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Research Article

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Solving the First-Order Linear Matrix Differential Equations Using Bernstein Matrix Approach

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Abstract

This paper uses a new framework for solving a class of linear matrix differential equations. For doing so, the operational matrix of the derivative based on the shifted Bernstein polynomials together with the collocation method are exploited to decrease the principal problem to system of linear matrix equations. An error estimation of this method is provided. Numerical experiments are reported to show the applicably and efficiency of the propounded method.

Keywords : Matrix differential equation; Bernstein polynomials; Operational matrix of derivative; Error estimation.

1 Introduction

In this paper, we focus on the following first-order linear matrix differential problem:

$$\begin{cases} P'(t) = A(t)P(t) + B(t), & a \leq t \leq b, \\ P(a) = P_0, \end{cases} \tag{1.1}$$

Where $P \in \mathbb{R}^{p\times q}$ is an unknoun matrix, the matrices $P_0 \in \mathbb{R}^{p\times q}$, $A : [a, b] \to \mathbb{R}^{p\times p}$ and $B : [a, b] \to \mathbb{R}^{p\times q}$ are given. Let us assume that in (1.1), $A, B \in C^s([a, b])$ for $s \geq 1$ which guarantees the existence of a unique and continuously differentiable solution $P(t)$ of (1.1); for more details see [15].

Considering the fundamental role of matrix differential models in the numerous areas of Mathematics, Chemistry, Physics and Engineering, development and implementation of the accurate methods for solving these equations are the subject of interest and have been examined intensively in the literature; see [1, 2, 10, 11, 16, 26] and the references therein.

The matrix differential equations are widely used for modelling the complex real world problems occurring in scientific and engineering applications. Thus these equations are an important aspect which finds many applications. So, we presented to solve the numerical method of this equations. It is clear that the collection methods are the specific type of spectral methods. These methods have been used for solving various types of the differential and integral

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equations. Simple usage and high accuracy are the two main advantages of the collocation methods. In fact, the collocation methods take advantage of an assumption of smoothness of the solution in random space to achieve quick convergence. For further details see [8, 9]. However, the numerical implementation of stochastic collocation is trivial, because it requires only repetitive runs of an existing deterministic solver, similar to Monte Carlo methods. The computational cost of the collocation methods depends on the choice of the collocation points, so we present two different collocation points. The main purpose of the current study is to implement the collocation method for evaluating a rough solution for the linear matrix differential equations given by (1.1). In this regard, first, each entry of the approximate solution $P(t)$ is expanded in terms of the Bernstein polynomials. Afterwards, with the appropriate collocation knots and properties of the Bernstein polynomials, we arrive at a system of linear matrix equations. Therefore, the computations can be simply solved and the unknown coefficients will be calculated by solving the matrix equations. In this paper results error are about 10^{-13}. The structure of this paper is organized as follows. Section 2, presents a brief survey on some preparatory definitions and concepts of the Bernstein polynomials which are required for our subsequent development. Then, we present the operational matrix of derivative of the Bernstein's polynomials. In Section 3, first, it reveals that how the Bernstein polynomials can be implemented to reduce solving (1.1) into resolving matrix equations. In addition, an upper error bound for the approximated solution obtained by our method is established. Section 4 is devoted to reporting some numerical examples which turn out the accuracy of the proposed numerical scheme for solving (1.1). Eventually, the paper is ended with a brief conclusion in Section 5.

2 An overview on Bernstein polynomials

In this section we will first outline some of the basic definitions and properties of Bernstein's polynomials. Then, we derive the operational matrix of derivative of the Bernstein's polynomials.

The Bernstein polynomials of m-th degree are defined on the interval $[0,1]$ as follows [7]

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \quad 0 \le i \le m,$$
$$\binom{m}{i} = \frac{m!}{i!\,(m-i)!}.$$

A recursive definition also can be used to generate the Bernstein polynomials over $[0,1]$, so that the i, m-th degree Bernstein polynomial can be written as

$$B_{i,m}(x) = (1-x)B_{i,m-1}(x) + xB_{i-1,m-1}(x).$$

It can be readily shown that each of the Bernstein polynomials is unity for all real x belonging to the interval $[0,1]$, that is, $\sum_{i=0}^{m} B_{i,m}(x) = 1$. In order to utilize these polynomials on an arbitrary interval $[a,b]$, we define the so-called shifted Bernstein polynomials by applying the change of variable

$$x = \frac{(t-a)}{h}, \qquad a \le t \le b,$$

where $h = b - a$. Consequently,

$$\begin{aligned} B_{i,m}(t) &= \binom{m}{i} \left(\frac{t-a}{h}\right)^i \left(1 - \left(\frac{t-a}{h}\right)\right)^{m-i} \\ &= \frac{1}{h^m} \binom{m}{i} (t-a)^i (b-t)^{m-i}. \end{aligned}$$

Presume that $H := L^2[a,b]$ and

$$Y = \operatorname{span}\{B_{0,m}, B_{1,m}, \cdots, B_{m,m}\},$$

where $m \in N \cup \{0\}$ and $B_{i,m}$'s are the Bernstein polynomials. Since $Y \subset H$ is a finite dimensional vector space, for every $u \in H$, there exists a unique $g \in Y$ such that

$$\|u - g\|_2 \le \|u - y\|_2 \qquad \forall y \in Y,$$

in which $\|u\|_2 = \sqrt{\langle u, u \rangle}$. Here, the function g is called the best approximation to u out of Y. As $g \in Y$, we may conclude that

$$u(t) \approx g(t) = \sum_{j=0}^{m} C_j B_{j,m}(t) = C^T \Psi(t),$$

where

$$\Psi^T(t) = \left(B_{0,m}(t), B_{1,m}(t), \cdots, B_{m,m}(t)\right), \quad (2.2)$$

and $C^T = (c_0, c_1, \cdots, c_m)$ such that C^T uniquely calculated by

$$C^T Q = \int_a^b u(t)\Psi(t)dt, \tag{2.3}$$

where Q is an $(m+1)\times(m+1)$ matrix and is called the dual matrix of $\Psi(t)$ and given by

$$Q = \int_a^b \Psi(t)\Psi^T(t)dt.$$

For more details about best approximation see [18]. The subsequent proposition is useful for the next sections. As in [9], the weighted $L^2_\omega[a,b]$ norm is defined as

$$\|u\|^2_{L^2_\omega[a,b]} = \int_a^b |u(t)|^2\omega(t)dt.$$

Definition 2.1 *A function, $u : [a,b] \to \mathbb{R}$, belongs to Sobolev space $H^{k,2}_\omega$, if its jth weak derivative, lies in $L^2_\omega[a,b]$ for $0 \le j \le k$ with the norm*

$$\|u\|^2_{H^{k,2}_\omega(a,b)} = \sum_{j=0}^{k} \|u^{(j)}\|^2_{L^2_\omega}.$$

Proposition 2.1 *([9]) Assume that $u \in H^{k,2}_\omega(a,b)$, $I_m u$ is the interpolation of u at Chebyshev-Gauss points and the weight function defined as $\omega(t) = (1-t^2)^{-1/2}$. Then*

$$\|u - I_m u\|_{L^2_\omega(a,b)} \le \widetilde{C} h^{\min(k,m)} m^{-k} |u|_{H^{k,2}_\omega(a,b)}, \tag{2.4}$$

where

$$|u|^2_{H^{k,2}_\omega(a,b)} = \sum_{j=\min(k,m+1)}^{k} \|u^{(j)}\|^2_{L^2_\omega},$$

and $\widetilde{C}$ is a constant independent of m and u.

Recently, Yousefi and Behroozifar in [25] have expanded the Bernstein polynomials on the interval $[0,1]$ in terms of Taylor basis. Then proposed a general method for forming operational matrices for these polynomials. We apply an analogous approach for these polynomials on an orbitrary interval $[a,b] \subseteq [0,1]$. By using bionomial expansion of $(1-x)^{m-i} = \sum_{k=0}^{m-i}(-1)^k\binom{m-i}{k}x^k$, we have

$$\begin{aligned} B_{i,m}(t) &= \binom{m}{i}\left(\frac{t-a}{h}\right)^i\left(1-\left(\frac{t-a}{h}\right)\right)^{m-i} \\ &= \binom{m}{i}\left(\frac{t-a}{h}\right)^i \\ &\quad \times\left(\sum_{k=0}^{m-i}(-1)^k\binom{m-i}{k}\left(\frac{t-a}{h}\right)^k\right) \\ &= \sum_{k=0}^{m-i}(-1)^k\binom{m}{i}\binom{m-i}{k}\left(\frac{t-a}{h}\right)^{i+k} \end{aligned}$$

For $i = 0, 1, \cdots, m$, we mention the vector V_{i+1} as follow:

$$V_{i+1} = \left[\overbrace{0, 0, \cdots, 0}^{i \text{ times}}, \frac{(-1)^0}{h^i}\binom{m}{i}, \frac{(-1)^1}{h^{i+1}}\binom{m}{i}\binom{m-i}{1}, \ldots, \frac{(-1)^{m-i}}{h^m}\binom{m}{i}\binom{m-i}{m-i}\right]. \tag{2.6}$$

The definition of the ith-order Bernstein polynomials implies that

$$B_{i,m}(t) = V_{i+1}T_m(t),$$

where

$$T_m(t) = \begin{bmatrix} 1 \\ t-a \\ \vdots \\ (t-a)^m \end{bmatrix}.$$

Let us define the matrix $V \in \mathbb{R}^{(m+1)\times(m+1)}$ such that

$$V = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{m+1} \end{bmatrix}, \tag{2.7}$$

Then above matrix is an upper triangular matrix and it can be verified that V is an invertible matrix. It is not difficult to see that

$$\Psi(t) = VT_m(t), \tag{2.8}$$

where $\Psi(t)$ defined in (2.2).

2.1 Bernstein polynomials operational matrix of differentiation

In this subsection, we want to determin e an explicit formula for Bernstein polynomials of m-th-degree operational matrix of differentiation. Suppose that $D \in \mathbb{R}^{(m+1)\times(m+1)}$ is an operational matrix of differentiation, then $\frac{d}{dt}\Psi(t) = D\Psi(t)$ where $a \leq t \leq b$.

From (2.8) we have,

$$\begin{aligned}\frac{d}{dt}\Psi(t) &= V\begin{bmatrix} 0 \\ 1 \\ 2(t-a) \\ \vdots \\ m(t-a)^{m-1}\end{bmatrix}, \\ &= V\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & m\end{bmatrix} \\ &\times \begin{bmatrix} 1 \\ (t-a) \\ (t-a)^2 \\ \vdots \\ (t-a)^{m-1}\end{bmatrix}, \\ &= V\Delta Z.\end{aligned}$$

where Δ is $(m+1)\times m$ matrix

$$\Delta = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & m\end{bmatrix},$$

and

$$Z = \begin{bmatrix} 1 \\ (t-a) \\ (t-a)^2 \\ \vdots \\ (t-a)^{m-1}\end{bmatrix}.$$

Now, we develop vector Z in terms of $\{B_{i,m}\}_{i=0}^{m}$. By (2.8), we have $T_m(t) = V^{-1}\Psi(t)$, then for $k = 0, 1, \cdots, m$

$$(t-a)^k = V_{k+1}^{-1}\Psi(t), \tag{2.9}$$

where V_{k+1}^{-1} is $(k+1)$-th row of V^{-1} for $k = 0, 1, \cdots, m$, that is

$$V^{-1} = \begin{bmatrix} V_1^{-1} \\ V_2^{-1} \\ \vdots \\ V_{m+1}^{-1}\end{bmatrix}.$$

By using (2.9), we can write $Z = G\Psi(t)$, where

$$G = \begin{bmatrix} V_1^{-1} \\ V_2^{-1} \\ \vdots \\ V_m^{-1}\end{bmatrix},$$

thus

$$\frac{d}{dt}\Psi(t) = V\Delta G\Psi(t),$$

and therefore we have operational matrix of derivative as

$$D = V\Delta G.$$

3 Proposing the main approach for solving the first-order linear matrix differential problem

Let us approximate each of the entries of $P(t) = [p_{ij}(t)]_{p\times q}$ in (1.1), on the interval $[a, b]$ by the Bernstein polynomials. Consequently, we have

$$P(t) \approx \overline{C}(I_q \otimes \Psi(t)). \tag{3.10}$$

where the notation $\otimes$ stands for the well-known Kronecker product, I_q defines the identity matrix of order q, $\Psi(t)$ is given by (2.2) and $\overline{C} \in \mathbb{R}^{p\times q(m+1)}$ is the unknown constant matrix to be determined. The definition of the operational matrix of derivative implies that

$$P'(t) \approx \overline{C}(I_q \otimes D\Psi(t)), \tag{3.11}$$

By substituting Eqs. (3.10) and (3.11) in (1.1), we derive

$$\begin{aligned}\overline{C}(I_q \otimes D\Psi(t)) &= A(t)\overline{C}(I_q \otimes \Psi(t)) \\ &+ B(t) + R_m(t).\end{aligned} \tag{3.12}$$

In order to calculate the unknown cofficients in (3.12), we collocate this equation at m collocation points, which is named t_i and $R_m(t_i) = 0$ for $i = 1, \cdots, m$. We chose the Chebyshev-Gauss nodes in $[a, b]$ as suitable collocation points. Therefore, by replacing the above knots in (3.12), we reach the following coupled linear matrix equations

$$\overline{C}A_i = D_i\overline{C}N_i + F_i, \quad i = 1, \cdots, m,$$

where $A_i = I_q \otimes D\Psi(t_i)$, $D_i = A(t_i)$, $N_i = I_q \otimes \Psi(t_i)$ and $F_i = B(t_i)$. In addition for $i = m+1$ from initial condition, we set $A_{m+1} = [0]_{q(m+1)\times q}$, $D_{m+1} = I_p$, $N_{m+1} = I_q \otimes \Psi(a)$ and $F_{m+1} = -P(a)$. Hence, in order to numerically solve the problem (1.1), we may solve the following coupled linear matrix equation

$$XA_i - D_iXN_i = F_i, \; i = 1, \cdots, m+1, \quad (3.13)$$

where A_i, D_i, N_i and F_i are constant matrices and the unknown matrix $X := \overline{C}$ is to be determined. By using the following relation (see [6])

$$vec(AXB) = (B^T \otimes A)vec(X),$$

it can be found that the coupled matrix equations (3.13) are equivalent to the following linear system $Ax = b$, with subsequent parametrs

$$\begin{pmatrix} A_1^T \otimes I_p - N_1^T \otimes D_1 \\ \vdots \\ A_{m+1}^T \otimes I_p - N_{m+1}^T \otimes D_{m+1} \end{pmatrix} vec(X)$$

$$= \begin{pmatrix} vec(F_1) \\ \vdots \\ vec(F_{m+1}) \end{pmatrix}, \quad (3.14)$$

The above linear system can be solved via the classical methods such as the GMRES or conjugate gradient method [20]. However, the size of the coefficient matrix of the system (3.14) is $pq(m+1)$ and it may become too large even for moderate values of p, q and m. This stimulates us to use an iterative method for solving the coupled linear matrix equations (3.13) rather than the linear system. In the literature, a large number of papers are devoted to applying different kinds of iterative algorithm for solving various linear coupled matrix equations, for more details see [3, 4, 5, 13, 14, 23] and the references therein.

3.1 Implementing the method

For solving (1.1), we use a step-by-step method. To do so, we first choose a step length $h \neq 0$ and consider the points $x_i = x_0 + ih$, $i = 0, 1, 2, \cdots$. Then, starting with the given initial values $x_0 := a$, $W_0 := P(x_0)$. Now, by using the approach described in the previous section we solve the following matrix differential equation

$$\begin{cases} W'(t) = A(t)W(t) + B(t), & x_i \le t \le x_{i+1}, \\ W(x_i) = W_i, \end{cases}$$

and successively compute the rough solution $W(t)$ to $P(t)$ on $[x_i, x_{i+1})$ for $i = 0, 1, \cdots, [\frac{1}{h}] - 1$. Afterward we set $W_{i+1} = W(x_{i+1})$, to compute the approximate solution $W(t)$ of $P(t)$ on the next subinterval.

3.2 Estimation of an upper error bound

In this subsection,we present an upper error bound analytically that reveals the spectral rate of convergence. In what follows, the (i, j)th entry of the matrix $P(t)$ is denoted by $p_{ij}(t)$.

Definition 3.1 *Let $U(t) = [u_{ij}(t)]$ be an arbitrary $p \times q$ matrix defined on the interval $[a, b]$ such that $u_{ij}(t) \in L^2[a, b]$. Then, we define*

$$\|U\|_\infty = \max_{i,j} \|u_{ij}\|_{L^2_\omega}, \quad 1 \le i \le p, \quad 1 \le j \le q.$$

Theorem 3.1 *Consider the problem (1.1) where $p_{ij} \in H^{k,2}_\omega(x_l, x_{l+1})$, $A(t) = [a_{ij}(t)]_{p\times p}$ and $B(t) = [b_{ij}(t)]_{p\times q}$ are given such that $a_{ij}(t)$ and $b_{ij}(t)$ are sufficiently smooth. In addition, assume that $P_m = \overline{C}(I_q \otimes \Psi)$ stands for the Bernstein collocation approximation of P. Furthermore, suppose that $M_1 = \max_{i,j} \max_{t\in(x_l,x_{l+1})} |a_{ij}(t)|$ and $\widetilde{C}_1$ and $\widetilde{C}_2$ are constants independent of m and u. Then $\lim_{m\to\infty} P_m(t) = P(t)$. Besides the following statement holds:*

$$\begin{aligned} \|P - P_m\|_\infty \le\; & \widetilde{C}_1 M_1 h^{\min(k,m)+1} m^{-1-k} \\ & \times \max_j \sum_{\nu=1}^{p} |p_{\nu j}|_{H^{k,2}_\omega(x_l,x_{l+1})} \\ & + \widetilde{C}_2 h^{\min(k,m)} m^{-k} \\ & \times \max_{ij} |p_{ij}|_{H^{k,2}_\omega(x_l,x_{l+1})}. \end{aligned}$$

Integrating (1.1) in $[x_l, t]$ results in

$$P(t)=\int_{x_l}^{t}(A(x)P(x)+B(x))dx+P(x_l). \quad (3.15)$$

Since we assume that $P_m(x_l)=P(x_l)$, we can rewrite (3.13) as follows:

$$\begin{aligned} P_m(\xi_n) &= \int_{x_l}^{\xi_n}(A(x)P_m(x)+B(x))dx \\ &+ P(x_l), \quad n=1,\cdots,m+1, \end{aligned} \quad (3.16)$$

where ξ_n, $n=1,\cdots,m$, Chebyshev-Gauss knots on the interval $[x_l, x_{l+1}]$ and $\xi_{m+1}=x_l$.

It follows from (3.16) that

$$\begin{aligned} P_m(\xi_n) &= \int_{x_l}^{\xi_n} A(x)H(x)dx \\ &+ \int_{x_l}^{\xi_n}(A(x)P(x)+B(x))dx \\ &+ P(x_l), \end{aligned} \quad (3.17)$$

such that $H=[h_{ij}]_{p\times q}=P_m-P$. Multiplying both sides of the n-th equation of (3.17) by Largrange interpolating polynomial, L_n and summing up over n from 1 to $m+1$ results in

$$\begin{aligned} \sum_{n=1}^{m+1} L_n(t)P_m(\xi_n) = & \\ & \sum_{n=1}^{m+1} L_n(t)\int_{x_l}^{\xi_n} A(x)H(x)dx \\ + & \sum_{n=1}^{m+1} L_n(t)(\int_{x_l}^{\xi_n}(A(x)P(x)+B(x))dx \\ + & P(x_l)). \end{aligned}$$

Subtracting from (3.15) yields

$$\begin{aligned} \sum_{n=1}^{m+1} L_n(t)P_m(\xi_n)-P(t) &= \int_{x_l}^{t} A(x)H(x)dx \\ &+ E_2(t)+E_1(t), \end{aligned} \quad (3.18)$$

where

$$E_1(t)=$$

$$\sum_{n=1}^{m+1} L_n(t)\int_{x_l}^{\xi_n} A(x)H(x)dx-\int_{x_l}^{t} A(x)H(x)dx,$$

and

$$E_2(t)=$$

$$\begin{aligned} & \sum_{n=1}^{m+1} L_n(t)(\int_{x_l}^{\xi_n}(A(x)P(x)+B(x))dx+P(x_l)) \\ - & \int_{x_l}^{t}(A(x)P(x)+B(x))dx-P(x_l). \end{aligned}$$

We may rewrite (3.18) in the following form

$$H(t)=\int_{x_l}^{t} A(x)H(x)dx+S(t), \quad (3.19)$$

By implying Gronwall inequality in [9] on (3.19) we have

$$\|H\|_\infty \leq C\|S\|_\infty. \quad (3.20)$$

Since we assume that the A and B are sufficiently smooth, for $E_1(t)$ and $E_2(t)$ we obtain the following results. From Definition (3.1)

$$\|E_1\|_\infty = \max_{ij}\|I_m u-u\|_{L^2_\omega},$$

in which $u(t)=\int_{x_l}^{t}\sum_{\nu=1}^{p} a_{i\nu}(x)h_{\nu j}(x)dx$. Using (2.4) for $k=1$, it can be deduced that

$$\begin{aligned} \|E_1\|_\infty &= \widetilde{C}_1 h m^{-1}\max_{ij}\|\sum_{\nu=1}^{p} a_{i\nu}h_{\nu j}\|_{L^2_\omega} \\ &\leq \widetilde{C}_1 M_1 h^{\min(k,m)+1} m^{-1-k} \\ & \quad \max_{j}\sum_{\nu=1}^{p}|p_{\nu j}|_{H^{k,2}_\omega(x_l,x_{l+1})}. \end{aligned} \quad (3.21)$$

Also, for $E_2(t)$, we derive that

$$\begin{aligned} \|E_2\|_\infty &= \max_{ij}\|I_m p_{ij}-p_{ij}\|_{L^2_\omega}, \\ &\leq \widetilde{C}_2 h^{\min(k,m)} m^{-k} \\ & \quad \max_{ij}|p_{ij}|_{H^{k,2}_\omega(x_l,x_{l+1})}. \end{aligned} \quad (3.22)$$

Now the assertion can be concluded from (3.20), (3.21) and (3.22).

4 Numerical experiments

In this section, several numerical examples are presented to demonstrate the capability and accuracy of the present method for solving (1.1). All of the numerical computations are performed using Mathematica 10 with a machine unit round off precision of around 10^{-16}. The computer specifications are Microsoft Windows 7 32-bit , Intel(R)Core(TM)i5 CPU 2.27GHz, with 4GB of RAM . The comparison results for $m = 3, 4, 5, 6$ between the method of [11, 12] and present method can be seen in Tables 1 to 5 . The following examples are chosen from [11, 12].

Example 4.1 *Consider the following first-order linear matrix differential problem*

$$\begin{cases} P'(t) = A(t)P(t) + B(t), \quad 0 \le t \le 1, \\ P(0) = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \end{cases} \tag{4.23}$$

where $A(t) = \begin{pmatrix} 1 & -1 \\ 1 & e^t \end{pmatrix}$, $B(t) = \begin{pmatrix} -3e^{-t} - 1 & 2 - 2e^{-t} \\ -3e^{-t} - 2 & 1 - 2\cosh(t) \end{pmatrix}$. The exact solution of equation (4.23) is equal $P(t) = \begin{pmatrix} 2e^{-t} + 1 & e^{-t} - 1 \\ e^{-t} & 1 \end{pmatrix}$.

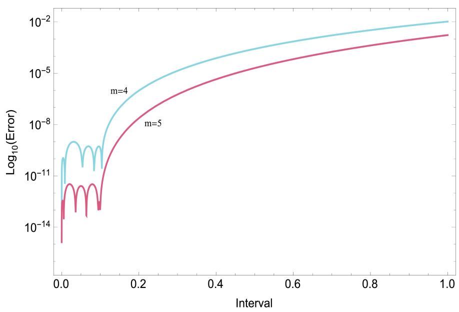

Figure 1: Approximated solutions of example 4.1 for Shifted Chebyshev points.

The computational results for various m have been reported in Table 1. The logarithmic absolute errors are computed in Fig . 1 . The time required to execute this algorithm, when $m = 5$, was 2 : 56 seconds. These results show that the results obtained from the method described in this paper are highly accurate .

Example 4.2 *Consider the following first-order linear matrix differential problem*

$$\begin{cases} P'(t) = \frac{1}{t^3 - t - 1} \begin{pmatrix} 2t^2 - 1 & t^2 - 2t - 1 \\ -t - 1 & t^3 + t^2 - t - 1 \end{pmatrix} P(t), \\ P(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0 \le t \le 1, \end{cases} \tag{4.24}$$

The exact solution of equation (4.24) is equal $P(t) = \begin{pmatrix} e^t \\ te^t \end{pmatrix}$.

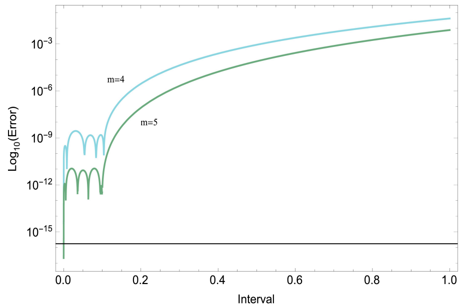

Figure 2: Approximated solutions of Example 4.2.

The computational results between the Spline method of [11] and present method have been reported in Table 2. The logarithmic absolute errors are computed in Fig . 2 . The time required to execute this algorithm , when $m = 5$, was 2 : 23 seconds .

Example 4.3 *Consider the following first-order linear matrix differential problem*

$$\begin{cases} P'(t) = A(t)P(t) + B(x), \quad 0 \le t \le 1, \\ P(0) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}, \end{cases} \tag{4.25}$$

Where

$$A(t) = \begin{pmatrix} -1 - t & 0 & -1 + e^t + t \\ e^t & -t & 1 \\ 0 & -1 & e^t \end{pmatrix}$$

and

$B(t) =$

$$\begin{pmatrix} -2 + (-3 + e^t)t & -1 - t - t^2 - e^t(2 + t) \\ t + e^t(1 + t) & e^{2t} + te^t - (5 + t)(1 + t^2) \\ -1 + te^t & 1 - t(5 + t) \end{pmatrix}$$

Table 1: The maximal absolute error of example 4.1

Interval	[0, 0.1]	[0.1, 0.2]	[0.2, 0.3]	[0.3, 0.4]	[0.4, 0.5]
Method [11] for ($m = 4$)	5.06×10^{-8}	1.02×10^{-7}	1.54×10^{-7}	2.10×10^{-7}	2.70×10^{-7}
Current method for ($m = 4$)	5.52×10^{-11}	4.79×10^{-8}	8.25×10^{-8}	6.87×10^{-9}	7.16×10^{-8}
Method [11] for ($m = 5$)	5.75×10^{-10}	1.36×10^{-9}	2.06×10^{-9}	2.80×10^{-9}	3.60×10^{-9}
Current method for ($m = 5$)	1.31×10^{-12}	6.57×10^{-10}	1.29×10^{-9}	6.60×10^{-11}	9.71×10^{-10}
Interval	[0.5, 0.6]	[0.6, 0.7]	[0.7, 0.8]	[0.8, 0.9]	[0.9, 1]
Method [11] for ($m = 4$)	3.38×10^{-7}	4.19×10^{-7}	5.21×10^{-7}	6.59×10^{-7}	8.51×10^{-7}
Current method for ($m = 4$)	3.71×10^{-6}	4.36×10^{-5}	2.62×10^{-5}	1.07×10^{-4}	3.41×10^{-4}
Method [11] for ($m = 5$)	4.50×10^{-9}	5.57×10^{-9}	6.93×10^{-9}	8.75×10^{-9}	1.13×10^{-8}
Current method for ($m = 5$)	1.30×10^{-7}	2.79×10^{-6}	2.66×10^{-5}	1.57×10^{-4}	6.83×10^{-4}

Table 2: The maximal absolute error of example 4.2

Interval	[0, 0.1]	[0.1, 0.2]	[0.2, 0.3]	[0.3, 0.4]	[0.4, 0.5]
Method [11] for ($m = 4$)	1.14×10^{-7}	2.62×10^{-7}	4.51×10^{-7}	6.89×10^{-7}	9.89×10^{-7}
Current method for ($m = 4$)	2.19×10^{-10}	1.61×10^{-7}	3.45×10^{-7}	3.61×10^{-8}	4.02×10^{-7}
Method [11] for ($m = 5$)	1.80×10^{-9}	4.09×10^{-9}	7.00×10^{-9}	1.07×10^{-8}	1.53×10^{-8}
Current method for ($m = 5$)	4.37×10^{-12}	2.65×10^{-9}	6.48×10^{-9}	3.97×10^{-10}	7.49×10^{-9}
Interval	[0.5, 0.6]	[0.6, 0.7]	[0.7, 0.8]	[0.8, 0.9]	[0.9, 1]
Method [11] for ($m = 4$)	1.36×10^{-6}	1.83×10^{-6}	2.37×10^{-6}	3.05×10^{-6}	3.86×10^{-7}
Current method for ($m = 4$)	2.65×10^{-5}	3.98×10^{-5}	2.99×10^{-5}	1.52×10^{-5}	6.04×10^{-5}
Method [11] for ($m = 5$)	2.10×10^{-8}	2.80×10^{-8}	3.65×10^{-8}	4.67×10^{-8}	5.90×10^{-8}
Current method for ($m = 5$)	1.09×10^{-7}	2.98×10^{-6}	3.55×10^{-6}	2.62×10^{-6}	1.41×10^{-6}

Table 3: The maximal absolute error of example 4.3

Interval	[0, 0.1]	[0.1, 0.2]	[0.2, 0.3]	[0.3, 0.4]	[0.4, 0.5]
Current method for ($m = 3$)	1.33×10^{-8}	1.68×10^{-6}	3.25×10^{-6}	4.74×10^{-7}	3.80×10^{-6}
Method [12] for ($m = 5$)	1.39×10^{-6}	1.39×10^{-6}	1.43×10^{-6}	1.43×10^{-6}	1.49×10^{-6}
Current method for ($m = 5$)	7.10×10^{-13}	4.29×10^{-10}	1.03×10^{-9}	5.32×10^{-11}	1.17×10^{-9}
Interval	[0.5, 0.6]	[0.6, 0.7]	[0.7, 0.8]	[0.8, 0.9]	[0.9, 1]
Current method for ($m = 3$)	1.14×10^{-5}	9.07×10^{-5}	4.19×10^{-5}	1.44×10^{-5}	4.09×10^{-5}
Method [12] for ($m = 5$)	1.49×10^{-6}	1.57×10^{-6}	1.57×10^{-6}	1.65×10^{-6}	1.65×10^{-6}
Current method for ($m = 5$)	4.53×10^{-6}	5.31×10^{-6}	3.86×10^{-6}	2.62×10^{-6}	7.67×10^{-7}

. The exact solution of equation (4.25) is equal $P(t) = \begin{pmatrix} 1+t & e^t + t \\ 0 & -1+5t+t^2 \\ t & 0 \end{pmatrix}$. The logarithmic absolute errors are plotted in Fig . 3 . The time required for this algorithm was 1 : 25 seconds . As the results in Table 3 show that, this method is a highly efficient and accurate method for solving the first-order linear matrix differential equations .

Example 4.4 *Consider the following first-order linear matrix differential problem*

$$\begin{cases} P'(t) = A(t)P(t) + B(t), \quad 0 \le t \le 1, \\ P(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{cases} \tag{4.26}$$

Where

$$A(t) = \begin{pmatrix} 0 & te^t \\ t & 0 \end{pmatrix}$$

and

Table 4: The approximate solution for example 4.4

Interval	Approximation value for $m = 4$
$[0, 0.1]$	$\begin{pmatrix} 1 - t + 0.4t^2 - 0.1t^3 + 0.1t^4 & -1.4 \times 10^{-15}t + 7.5 \times 10^{-14}t^2 - 1.1 \times 10^{-12}t^3 + 4.9 \times 10^{-12}t^4 \\ t - 1.4 \times 10^{-9}t^2 + 3.1 \times 10^{-8}t^3 - 1.6 \times 10^{-7}t^4 & 1 + 1.4 \times 10^{-14}t - 4.5 \times 10^{-13}t^2 + 5.4 \times 10^{-12}t^3 - 1.1 \times 10^{-11}t^4 \end{pmatrix}$
$[0.1, 0.2]$	$\begin{pmatrix} 1 - 0.9t + 0.4t^2 - 0.1t^3 + 0.1t^4 & 8.8 \times 10^{-14}t - 9.1 \times 10^{-13}t^2 + 4.1 \times 10^{-12}t^3 - 7.1 \times 10^{-12}t^4 \\ -1.8 \times 10^{-10} + t - 5.7 \times 10^{-8}t^2 + 2.6 \times 10^{-7}t^3 - 4.4 \times 10^{-7}t^4 & 1 - 4.5 \times 10^{-13}t + 2.7 \times 10^{-12}t^2 - 1.8 \times 10^{-11}t^3 + 2.5 \times 10^{-11}t^4 \end{pmatrix}$
$[0.2, 0.3]$	$\begin{pmatrix} 0.9 - 0.9t + 0.4t^2 - 0.1t^3 + 0.1t^4 & -6.4 \times 10^{-13}t + 3.9 \times 10^{-12}t^2 - 1.1 \times 10^{-11}t^3 + 1.1 \times 10^{-11}t^4 \\ -2.4 \times 10^{-9} + t - 2.4 \times 10^{-7}t^2 + 6.7 \times 10^{-7}t^3 - 6.7 \times 10^{-7}t^4 & 1 + 4.5 \times 10^{-13}t + 3.6 \times 10^{-12}t^2 + 7.2 \times 10^{-12}t^4 \end{pmatrix}$
$[0.3, 0.4]$	$\begin{pmatrix} 0.9 - 0.9t + 0.4t^2 - 0.1t^3 + 0.1t^4 & -2.4 \times 10^{-14}t + 3.8 \times 10^{-13}t^2 - 6.1 \times 10^{-13}t^3 + 4.7 \times 10^{-13}t^4 \\ -1.2 \times 10^{-8} + t - 6.2 \times 10^{-7}t^2 + 1.1 \times 10^{-6}t^3 - 8.5 \times 10^{-7}t^4 & 1 + 3.6 \times 10^{-12}t - 1.1 \times 10^{-11}t^2 + 1.4 \times 10^{-11}t^3 - 1.8 \times 10^{-11}t^4 \end{pmatrix}$
$[0.4, 0.5]$	$\begin{pmatrix} 0.9 - 0.9t + 0.4t^2 - 0.1t^3 + 0.1t^4 & 4.8 \times 10^{-13}t - 1.5 \times 10^{-12}t^2 + 2.4 \times 10^{-12}t^3 - 1.4 \times 10^{-12}t^4 \\ -3.9 \times 10^{-8} + t - 1.2 \times 10^{-6}t^2 + 1.7 \times 10^{-6}t^3 - 9.9 \times 10^{-7}t^4 & 1 + 1.8 \times 10^{-12}t - 7.2 \times 10^{-12}t^2 + 2.9 \times 10^{-11}t^3 - 7.2 \times 10^{-12}t^4 \end{pmatrix}$
$[0.5, 0.6]$	$\begin{pmatrix} 0.9 - 0.9t + 0.4t^2 - 0.1t^3 + 0.1t^4 & 3.1 \times 10^{-13}t + 2.6 \times 10^{-13}t^2 + 3.3 \times 10^{-13}t^3 - 2.8 \times 10^{-13}t^4 \\ -9.9 \times 10^{-8} + t - 1.9 \times 10^{-6}t^2 + 2.4 \times 10^{-6}t^3 - 1.1 \times 10^{-6}t^4 & 1 - 7.2 \times 10^{-12}t + 3.6 \times 10^{-11}t^2 - 1.4 \times 10^{-11}t^3 + 1.1 \times 10^{-11}t^4 \end{pmatrix}$
$[0.6, 0.7]$	$\begin{pmatrix} 0.9 - 0.9t + 0.4t^2 - 0.1t^3 + 0.1t^4 & -7.4 \times 10^{-13}t + 6.7 \times 10^{-12}t^2 - 4.9 \times 10^{-12}t^3 + 1.5 \times 10^{-12}t^4 \\ -2.1 \times 10^{-7} + t - 2.9 \times 10^{-6}t^2 + 3.1 \times 10^{-6}t^3 - 1.1 \times 10^{-6}t^4 & 1 + 2.9 \times 10^{-11}t - 2.9 \times 10^{-11}t^2 + 2.9 \times 10^{-11}t^3 - 1.4 \times 10^{-11}t^4 \end{pmatrix}$
$[0.7, 0.8]$	$\begin{pmatrix} 0.9 - 0.9t + 0.4t^2 - 0.1t^3 + 0.1t^4 & 1.9 \times 10^{-12}t + 5.8 \times 10^{-12}t^2 - 2.5 \times 10^{-12}t^3 + 3.1 \times 10^{-13}t^4 \\ -3.8 \times 10^{-7} + t - 4.1 \times 10^{-6}t^2 + 3.6 \times 10^{-6}t^3 - 1.2 \times 10^{-6}t^4 & 1 + 2.9 \times 10^{-11}t - 1.4 \times 10^{-11}t^2 + 4.3 \times 10^{-11}t^3 - 1.4 \times 10^{-11}t^4 \end{pmatrix}$
$[0.8, 0.9]$	$\begin{pmatrix} 0.9 - 0.9t + 0.4t^2 - 0.1t^3 + 0.1t^4 & 5.8 \times 10^{-12}t + 1.1 \times 10^{-11}t^2 - 5.1 \times 10^{-12}t^3 + 6.8 \times 10^{-13}t^4 \\ -6.5 \times 10^{-7} + t - 5.4 \times 10^{-6}t^2 + 4.2 \times 10^{-6}t^3 - 1.2 \times 10^{-6}t^4 & 1 - 1.1 \times 10^{-10}t + 1.7 \times 10^{-10}t^2 - 1.1 \times 10^{-10}t^3 + 3.2 \times 10^{-11}t^4 \end{pmatrix}$
$[0.9, 1]$	$\begin{pmatrix} 0.9 - 0.9t + 0.4t^2 - 0.1t^3 + 0.1t^4 & -9.4^{-13}t + 2.4 \times 10^{-11}t^2 - 1.4 \times 10^{-11}t^3 + 3.2 \times 10^{-12}t^4 \\ -1.03 \times 10^{-6} + t - 6.9 \times 10^{-6}t^2 + 4.8 \times 10^{-6}t^3 - 1.2 \times 10^{-6}t^4 & 1 - 1.4 \times 10^{-11}t + 5.8 \times 10^{-11}t^2 - 1.4 \times 10^{-11}t^3 + 7.2 \times 10^{-12}t^4 \end{pmatrix}$

Table 5: The maximal absolute error of example 4.4

Interval	$[0, 0.1]$	$[0.1, 0.2]$	$[0.2, 0.3]$	$[0.3, 0.4]$	$[0.4, 0.5]$
Current method for $(m = 5)$	6.57×10^{-13}	3.28×10^{-10}	6.48×10^{-10}	3.66×10^{-11}	5.74×10^{-10}
Current method for $(m = 6)$	3.58×10^{-15}	3.97×10^{-12}	9.13×10^{-12}	9.62×10^{-11}	4.01×10^{-10}
Interval	$[0.5, 0.6]$	$[0.6, 0.7]$	$[0.7, 0.8]$	$[0.8, 0.9]$	$[0.9, 1]$
Current method for $(m = 5)$	6.51×10^{-8}	1.39×10^{-6}	1.33×10^{-5}	7.88×10^{-5}	3.42×10^{-5}
Current method for $(m = 6)$	2.68×10^{-9}	6.90×10^{-8}	1.14×10^{-6}	9.63×10^{-6}	5.70×10^{-6}

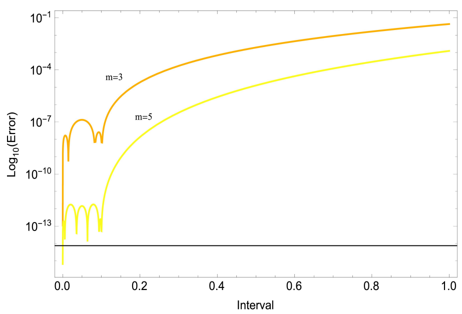

Figure 3: Approximated solutions of Example 4.3.

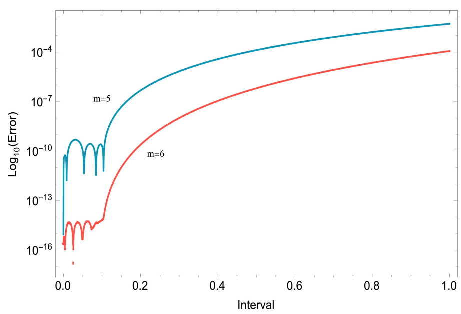

Figure 4: Approximated solutions of Example 4.4.

$$B(t) = \begin{pmatrix} -e^t(1+t^2) & -te^{-t} \\ 1 - te^{-t} & 0 \end{pmatrix}$$

.

The exact solution of equation (4.26) is equal $P(t) = \begin{pmatrix} e^{-t} & 0 \\ t & 1 \end{pmatrix}$.

In Table 4, we report the approximate solution to the exact solution of the fourth example computed by present method. The maximal absolute error have been reported in Table 5. The logarithmic absolute errors are computed in Fig. 2 .

5 Conclusion

The properties of the Bernstein polynomials and their operational matrices of derivative have been utilized to numerically solve a class of the first-order matrix differential problems. The proposed approach reduces the main problem to a linear coupled matrix equations. An upper bound for error of offered method was presented. Numerical examples have illustrated to demonstrate the efficiency and applicably of our approach. Finally, we showed that the proposed new strategy can be examined for more complicated types of matrix differential models.

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