and integral condition. For other problems with nonlocal conditions, related to other equations, we refer to [28, 29, 30, 33] and references therein.

2 Basic Definitions

For implementation of this method, we need the following definitions.

Definition 2.1 ***Radial basis functions*** *Considering a finite set $X \subseteq R^d$ and a function $u : X \to R^d$, according to the process of interpolation using radial basis functions [6], the interpolant of u is constructed in the following form:*

$$(Su)(\boldsymbol{x}) = \sum_{i=1}^{M} \lambda_i \varphi(\|\boldsymbol{x} - \boldsymbol{x}_i\|) + p(\boldsymbol{x}), \quad \boldsymbol{x} \in R^d,$$

where $\|.\|$ is the Euclidean norm and $\varphi(\|.\|)$ is a radial function.

Also, $p(\mathbf{x})$ is a linear combination of polynomials on R^d of total degree at most $m-1$ as follows:

$$p(\mathbf{x}) = \sum_{j=M+1}^{M+l} \lambda_j q_j(\mathbf{x}), \quad l = \binom{m+d-1}{d}.$$

Moreover, the interpolant Su and additional conditions must be determined to satisfy the system:

$$\begin{cases} (Su)(\mathbf{x}_i) = u(\mathbf{x}_i) \quad , & i = 1, 2, \ldots, M \\ \sum_{i=1}^{M} \lambda_i q_j(\mathbf{x}_i) = 0, \quad , & \forall q_j \in \Pi_{m-1}^d \end{cases} \tag{2.5}$$

where Π_{m-1}^d denotes the space of all polynomials on R^d of total degree at most $m-1$. Now we have a unique interpolant (Su) of u if $\varphi(r)$ is a conditionally positive definite radial basis function of order m. Some of the most important radial basis functions are shown in Table (1) (c is shape parameter). We will use the generalized thin plate splines(GTPS) which have the following form:

$$\varphi(\|\mathbf{x}-\mathbf{x}_i\|)=\varphi(r_i)=r_i^{2m}log(r_i), \\ ,i=1,2,3,\ldots,m=1,2,3,\ldots, \tag{2.6}$$

that in two-dimensional case we have

$$r_{ij}=\sqrt{(x_i-x_j)^2+(y_i-y_j)^2}.$$

We note that φ is $\mathbf{C}^{2m-1}$ continuous. Therefore, the higher order of partial differentials needs the higher order of thin plate splines. As $u(\mathbf{x})$ can be approximated by

$$u(\mathbf{x})\cong\sum_{\mathbf{x}_i\in X}\lambda_i\varphi(\|\mathbf{x}-\mathbf{x}_i\|)+p(\mathbf{x}),\mathbf{x}\in R^d, \tag{2.7}$$

For any partial differential operator L, Lu can be represented by:

$$Lu(\mathbf{x})=\sum_{\mathbf{x}_i\in X}\lambda_i L\varphi(\|\mathbf{x}-\mathbf{x}_i\|)+Lp(\mathbf{x}),\mathbf{x}\in R^d. \tag{2.8}$$

The coefficients λ_i will be obtained by solving the system of linear equations.

Remark 2.1 Note that $\theta=0$ gives the explicit scheme, $\theta=\frac{1}{2}$ the Crank-Nicolson, and $\theta=1$ a fully implicit backward time-difference method.

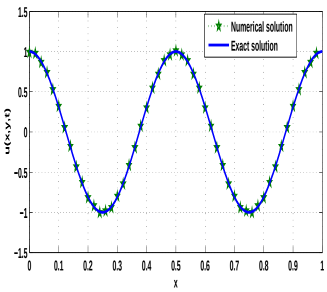

Figure 1: Numerical solution and exact solution at time $t=1s$ and $y_l=1$ for Example 4.1. with $\Delta t=\frac{1}{50}$ and $h=\frac{1}{50}$.

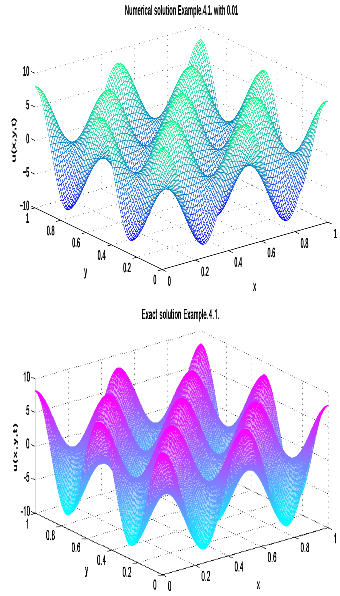

Figure 2: Comparison between exact and numerical solutions for the Example 4.1 with $M=100,\theta=1$.

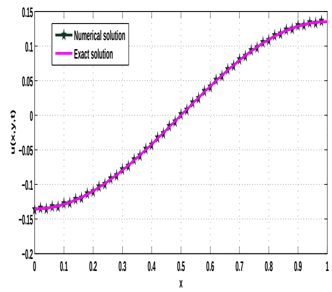

Figure 3: Numerical solution and exact solution at time $t=2s$ and $y_l=1$ for Example 4.2 with $\Delta t=\frac{1}{50}$ and $h=\frac{1}{50}$.

3 Discretization of the equation

Consider the two-dimensional equation

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+f(x,y,t),$$

with $u(x,y,t)$ in the region $\Omega=\{(x,y):0<x<l\,,\ 0<y<l\,\}$, for $t>0$ and mentioned initial conditions and nonlocal (integral) boundary conditions.

According to definitions (2.1) and remark (2.1), from eq. (1.1) and θ-method we get:

$$\frac{\partial u(x,y,t^{n+1})}{\partial t^2}=[\theta\nabla^2u^{n+1}+(1-\theta)\nabla^2u^n], \\ +f^{n+1}, \tag{3.9}$$

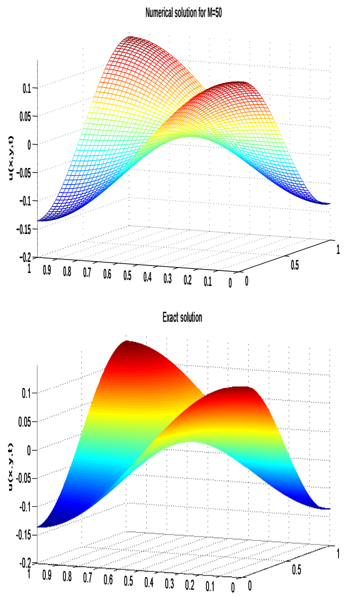

Figure 4: Comparison between exact and numerical solutions for the Example 4.2 with $M = 50, \theta = 0.5$.

where ∇^2 is the laplacian operator and $0 \leq \theta \leq 1$. In the current work, we employ a time-stepping scheme to approximate the time derivative. For this purpose, the following finite difference approximation can be used:

$$\frac{\partial^2 u(\mathbf{x},t)}{\partial t^2} \cong \frac{1}{(\Delta t)^2}\left(u^{n+1}(\mathbf{x}) - 2u^n(\mathbf{x}) + u^{n-1}(\mathbf{x})\right). \tag{3.10}$$

By substituting finite difference for left hand into (3.9) we have:

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{(\Delta t)^2} = [\theta\nabla^2 u^{n+1} + (1-\theta)\nabla^2 u^n] + f^{n+1}, \tag{3.11}$$

that Δt is the time step size and

$$u^n = u(x,y,t^n), t^{n+1} = t^n + \Delta t,$$

$$f^{n+1} = f(x,y,t^{n+1}),$$

and for $(\Delta t)^2 = \mu$:

$$u^{n+1} - \mu\theta\nabla^2 u^{n+1} = 2u^n - u^{n-1} + \mu(1-\theta)\nabla^2 u^n + \mu f^{n+1}. \tag{3.12}$$

In other words, we get:

$$(1 - \mu\theta\nabla^2)u^{n+1} = 2u^n - u^{n-1} + \mu(1-\theta)\nabla^2 u^n + \mu f^{n+1}. \tag{3.13}$$

From (3.11) we have:

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{(\Delta t)^2} = \theta\left(\frac{\partial^2 u^{n+1}(\mathbf{x})}{\partial x^2} + \frac{\partial^2 u^{n+1}(\mathbf{x})}{\partial y^2}\right) + (1-\theta)\left(\frac{\partial^2 u^n(\mathbf{x})}{\partial x^2} + \frac{\partial^2 u^n(\mathbf{x})}{\partial y^2}\right) + f^{n+1}, \tag{3.14}$$

and for $(\Delta t)^2 = \mu$:

$$u^{n+1} - \mu\theta\left(\frac{\partial^2 u^{n+1}(\mathbf{x})}{\partial x^2} + \frac{\partial^2 u^{n+1}(\mathbf{x})}{\partial y^2}\right) = 2u^n + \mu(1-\theta)\left(\frac{\partial^2 u^n(\mathbf{x})}{\partial x^2} + \frac{\partial^2 u^n(\mathbf{x})}{\partial y^2}\right) - u^{n-1} + \mu f^{n+1}. \tag{3.15}$$

Now, according to the mentioned method in two-dimensional case, we have:

$$u^{n+1}(x,y) = \sum_{j=1}^{M-1} \lambda_j^{n+1}\varphi(r_j) + \lambda_M^{n+1}x + \lambda_{M+1}^{n+1}y + \lambda_{M+2}^{n+1}, \tag{3.16}$$

and the collocation method is used at every point $(x_i,y_i), i = 1,2,...,M-1$,we have

$$u^{n+1}(x_i,y_i) = \sum_{j=1}^{M-1} \lambda_j^{n+1}\varphi(r_{ij}) + \lambda_M^{n+1}x_i + \lambda_{M+1}^{n+1}y_i + \lambda_{M+2}^{n+1}. \tag{3.17}$$

Three additional conditions can be described as:

$$\sum_{j=1}^{M-1} \lambda_j^{n+1} = \sum_{j=1}^{M-1} \lambda_j^{n+1}x_j = \sum_{j=1}^{M-1} \lambda_j^{n+1}y_j = 0. \tag{3.18}$$

Finally, by combining equations (3.17), (3.18), we obtain a matrix form:

$$[u]^{n+1} = A[\lambda]^{n+1}, \tag{3.19}$$

where:

$$[u]^{n+1} = [u_1^{n+1}, u_2^{n+1}, \dots, u_{M-1}^{n+1}, 0, 0, 0]^T,$$

and

$$[\lambda]^{n+1} = [\lambda_1^{n+1}, \lambda_2^{n+1}, \dots, \lambda_{M+2}^{n+1}]^T,$$

and the matrix $A = (a_{ij})_{(M+2)\times(M+2)}$ is given by:

$$A = \begin{bmatrix} \Phi & P \\ P^T & O_3 \end{bmatrix}. \tag{3.20}$$

By substituting (3.16) into (3.11), (3.12) and considering (3.18) and initial and boundary conditions we obtain a matrix form:

$$[c]^{n+1} = B[\lambda]^{n+1}, \tag{3.21}$$

where:

$$[c]^{n+1} = [c_1^{n+1}, c_2^{n+1}, \dots, c_{M-1}^{n+1}, 0, 0, 0]^T$$

and the matrices A and B are given in detailed after Table 2. In other form

$$B = \begin{bmatrix} L(\Phi) & L(P) \\ P^T & O_3 \end{bmatrix}. \tag{3.22}$$

where L represents an operator given by

$$L(*) = \begin{cases} (1-\mu\theta\nabla^2)(*), & 1 < i < M-1, \\ (*), & i = 1 or i = M-1, \end{cases} \tag{3.23}$$

and

$$c_i^{n+1} = \begin{cases} 2u^n - u^{n-1} + \mu(1-\theta)\nabla^2 u^n \\ +\mu f^{n+1},\ n \geq 0, 1 < i < M-1, \\ g(\mathbf{x}_i, t^{n+1})\ i = 1 \text{ or } i = M-1. \end{cases} \tag{3.24}$$

For solving the system (3.21) we need other necessary equations that discuss below and then we find $(M+2)$ unknowns λ_j^{n+1} then with (3.16) we approximate the value of u.
For purely integral conditions, we have:

$$\int_0^l u(x,y,t)dx = 0, \int_0^l xu(x,y,t)dx = 0, \tag{3.25}$$

and

$$\int_0^l u(x,y,t)dy = 0, \int_0^l yu(x,y,t)dy = 0, \tag{3.26}$$

that with discretization:

$$\begin{gathered} \int_0^l u^n(x,y,t)dx = 0, \\ \int_0^l xu^n(x,y,t)dx = 0, 0 < x < l, \end{gathered} \tag{3.27}$$

and

$$\begin{gathered} \int_0^l u^n(x,y,t)dy = 0, \\ \int_0^l yu^n(x,y,t)dy = 0, 0 < y < l, \end{gathered} \tag{3.28}$$

We have four equations that with numerical integration methods we obtain to values of u on the boundary.

$$A = \begin{array}{cccc} \varphi_{11} & \cdots & \varphi_{1j} & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ \varphi_{i1} & \cdots & \varphi_{ij} & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ \varphi_{(M-1)1} & \cdots & \varphi_{(M-1)j} & \cdot \\ x_1 & \cdots & x_j & \cdots \\ y_1 & \cdots & y_j & \cdots \\ 1 & \cdots & 1 & \cdots \end{array} \tag{3.29}$$

$$\begin{array}{cccc} \varphi_{1(M-1)} & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_{i(M-1)} & x_i & y_i & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_{(M-1)(M-1)} & x_{M-1} & y_{M-1} & 1 \\ x_{M-1} & 0 & 0 & 0 \\ y_{M-1} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}.$$

$$B = \begin{array}{cccc} L(\varphi_{11}) & \cdots & L(\varphi_{1j}) & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ L(\varphi_{i1}) & \cdots & L(\varphi_{ij}) & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots \\ L(\varphi_{(M-1)1}) & \cdots & L(\varphi_{(M-1)j}) & \cdots \\ x_1 & \cdots & x_j & \cdots \\ y_1 & \cdots & y_j & \cdots \\ 1 & \cdots & 1 & \cdots \end{array} \tag{3.30}$$

$$\begin{array}{cccc} L(\varphi_{1(M-1)}) & L(x_1) & L(y_1) & L(1) \\ \vdots & \vdots & \vdots & \vdots \\ L(\varphi_{i(M-1)}) & L(x_i) & L(y_i) & L(1) \\ \vdots & \vdots & \vdots & \vdots \\ L(\varphi_{(M-1)(M-1)}) & L(x_{M-1}) & L(y_{M-1}) & L(1) \\ x_{M-1} & 0 & 0 & 0 \\ y_{M-1} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}.$$

Remark 3.1 At the first time level, when $n = 0$, according to the initial conditions that were introduced in Eq. (1.2), we apply the following assumptions:

$$u^{(0)} = u(x,y,0) = g(x,y),$$

and

$$u^{-1} = u^1 - 2(\Delta t)h(x,y),$$

Table 1: Some types of RBF functions.

Name	Abbreviation	Formula
Cubic	CU	$\varphi(r) = r^3$
Generalized Thin plate splines	GTPS	$\varphi(r) = r^{2m}\log(r), \quad m \in N$
Inverse quadrics(or Cauchy)	IQ	$\varphi(r) = \dfrac{1}{c^2 + r^2}$
Multi-quadrics	MQ	$\varphi(r) = \sqrt{c^2 + r^2}$
Inverse Multi-quadrics	IMQ	$\varphi(r) = \dfrac{1}{\sqrt{c^2 + r^2}}$
Gaussian RBF	GA	$\varphi(r) = exp(-r^2/c^2)$

Table 2: The L^1, L^2 and L^∞ errors for Example 4.1. with different $\Delta x, \Delta y$ and Δt at time $t = 1s$.

Δt	h	$\parallel e \parallel_\infty$	$\parallel e \parallel_2$	$\parallel e \parallel_1$
$\frac{1}{10}$	$\frac{1}{10}$	$3.123345e-01$	$4.741428e-01$	$6.321501e-01$
$\frac{1}{20}$	$\frac{1}{20}$	$2.612115e-02$	$8.307718e-02$	$2.453231e-01$
$\frac{1}{50}$	$\frac{1}{50}$	$3.604830e-03$	$1.005214e-02$	$4.482354e-02$
$\frac{1}{100}$	$\frac{1}{100}$	$2.761684e-04$	$6.338475e-04$	$3.758741e-03$

Table 3: The L^1, L^2 and L^∞ errors for Example 4.2 with different $\Delta x, \Delta y$ and Δt at time $t = 2s$.

Δt	h	$\parallel e \parallel_\infty$	$\parallel e \parallel_2$	$\parallel e \parallel_1$
$\frac{1}{10}$	$\frac{1}{10}$	$5.345578e-02$	$8.963640e-02$	$3.337215e-01$
$\frac{1}{20}$	$\frac{1}{20}$	$4.834327e-03$	$9.521487e-03$	$4.678523e-02$
$\frac{1}{50}$	$\frac{1}{50}$	$5.826059e-04$	$2.143351e-03$	$6.604587e-03$
$\frac{1}{100}$	$\frac{1}{100}$	$4.2145783e-05$	$8.521543e-05$	$5.125897e-04$

because

$$u_t^0 = h(x,y) \cong \frac{u^1 - u^{-1}}{2(\Delta t).}$$

Remark 3.2 The Laplacian operator ∇^2 for φ function is given by

$$\nabla^2(\varphi(r)) = \frac{\partial\varphi}{\partial r}\left(\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right) + \frac{\partial^2\varphi}{\partial r^2}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right], \tag{3.31}$$

Remark 3.3 The derivatives of $u(\mathbf{x})$ are easily obtained and discretization of ∇^2 is as follows:

$$\begin{aligned} \frac{\partial^2 u(\mathbf{x})}{\partial x^2} &= \sum_{j=1}^{M-1} \lambda_j^{n+1} \frac{\partial^2\varphi(r_j)}{\partial x^2}, \\ \frac{\partial^2 u(\mathbf{x})}{\partial y^2} &= \sum_{j=1}^{M-1} \lambda_j^{n+1} \frac{\partial^2\varphi(r_j)}{\partial y^2}. \end{aligned} \tag{3.32}$$

therefore,

$$\begin{aligned} \nabla^2 u(\mathbf{x}) &= \frac{\partial^2 u(\mathbf{x})}{\partial x^2} + \frac{\partial^2 u(\mathbf{x})}{\partial y^2} = \\ &\sum_{j=1}^{M-1} \lambda_j^{n+1}\left(\frac{\partial^2\varphi(r_j)}{\partial x^2} + \frac{\partial^2\varphi(r_j)}{\partial y^2}\right) \\ &= \sum_{j=1}^{M-1} \psi(r_j), \end{aligned} \tag{3.33}$$

Thus, substituting the collocation points gives

$$\nabla^2 u(\mathbf{x}_i) = \sum_{j=1}^{M-1} \psi(r_{ij}), \tag{3.34}$$

4 Numerical examples

In this section the numerical results obtained from application of the method for solving the two-dimensional equation with purely integral conditions are presented.

Example 4.1 *In the domain* $D = \{(x, y, t) : 0 < x < 1, 0 < y < 1, 0 < t < T\}$, *we consider eq.* (1.1) *with initial conditions*

$$u(x, y, 0) = g(x, y) = 0, \quad \frac{\partial u}{\partial t}(x, y, 0) = h(x, y) = 0, \tag{4.35}$$

and nonlocal (integral) boundary conditions

$$\int_0^1 u(x, y, t)dx = 0, \int_0^1 xu(x, y, t)dx = 0, \tag{4.36}$$

$$\int_0^1 u(x, y, t)dy = 0, \int_0^1 yu(x, y, t)dy = 0, \tag{4.37}$$

where the exact solution is given by:

$$u(x, y, t) = t^3 cos(4\pi x)cos(4\pi y),$$

$$0 < x < 1, 0 < y < 1, 0 < t < T.$$

The function f with the initial and purely integral conditions can be obtained by using the exact solution, where $f(x, y, t)$ is defined:

$$f(x, y, t) = (6t + 32\pi^2 t^3)cos(4\pi x)cos(4\pi y),$$

Example 4.2 *In the domain* $D = \{(x, y, t) : 0 < x < 1, 0 < y < 1, 0 < t < T\}$, *we consider eq.* (1.1) *with initial conditions*

$$u(x, y, 0) = g(x, y) = cos(\pi x)cos(\pi y), \quad \frac{\partial u}{\partial t}(x, y, 0) = h(x, y) = -cos(\pi x)cos(\pi y), \tag{4.38}$$

and nonlocal (integral) boundary conditions

$$\int_0^1 u(x, y, t)dx = 0, \int_0^1 xu(x, y, t)dx = 0, \tag{4.39}$$

$$\int_0^1 u(x, y, t)dy = 0, \int_0^1 yu(x, y, t)dy = 0, \tag{4.40}$$

where the exact solution is given by:

$$u(x, y, t) = exp(-t)cos(\pi x)cos(\pi y),$$

$$0 < x < 1, 0 < y < 1, 0 < t < T.$$

The function f with the initial and purely integral conditions can be obtained by using the exact solution, where $f(x, y, t)$ is defined:

$$f(x, y, t) = exp(-t)(1 + 2\pi^2)cos(\pi x)cos(\pi y),$$

5 Conclusions

In this paper we have presented a numerical scheme based on meshless collocation radial basis function (so-called Kansa's method) to solve two dimensional equation with purely integral conditions. The $\theta-$method has been applied to derivative. The method has been tested on two illustrative numerical examples. The computational results are found to be in good agreement with the exact solutions. In the current work, to demonstrate the accuracy and usefulness of this method, two numerical examples have been presented. As demonstrated by the computational results, it is very easy to implement the proposed method for similar problems.

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Dr. Elyas Shivanian was born in Zanjan province, Iran in August 26, 1982. He started his master course in applied mathematics in 2005 at Amirkabir University of Technology and has finished MSc. thesis in the field of fuzzy linear programming in 2007. After his Ph.D. in the field of prediction of multiplicity of solutions to the boundary value problems, at Imam Khomeini International University in 2012, he became an Assistant Professor at the same university. His research interests are analytical and numerical solutions of ODEs, PDEs and IEs. He has published several papers on these subjects. He also has published some papers in other fields, for more information see please http://scholar.google.com/citations?user=MFncks8AAAAJ&hl=en

Mohammad Aslefallah is currently a PhD candidate in Numerical Analysis at the Department of Applied Mathematics, Imam Khomeini International University, Qazvin, Iran. He received his BS and MSc degree in applied mathematics at Imam Khomeini International University. Main research interest includes Numerical analysis, Differential equations, Meshless methods, Fractional calculus and so on. He has several papers on these subjects. More details about his papers may be found online at http://scholar.google.com/citations?user=ryuYG2IAAAAJ&hl=en