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Research Article

Inverse DEA Using Enhanced Russell Measure in the Presence of Fuzzy Data

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Abstract

The present study deals with the inverse DEA using the non-radial Enhanced Russell (ER)-measure in the presence of fuzzy data. This paper proposes a technique to treat the fuzzy data in the problem of simultaneous estimation of input-output levels. Necessary and sufficient conditions are provided for ER-measure maintaining in the presence of fuzzy data. A numerical example with real data is presented to show the accuracy of the proposed method.

Keywords : Data Envelopment Analysis (DEA); Inverse DEA; Fuzzy Decision Making; Multiple-Objective Linear Programming(MOLP); Enhanced Russell(ER)-Measure.

1 Introduction

Data Envelopment Analysis is a useful non-parametric technique to estimate relative efficiencies of a set of decision making units (DMUs) in a multiple-input multiple-output production technology. In traditional DEA, it is assumed that all input and output data are exactly known. DEA technique was initially suggested by Charnes et al. [3] (CCR model) and was extended by many scholars, see e.g. [5, 6, 18] for some reviews.

The idea of the inverse DEA was first introduced by Zhang and Cui [37]. They measured the input increase of a particular DMU under its given output increase and preserving the CCR efficiency. After introducing inverse DEA, some scholars investigated the problem more and established various results in different frameworks, see, e.g. [1, 4, 7, 10, 27, 35]. The problem of output-estimation under preserving the efficiency is studied by Wei et al. [34] (The first Question in inverse DEA). The problem of input-estimation is discussed by Hadi-Vencheh et al. [16, 17], provided that the DMU maintains its current efficiency level (The second Question in inverse DEA). Both problems, input-estimation and output-estimation, provided that the DMU maintains or improves its current efficiency level, are studied by Jahanshahloo et al. [20, 21]. Both problems are investigated under inter-temporal dependence by Jahanshahloo et al. [24]. They used MOLP models and established necessary and sufficient conditions for input/output estimation. In addition, the problem of simultaneous estimation of input-output levels is studied by Jahanshahloo et al. [22] (The third Question in inverse DEA). They used multiple-objective lin-

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ear programming tools for input-output estimating under preserving the efficiency score. This Question was studied by Ghobadi under improving the efficiency score [11]. The aim of the paper is possible extensions and applications of the existing approaches in a different framework.

Although traditional DEA models evaluate the DMUs with the exactly known data, in recent efforts, some researchers have dealt with assessing the DMUs in the presence of fuzzy data, using an analytical framework of DEA: fuzzy DEA. Fuzzy DEA is one of the most noticeable subjects for performance analysis under imprecise data both practically and theoretically. Emrouznejad and Tavana [9] reviewed the literature on the fuzzy DEA methods and provided a classification scheme with six categories: (i) The tolerance approach; (ii) The q-level-based approach; (iii) The fuzzy ranking approach; (iv) The possibility approach; (v) The fuzzy arithmetic; and (vi) The fuzzy random/type-2 fuzzy set. The use of fuzzy set theory, fuzzy decision making, and fuzzy hybrid solutions in various models for evaluating and selecting suppliers in a 50-year period have been investigated by Dragan Simic et al. [31]. Kao and Liu[25, 26] applied the notion of fuzziness and converted a fuzzy DEA model to a family of traditional DEA models by applying the α-cut approach and Zadeh's extension principle [36]. Some scholars have suggested fuzzy DEA models, using the concept of comparing fuzzy numbers, including [14, 32]. Hougaard [19] proposed the application of efficiency aggregated, using the value judgments or manager's opinions. Guo and Tanaka [15] extended the CCR efficiency score in traditional DEA to fuzzy DEA and studied the relationship between DEA and regression analysis. Soleimani-damaneh [32] studied the additive model as a free coordinate, orientation-less, and slack-based model in traditional DEA and extended this model to be a fuzzy DEA model, utilizing a fuzzy signed distance and fuzzy upper bound concept. The production possibility set is extended by Allahviranloo et al. to the fuzzy production possibility set, using extension principle[2]. They proposed the CCR fuzzy model which satisfies the initial concepts with crisp data. The problem of input-estimation in traditional DEA was extended to fuzzy DEA by Ghobadi and Jahangiri [12]; they provided a sufficient condition for efficiency maintaining in the presence of fuzzy data. Both problems, input-estimation and output-estimation, were extended under inter-temporal dependence in the presence of fuzzy data by Ghobadi et al. [13]. In this paper, we extend the third Question in inverse DEA, which has been provided by Jahanshahloo et al. [22], to a fuzzy framework. For this purpose, the non-radial ER-measure model [30] is extended to the fuzzy framework. The technique proposed to treat the fuzzy data in the problem of simultaneous estimation of input-output levels is using fuzzy ER-measure model. To answer the third Question in inverse DEA with fuzzy data, necessary and sufficient conditions are proposed, utilizing Pareto and weak Pareto solutions of fuzzy MOLP problems. A numerical example with real data is provided to confirm the credibility and applicability of our method. The given results are important theoretically, because these provide some theoretical extensions of inverse DEA theory in the presence of fuzzy data, which can expand the application area of inverse DEA. They might be practically significant as well because they can be used to make better decisions in order to extend DMUs with fuzzy data.

The structure of the paper is as follows: Some preliminaries from fuzzy decision-making theory and inverse DEA are presented in Section 2. Section 3 is devoted to the main results of the paper. This section provides some theoretical extensions of inverse DEA theory in the presence of fuzzy data, which can expand the application area of inverse DEA. Necessary and sufficient conditions for simultaneous estimation of input-output levels are proposed in the presence of fuzzy data. Moreover, a numerical example with real data is presented to illustrate the purpose of this research. Finally, concluding remarks and directions for future research are given in Section 4.

2 Preliminaries

2.1 Fuzzy Numbers in Fuzzy Decision Making

The concept of the fuzzy sets first introduced by Zadeh [36]. He suggested fuzzy sets as sets with boundaries that were not precise. In this subsection, we review some of the basic concept of the fuzzy numbers needed through the paper.

Let X be a nonempty topological space. The set of all fuzzy subsets $\tilde{A}$ of X, we denote by $F(X)$, where every fuzzy subsets $\tilde{A}$ of X is uniquely determined by the membership function $\mu_{\tilde{A}} : X \longrightarrow [0,1]$, and $[0,1] \subset \mathbb{R}$ is a unit interval, $\mathbb{R}$ is the Euclidean space of real number. For each $\alpha \in [0,1]$, the α-cut set of $\tilde{A}$ is defined as follows:

$$[\tilde{A}]_\alpha = \begin{cases} \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\} \ if \ \alpha \in (0,1], \\ \\ cl\{x \in X \mid \mu_{\tilde{A}}(x) > \alpha\} \ if \ \alpha = 0, \end{cases} \tag{2.1}$$

where clB means a topological closure of B, $B \subset X$. $[\tilde{A}]_\alpha^L$ and $[\tilde{A}]_\alpha^U$ are represented the lower and upper endpoint of any α-cut set of $\tilde{A}$, respectively. Also, $[\tilde{A}]_0$ is called support of $\tilde{A}$ and denoted by $supp\tilde{A}$. Furthermore, if $[\tilde{A}]_\alpha$ for every $\alpha \in [0,1]$, are closed, bounded, compact or convex, then fuzzy subset $\tilde{A}$ of X is closed, bounded, compact or convex, respectively. Moreover, if $[\tilde{A}]_\alpha \neq \emptyset$ then $\tilde{A}$ is called normal. A fuzzy number is defined as follows:

Definition 2.1 *fuzzy set $\tilde{A}$ of $\mathbb{R}$ is called a fuzzy number, satisfying three conditions:*

(a) $\mu_{\tilde{A}}$ be an upper semi-continuous function on $\mathbb{R}$,

(b) $supp\tilde{A}$ be a compact interval,

(c) if $supp\tilde{A} = [a,b]$ then there exist c,d, in which, $a \leq c \leq d \leq b$ and $\mu_{\tilde{A}}$ is non-decreasing on the interval $[a,c]$, equal to 1 on the interval $[c,d]$, and non-increasing on the interval $[d,b]$.

If $[\tilde{A}]_0^L > 0$, then fuzzy number $\tilde{A}$ is called a positive fuzzy number. Let $L, R : [0,1] \longrightarrow [0,1]$, with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$, are non-increasing, continuous shape functions. Also, let $F(\mathbb{R})$ be the family of fuzzy numbers on $\mathbb{R}$. An L-R fuzzy number $\tilde{A}$ is denoted as $(a,\beta,\gamma)_{L-R}$ and defined with the following membership function:

$$\mu_{\tilde{A}}(x) = \begin{cases} L(\frac{a-x}{\beta}) & if \quad a-\beta \leq x \leq a, \\ \\ 1 & if \quad x = a \\ \\ R(\frac{x-a}{\gamma}) & if \quad a \leq x \leq a+\gamma, \\ \\ 0 & otherwise, \end{cases}$$

where $\beta, \gamma > 0$ are positive scalers and "a" is the center of $\tilde{A}$. The set all L-R fuzzy numbers denote by $FNLR(\mathbb{R})$. For each $\alpha \in [0,1]$, it is easy see that

$$[\tilde{A}]_\alpha = [a - L^{-1}(\alpha)\beta, \quad a + R^{-1}(\alpha)\gamma]. \tag{2.2}$$

Let $\tilde{A} = (a,\beta,\gamma)_{L-R}$ and $\tilde{B} = (b,\eta,\mu)_{L-R}$ be two L-R fuzzy numbers and λ a non-negative real number. Then:

$$\begin{cases} \tilde{A} \oplus \tilde{B} = (a+b, \beta+\eta, \gamma+\mu)_{L-R}, \\ \\ \lambda \cdot \tilde{A} = (\lambda a, \lambda\beta, \lambda\gamma)_{L-R}. \end{cases} \tag{2.3}$$

The most used operator in the context of fuzzy decision making, is the min T-norm operator. Considering this operator to evaluate a linear combination of fuzzy quantities $\lambda_1 \cdot \tilde{A}_1 \oplus \lambda_2 \cdot \tilde{A}_2 \oplus \ldots \oplus \lambda_n \cdot \tilde{A}_n$, where $\tilde{A}_j = (a_j, \beta_j, \gamma_j)_{L-R}$; $j = 1,\ldots,n$ are L-R fuzzy numbers with common shape functions (L and R) and λ_j; $j = 1,\ldots,n$ are positive scalars, we have

$$\begin{aligned} \lambda_1 \cdot \tilde{A}_1 \oplus \lambda_2 \cdot \tilde{A}_2 \oplus \ldots \oplus \lambda_n \cdot \tilde{A}_n \\ = \sum_{j=1}^{n} \lambda_j \cdot \tilde{A}_j \\ = (\sum_{j=1}^{n} \lambda_j a_j, \sum_{j=1}^{n} \lambda_j \beta_j, \sum_{j=1}^{n} \lambda_j \gamma_j)_{L-R}. \end{aligned} \tag{2.4}$$

Therefore, for each $\alpha \in [0,1]$, we get

$$\begin{aligned} [\sum_{j=1}^{n} \lambda_j \tilde{A}_j]_\alpha = [\sum_{j=1}^{n} \lambda_j (a_j - L^{-1}(\alpha)\beta_j), \\ \sum_{j=1}^{n} \lambda_j (a_j + R^{-1}(\alpha)\gamma_j)]. \end{aligned} \tag{2.5}$$

For ranking fuzzy numbers different methods have been proposed by many researchers, including [14, 15, 32]. Soleimani-damaneh [32] defined the weighted signed distance of $\tilde{A}$ and $\tilde{B}$ as follows:

$$d(\tilde{A}, \tilde{B}) = \int_0^1 ([\tilde{A}]_\alpha^L + [\tilde{A}]_\alpha^U - [\tilde{B}]_\alpha^L - [\tilde{B}]_\alpha^U) d\alpha. \tag{2.6}$$

A decision maker can rank a pair of fuzzy numbers, $\tilde{A}$ and $\tilde{B}$, using $d(\tilde{A},\tilde{B})$ based on the following rules:

$$\begin{aligned} &a)\ \tilde{A} \prec \tilde{B}\ \ iff\ \ d(\tilde{A},\tilde{B}) < 0,\\ &b)\ \tilde{A} \succ \tilde{B}\ \ iff\ \ d(\tilde{A},\tilde{B}) > 0,\\ &c)\ \tilde{A} \approx \tilde{B}\ \ iff\ \ d(\tilde{A},\tilde{B}) = 0. \end{aligned} \tag{2.7}$$

If $\tilde{A} = (a,\beta,\gamma)_{L-R}$ and $\tilde{B} = (b,\eta,\mu)_{L-R}$ are L-R fuzzy numbers with the same shape functions, then we obtained:

$$\begin{aligned} d(\tilde{A},\tilde{B}) &= (2a-2b)\\ &+(\eta-\beta)\int_0^1 L^{-1}(\alpha)d\alpha\\ &+(\gamma-\mu)\int_0^1 R^{-1}(\alpha)d\alpha. \end{aligned} \tag{2.8}$$

Let $\tilde{A} = (a,\beta,\gamma)_{L-R}$ be a $L-R$ fuzzy number. If $L(x) = R(x) = 1-x$, then $\tilde{A}$ is called a triangular fuzzy number. In addition, if $L(x) = R(x) = 1-x$ and $\beta = \gamma$, then $\tilde{A}$ is called a symmetric triangular fuzzy number and denoted as $\tilde{A} = (a,\beta)$.

2.2 Inverse DEA

In this subsection, we review the problem provided by Jahanshahloo et al. [22] in the inverse DEA field. Let us to consider a set of n DMUs, $\{DMU_j : j = 1,\dots,n\}$, in which DMU_j produce multiple positive outputs $y_{rj}(\ r = 1,\dots,s)$, by utilizing multiple positive inputs $x_{ij}(\ i = 1,\dots,m)$. Suppose that the input and output for DMU_j be denoted by $X_j = (x_{1j},x_{2j},\dots,x_{mj})^t$ and $Y_j = (y_{1j},y_{2j},\dots,y_{sj})^t$, respectively. The ER-measure model [30] is considered for measuring the relative efficiency of the unit under assessment DMU_o, $o \in \{1,2,\dots,n\}$, as follows:

$$\begin{aligned} \rho_o^* = \min\ & \frac{\frac{1}{m}\sum_{i=1}^m \theta_i}{\frac{1}{s}\sum_{r=1}^s \varphi_r} \\ s.t.\ & \sum_{j=1}^n \lambda_j x_{ij} \le\ \theta_i x_{io}, \qquad \forall i\\ & \sum_{j=1}^n \lambda_j y_{rj} \ge \varphi_r y_{ro}, \qquad \forall r\\ & \theta_i \le 1, \qquad i = 1,\dots,m,\\ & \varphi_r \ge 1, \qquad r = 1,\dots,s,\\ & \lambda \in \Omega, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} \Omega = \{\lambda|\lambda = (\lambda_1,\dots,\lambda_n),\ \delta_1(\sum_{j=1}^n \lambda_j\\ +\delta_2(-1)^{\delta_3}\nu) = \delta_1,\ \nu \ge 0,\ \lambda_j \ge 0, \forall j\}. \end{aligned} \tag{2.10}$$

In the above model δ_1, δ_2, and δ_3 are parameters with $0-1$ values. It is easy to see that: If $\delta_1 = 0$, then model (2.9) is under a constant returns to scale (CRS); If $\delta_1 = 1$ and $\delta_2 = 0$, then the above model is under a variable returns to scale (VRS); If $\delta_1 = \delta_2 = 1$ and $\delta_3 = 0$, then model (2.9) is under a non-increasing returns to scale (NIRS); and If $\delta_1 = \delta_2 = \delta_3 = 1$, then the above model is under a non-decreasing returns to scale (NDRS) assumption of the production technology.

Definition 2.2 *(ER-measure) [30] The optimal value ρ_o^* of the model (2.9) is called the ER-measure of DMU_o. DMU_o is ER-efficient, if and only if, $\rho_o^* = 1$(this condition is equivalent to $\theta_i^* = 1$ and $\varphi_r^* = 1$ for each $i = 1,\dots,m$, $r = 1,\dots,s$ in any optimal solution).*

Inverse DEA concept first was studied by Zhang and Cui [37]. Since then this problem has allocated to itself some of researches in DEA field and provided various results in different frameworks. Jahanshahloo et al. [22] proposed the following important question in inverse DEA field:

Question. If the efficiency score ρ_o^* remains unchanged, but the decision maker is required to increase input-output levels, how much should the input-output levels of DMU_o increase?

The aim of Question is estimating the minimum increase of input vector (α_o^*) and the maximum increase of output vector (β_o^*) provided that the efficiency score of DMU_o is still ρ_o^*. In fact,

$$\begin{aligned} \alpha_o^* &= (\alpha_{1o}^*,\alpha_{2o}^*,\dots,\alpha_{mo}^*)^t\\ &= X_o + \Delta X_o,\quad \Delta X_o \gneqq 0,\\ \beta_o^* &= (\beta_{1o}^*,\beta_{2o}^*,\dots,\beta_{so}^*)^t\\ &= Y_o + \Delta Y_o,\quad \Delta Y_o \gneqq 0. \end{aligned}$$

For convenience, assume that DMU_{n+1} represents DMU_o after changing the input and output vectors. The following model is proposed to estimate the ER-measure of DMU_{n+1}:

$$\rho_o^{+*} = \min \frac{\frac{1}{m}\sum_{i=1}^{m}\theta_i}{\frac{1}{s}\sum_{r=1}^{s}\varphi_r} \tag{2.11}$$
$$s.t. \sum_{j=1}^{n}\lambda_j x_{ij} + \lambda_{n+1}\alpha_{io}^* \le \theta_i\alpha_{io}^*, \ \forall i,$$
$$\sum_{j=1}^{n}\lambda_j y_{rj} + \lambda_{n+1}\beta_{ro}^* \ge \varphi_r\beta_{ro}^*, \ \forall r,$$
$$\theta_i \le 1, \quad i = 1,\dots,m,$$
$$\varphi_r \ge 1, \quad r = 1,\dots,s,$$
$$\lambda \in \Omega^+,$$

where

$$\Omega^+ = \{\lambda | \lambda = (\lambda_1,\dots,\lambda_{n+1}), \delta_1(\sum_{j=1}^{n+1}\lambda_j + \delta_2$$
$$(-1)^{\delta_3}\nu) = \delta_1, \ \nu \ge 0, \ \lambda_j \ge 0, j = 1,\dots,n+1\}.$$

The variables of the above model are $\theta_1, \theta_2, ..., \theta_m, \varphi_1, \varphi_2, ..., \varphi_s, \lambda_1, \lambda_2, ..., \lambda_{n+1}$. If the optimal values of problems (2.9) and (2.11) are equal, we say that the ER-measure unchanged, i.e., $eff(\alpha_o^*, \beta_o^*) = eff(X_o, Y_o)$.[11]
To answer the above question, Jahanshahloo et al. [22] proposed the following MOLP problem:

$$\min \ (\alpha_{1o},\dots,\alpha_{mo}) \tag{2.12}$$
$$\max \ (\beta_{1o},\dots,\beta_{so})$$
$$s.t. \ \sum_{j=1}^{n}\lambda_j x_{ij} \le \ \theta_i^*\alpha_{io}, \ \forall i,$$
$$\sum_{j=1}^{n}\lambda_j y_{rj} \ge \ \varphi_r^*\beta_{ro}, \ \forall r,$$
$$\alpha_{io} \ge x_{io}, \quad i = 1,\dots,m$$
$$\beta_{ro} \ge y_{ro}, \quad r = 1,\dots,s$$
$$\alpha_o \in \Lambda, \beta_o \in \Gamma,$$
$$\lambda \in \Omega,$$

where $(\theta^* = (\theta_1^*,\dots,\theta_m^*), \varphi^* = (\varphi_1^*,\dots,\varphi_s^*))$ is the optimal solution of model (2.9). The variables vector in MOLP (2.12) is $(\lambda, \alpha_o, \beta_o)$. Λ and Γ are bounded sets and represent the increasing variation rate of input-output levels of DMU_o which are considered by the decision maker.

Theorems 2.1 and 2.2 are contains some of the main results of Jahanshahloo et al. [22].

[11] we use the notation $eff(X_oY_o)$ instead of measure efficiency of DMU_o.

Theorem 2.1 *[22] Suppose that DMU_o is ER-efficient and $(\lambda^*, \ \theta^*, \ \varphi^*)$ is an optimal solution to problem (2.9). Let $(\hat{\lambda}^*, \hat{\alpha}_o^*, \hat{\beta}_o^*)$ be a Pareto solution to MOLP(2.12) such that $\hat{\alpha}_o^* > X_o$. If the input-output levels of DMU_o are increased to $\hat{\alpha}_o^*$ and $\hat{\beta}_o^*$, respectively, then $eff(\hat{\alpha}_o^*, \hat{\beta}_o^*) = eff(X_o, Y_o)$.*

Theorem 2.2 *[22] Suppose that $(\lambda^*, \ \theta^*, \ \varphi^*)$ is an optimal solution to problem (2.9). Let $(\bar{\lambda}, \bar{\alpha}_o, \bar{\beta}_o)$ be a feasible solution to MOLP (2.12). If $eff(\bar{\alpha}_o, \bar{\beta}_o) = eff(X_o, Y_o)$, then $(\bar{\lambda}, \bar{\alpha}_o, \bar{\beta}_o)$ must be a weak Pareto solution to MOLP (2.12).*

3 Inverse DEA with Fuzzy Data

In this section, we extend Question (the problem simultaneous estimation of input-output levels, which has been provided by Jahanshahloo et al. [22]) to a fuzzy framework. In other words, we provides some theoretical extensions of inverse DEA theory in the presence of fuzzy data, which can expand the application area of inverse DEA. The technique proposed to treat the fuzzy data in the problem of simultaneous estimation of input-output levels is using fuzzy ER-measure model.

Let us to consider a set of n DMUs, $\{DMU_j : j = 1,\dots,n\}$, in which DMU_j consumes multiple positive fuzzy inputs $\tilde{x}_{ij}$ to produce multiple positive fuzzy outputs $\tilde{y}_{rj}$. Suppose that the inputs and outputs of DMU_j be denoted by $\tilde{x}_j = (\tilde{x}_{1j}, \tilde{x}_{2j}, \dots, \tilde{x}_{mj})^t$ and $\tilde{y}_j = (\tilde{y}_{1j}, \tilde{y}_{2j}, \dots, \tilde{y}_{sj})^t$, respectively. The input-output levels are considered as L-R fuzzy numbers as

$$\tilde{x}_{ij} = (x_{ij}, \nu_{ij}, \gamma_{ij})_{L_{ij}-R_{ij}}, \quad i = 1,2,\dots,m,$$
$$\tilde{y}_{rj} = (y_{rj}, \eta_{rj}, \mu_{rj})_{L'_{ij}-R'_{ij}}, \quad r = 1,2,\dots,s,$$

satisfying

$$L_{i1} = \ldots = L_{in} = L_i, \quad i = 1,2,\dots,m,$$
$$R_{i1} = \ldots = R_{in} = R_i, \quad i = 1,2,\dots,m,$$
$$L'_{r1} = \ldots = L'_{rn} = L'_r, \quad r = 1,2,\dots,s,$$
$$R'_{r1} = \ldots = R'_{rn} = R'_r, \quad r = 1,2,\dots,s.$$

As mentioned in [12] these conditions are not too restrictive, as we are simply requiring that, for any factor, the corresponding n data can be described by means of L-R fuzzy numbers of the same type. For instance, if these are trapezoidal or triangular fuzzy numbers, then the above conditions hold.

To measure the relative efficiency of the unit under assessment of DMU_o, $o \in \{1,2,\dots,n\}$, the non-radial ER-measure model (2.9) can be naturally extended to be the following fuzzy ER-measure model:

$$\begin{aligned} \rho_o^* = \min\ & \frac{\frac{1}{m}\sum_{i=1}^m \theta_i}{\frac{1}{s}\sum_{r=1}^s \varphi_r} \\ s.t.\ & \sum_{j=1}^n \lambda_j \tilde{x}_{ij} \preccurlyeq \theta_i \tilde{x}_{io}, \quad \forall i, \\ & \sum_{j=1}^n \lambda_j \tilde{y}_{rj} \succcurlyeq \varphi_r \tilde{y}_{ro}, \quad \forall r, \\ & \theta_i \le 1, \quad i=1,\dots,m, \\ & \varphi_r \ge 1, \quad r=1,\dots,s, \\ & \lambda \in \Omega. \end{aligned} \tag{3.13}$$

Using relations (2.4)-(2.7), the above model can be converted to the following optimization problem:

$$\begin{aligned} \rho_o^* = \min\ & \frac{\frac{1}{m}\sum_{i=1}^m \theta_i}{\frac{1}{s}\sum_{r=1}^s \varphi_r} \\ s.t.\ & d(\theta_i \tilde{x}_{io}, \sum_{j=1}^n \lambda_j \tilde{x}_{ij}) \ge 0, \quad \forall i, \\ & d(\sum_{j=1}^n \lambda_j \tilde{y}_{rj}, \varphi_r \tilde{y}_{ro}) \ge 0, \quad \forall r, \\ & \theta_i \le 1, \quad i=1,\dots,m, \\ & \varphi_r \ge 1, \quad r=1,\dots,s, \\ & \lambda \in \Omega. \end{aligned} \tag{3.14}$$

Now, we devoted to extending Question, provided by Jahanshahloo et al. [22]. The aim of the study is estimating the minimum increase of input vector $\tilde{\alpha}_o^* = (\alpha_o^*, \nu_o^*, \gamma_o^*)_{L-R}$ and the maximum increase of output vector $\tilde{\beta}_o^* = (\beta_o^*, \eta_o^*, \mu_o^*)_{L-R}$ provided that the DMU$_o$, with respect to other units, maintains its current efficiency level. In fact,

$$\begin{aligned} \tilde{\alpha}_o^* &= (\tilde{\alpha}_{1o}^*, \tilde{\alpha}_{2o}^*, \dots, \tilde{\alpha}_{mo}^*)^t \\ &= \tilde{x}_o + \Delta\tilde{x}_o, \quad \Delta\tilde{x}_o \in (FNLR(\mathbb{R})_+)^m, \\ \tilde{\beta}_o^* &= (\tilde{\beta}_{1o}^*, \tilde{\beta}_{2o}^*, \dots, \tilde{\beta}_{so}^*)^t \\ &= \tilde{y}_o + \Delta\tilde{y}_o, \quad \Delta\tilde{y}_o \in (FNLR(\mathbb{R})_+)^s, \end{aligned}$$

where $FNLR(\mathbb{R})_+$ is the family of all non-negative fuzzy numbers.

Suppose DMU_{n+1} represents DMU_o after changing the input-output levels. The following model is proposed to estimate the ER-measure of DMU_{n+1}:

$$\begin{aligned} \rho_o^{+*} = \min\ & \frac{\frac{1}{m}\sum_{i=1}^m \theta_i}{\frac{1}{s}\sum_{r=1}^s \varphi_r} \\ s.t.\ & \sum_{j=1}^n \lambda_j \tilde{x}_{ij} + \lambda_{n+1}\tilde{\alpha}_{io}^* \preccurlyeq \theta_i \tilde{\alpha}_{io}^*, \ \forall i, \\ & \sum_{j=1}^n \lambda_j \tilde{y}_{rj} + \lambda_{n+1}\tilde{\beta}_{ro}^* \succcurlyeq \varphi_r \tilde{\beta}_{ro}^*, \ \forall r, \\ & \theta_i \le 1, \quad i=1,\dots,m, \\ & \varphi_r \ge 1, \quad r=1,\dots,s, \\ & \lambda \in \Omega^+. \end{aligned} \tag{3.15}$$

If the optimal values of Models (3.13) and (3.15) are equal, we say that the ER-measure unchanged, i.e., $eff(\tilde{\alpha}_o^*, \tilde{\beta}_o^*) = eff(\tilde{x}_o, \tilde{y}_o)$. To answer Question, the following fuzzy MOLP problem is considered:

$$\begin{aligned} \min\ & (\tilde{\alpha}_{1o}, \dots, \tilde{\alpha}_{mo}) \\ \max\ & (\tilde{\beta}_{1o}, \dots, \tilde{\beta}_{so}) \\ s.t.\ & \sum_{j=1}^n \lambda_j \tilde{x}_{ij} \preccurlyeq \theta_i^* \tilde{\alpha}_{io}, \ \forall i, \\ & \sum_{j=1}^n \lambda_j \tilde{y}_{rj} \succcurlyeq \varphi_r^* \tilde{\beta}_{ro}, \ \forall r, \\ & \tilde{\alpha}_{io} \succcurlyeq \tilde{x}_{io}, \quad i=1,\dots,m, \\ & \tilde{\beta}_{ro} \succcurlyeq \tilde{y}_{ro}, \quad r=1,\dots,s, \\ & \tilde{\alpha}_o \in \tilde{\Lambda}, \tilde{\beta}_o \in \tilde{\Gamma}, \\ & \lambda \in \Omega, \end{aligned} \tag{3.16}$$

where $(\theta^* = (\theta_1^*, \dots, \theta_m^*), \varphi^* = (\varphi_1^*, \dots, \varphi_s^*))$ is an optimal solution of Model (3.13). $\tilde{\Lambda}$ and $\tilde{\Gamma}$ are bounded fuzzy sets and represent the increasing variation rate of inputs and outputs of the DMU_o which are considered by the decision maker. In this model, $(\lambda, \tilde{\alpha}_o, \tilde{\beta}_o) \in FNLR(\mathbb{R})^n \times FNLR(\mathbb{R})^m \times FNLR(\mathbb{R})^s$ is the variable vector.

Theorem 3.1 shows how the above fuzzy MOLP can be applied for input-output levels estimation.

Theorem 3.1 *Suppose that DMU_o is ER-efficient and $(\bar{\lambda}^*, \theta^*, \varphi^*)$ is an optimal solution of*

Model(3.13). Let $(\lambda^*, \tilde{\alpha}_o^* = (\alpha_o^*, \nu_o^*, \gamma_o^*)_{L-R}, \tilde{\beta}_o^* = (\beta_o^*, \eta_o^*, \mu_o^*)_{L-R})$ be a Pareto solution to fuzzy MOLP(3.16). If one of the following assumptions holds:

i) $\tilde{\alpha}_o^* \approx \tilde{x}_o$,

ii) $\tilde{\alpha}_o^* \succ \tilde{x}_o$.

Then

$$eff(\tilde{\alpha}_o^*, \tilde{\beta}_o^*) = eff(\tilde{x}_o, \tilde{y}_o).$$

Proof. To prove the theorem, $\rho_o^{+*} = \rho_o^* = 1$ should be shown. Because $(\lambda^*, \tilde{\alpha}_o^*, \tilde{\beta}_o^*)$ is a feasible solution for model (3.16), we have

$$\sum_{j=1}^{n} \lambda_j^* \tilde{x}_{ij} \preccurlyeq \ \theta_i^* \tilde{\alpha}_{io}^* \approx \tilde{\alpha}_{io}^*, \quad \forall i, \tag{3.17}$$

$$\sum_{j=1}^{n} \lambda_j^* \tilde{y}_{rj} \succcurlyeq \ \varphi_r^* \tilde{\beta}_{ro}^* \approx \tilde{\beta}_{ro}^*, \quad \forall r, \tag{3.18}$$

$$\tilde{\alpha}_{io}^* \succcurlyeq \tilde{x}_{io}, \qquad \forall i, \tag{3.19}$$

$$\tilde{\beta}_{ro}^* \succcurlyeq \tilde{y}_{ro}, \qquad \forall r, \tag{3.20}$$

$$\tilde{\beta}_o^* \in \tilde{\Gamma}, \tilde{\alpha}_o^* \in \tilde{\Lambda}, \tag{3.21}$$

$$\lambda^* \in \Omega. \tag{3.22}$$

Let $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n, \bar{\lambda}_{n+1})$, in which $\bar{\lambda}_j = \lambda_j^*$ for each $j = 1, \ldots, n$ and $\bar{\lambda}_{n+1} = 0$. It is clear that $\bar{\lambda} \in \Omega^+$. Since $\bar{\lambda} \in \Omega^+$, because of (3.17) and (3.18), $(\bar{\lambda}, \ \theta^*, \ \varphi^*) \in \mathbb{R}^{n+1} \times \mathbb{R}^m \times \mathbb{R}^s$ is a feasible solution to problem (3.15). Therefore, $\rho_o^{+*} \leq \rho_o^* = 1$.

Let $(\lambda^{+*} = (\lambda_1^{+*}, \ldots, \lambda_{n+1}^{+*}), \ \theta^{+*} = (\theta_1^{+*}, \ldots, \theta_m^{+*}), \ \varphi^{+*} = (\varphi_1^{+*}, \ldots, \varphi_s^{+*}))$ be an optimal solution to problem (3.15). The inequalities (3.17) and (3.18) will be used in problem (3.15), the following results are obtained:

$$\theta_i^{+*} \tilde{\alpha}_{io}^* \succcurlyeq \sum_{j=1}^{n} \lambda_j^{+*} \tilde{x}_{ij} + \lambda_{n+1}^{+*} \tilde{\alpha}_{io}^*$$

$$\succcurlyeq \sum_{j=1}^{n} \lambda_j^{+*} \tilde{x}_{ij} + \lambda_{n+1}^{+*} (\sum_{j=1}^{n} \lambda_j^* \tilde{x}_{ij}),$$

$$\theta_i^{+*} \tilde{\alpha}_{io}^* \succcurlyeq \sum_{j=1}^{n} (\lambda_j^{+*} + \lambda_{n+1}^{+*} \lambda_j^*) \tilde{x}_{ij}, \ \forall i, \tag{3.23}$$

$$\varphi_r^{+*} \tilde{\beta}_{ro}^* \preccurlyeq \sum_{j=1}^{n} \lambda_j^{+*} \tilde{y}_{rj} + \lambda_{n+1}^{+*} \tilde{\beta}_{ro}^*$$

$$\preccurlyeq \sum_{j=1}^{n} \lambda_j^{+*} \tilde{y}_{rj} + \lambda_{n+1}^{+*} (\sum_{j=1}^{n} \lambda_j^* \tilde{y}_{rj}),$$

$$\varphi_r^{+*} \tilde{\beta}_{ro}^* \preccurlyeq \sum_{j=1}^{n} (\lambda_j^{+*} + \lambda_{n+1}^{+*} \lambda_j^*) \tilde{y}_{rj}, \ \forall r. \tag{3.24}$$

Set $\hat{\lambda}_j := \lambda_j^{+*} + \lambda_{n+1}^{+*} \lambda_j^*$ for each $j = 1, 2, \ldots, n$. It is easily seen that $\hat{\lambda} = (\hat{\lambda}_j, \ldots, \hat{\lambda}_n) \in \Omega$.

By contradiction assume that $\rho_o^{+*} < \rho_o^* = 1$. If assumption (i) holds, then by (3.23) and (3.24), we get

$$\sum_{j=1}^{n} \hat{\lambda}_j \tilde{x}_{ij} \preccurlyeq \theta_i^{+*} \tilde{\alpha}_{io}^* \approx \theta_i^{+*} \tilde{x}_{io}, \ \forall i, \tag{3.25}$$

$$\sum_{j=1}^{n} \hat{\lambda}_j \tilde{y}_{rj} \succcurlyeq \varphi_r^{+*} \tilde{\beta}_{ro}^* \succcurlyeq \varphi_r^{+*} \tilde{y}_{ro}, \ \forall r. \tag{3.26}$$

Hence, $\hat{\lambda} \in \Omega$, (3.25) and (3.26) imply that $(\hat{\lambda}, \theta^{+*}, \varphi^{+*})$ is a feasible solution to model (3.13). This implies $\rho_o^* \leq \frac{\frac{1}{m}\sum_{i=1}^{m} \theta_i^{+*}}{\frac{1}{s}\sum_{r=1}^{s} \varphi_r^{+*}} < 1$, which contradicts the ER-measure of DMU_o and completes the proof under assumption (i).

To prove the theorem under assumption (ii), we should show that $\rho_o^{+*} = \rho_o^* = 1$. By contradict assumption, since $\rho_o^{+*} < \rho_o^* = 1$, then

$$\frac{1}{m}\sum_{i=1}^{m} \theta_i^{+*} < 1 \quad \text{or} \quad \frac{1}{s}\sum_{r=1}^{s} \varphi_r^{+*} > 1.$$

Therefore, there are two cases:
Case(a) If $\frac{1}{m}\sum_{i=1}^{m} \theta_i^{+*} < 1$, then there exists at least one t, $1 \leq t \leq m$, such that $\theta_t^{+*} < 1$. Since $\tilde{\alpha}_{to}^*$ is a positive fuzzy number, for all $\alpha \in [0, 1]$ the following inequality is obtained:

$$\alpha_{to}^* - \nu_{to}^* \int_0^1 L^{-1}(\alpha) d\alpha > 0,$$

$$\alpha_{to}^* + \gamma_{to}^* \int_0^1 R^{-1}(\alpha) d\alpha > 0.$$

Therefore,

$$2\alpha_{to}^* - \nu_{to}^* \int_0^1 L^{-1}(\alpha) d\alpha + \gamma_{to}^* \int_0^1 R^{-1}(\alpha) d\alpha > 0. \tag{3.27}$$

By (3.23), we have

$$\theta_t^{+*}(2\alpha_{to}^* - \nu_{to}^* \int_0^1 L^{-1}(\alpha)d\alpha + \gamma_{to}^* \int_0^1 R^{-1}(\alpha)d\alpha) \geq 2\sum_{j=1}^n \hat{\lambda}_j x_{tj} - (\sum_{j=1}^n \hat{\lambda}_j \nu_{tj}) \int_0^1 L^{-1}(\alpha)d\alpha + (\sum_{j=1}^n \hat{\lambda}_j \gamma_{tj}) \int_0^1 R^{-1}(\alpha)d\alpha. \quad (3.28)$$

By (3.27), (3.28), and $\theta_t^{+*} < 1$, we get

$$2\alpha_{to}^* - \nu_{to}^* \int_0^1 L^{-1}(\alpha)d\alpha + \gamma_{to}^* \int_0^1 R^{-1}(\alpha)d\alpha) > 2\sum_{j=1}^n \hat{\lambda}_j x_{tj} - (\sum_{j=1}^n \hat{\lambda}_j \nu_{tj}) \int_0^1 L^{-1}(\alpha)d\alpha + (\sum_{j=1}^n \hat{\lambda}_j \gamma_{tj}) \int_0^1 R^{-1}(\alpha)d\alpha. \quad (3.29)$$

In other words,

$$\sum_{j=1}^n \hat{\lambda}_j \tilde{x}_{tj} \prec \tilde{\alpha}_{to}^*. \quad (3.30)$$

If assumption (ii) holds, then $\tilde{\alpha}_{io}^* \succ \tilde{x}_{io}$ for each $i = 1, \ldots, m$. Considering $i = t$, we have

$$2\alpha_{to}^* - \nu_{to}^* \int_0^1 L^{-1}(\alpha)d\alpha + \gamma_{to}^* \int_0^1 R^{-1}(\alpha)d\alpha > 2x_{to} - \nu_{to} \int_0^1 L^{-1}(\alpha)d\alpha + \gamma_{to} \int_0^1 R^{-1}(\alpha)d\alpha. \quad (3.31)$$

Now, we define

$$\kappa_{to}^1 = 2\sum_{j=1}^n \hat{\lambda}_j x_{ij} - (\sum_{j=1}^n \hat{\lambda}_j \nu_{tj}) \int_0^1 L^{-1}(\alpha)d\alpha + (\sum_{j=1}^n \hat{\lambda}_j \gamma_{tj}) \int_0^1 R^{-1}(\alpha)d\alpha,$$

$$\kappa_{to}^2 = \rho_{to}^1 = 2\alpha_{to}^* - \nu_{to}^* \int_0^1 L^{-1}(\alpha)d\alpha + \gamma_{to}^* \int_0^1 R^{-1}(\alpha)d\alpha,$$

$$\rho_{to}^2 = 2x_{to} - \nu_{to} \int_0^1 L^{-1}(\alpha)d\alpha + \gamma_{to} \int_0^1 R^{-1}(\alpha)d\alpha.$$

If

$$\epsilon = \min\left\{\frac{\kappa_{to}^2 - \kappa_{to}^1}{2}, \frac{\rho_{to}^1 - \rho_{to}^2}{2}\right\}, \quad (3.32)$$

then $\epsilon > 0$, because $\kappa_{to}^2 - \kappa_{to}^1 > 0$ and $\rho_{to}^1 - \rho_{to}^2 > o$. Now, define $\hat{\tilde{\beta}}_o \approx \tilde{\beta}_o^*$ and

$$\hat{\tilde{\alpha}}_{io} \approx \begin{cases} (\alpha_{io}^* - 2\epsilon, \nu_{io}^*, \gamma_{io}^*)_{L-R} & if\ i = t, \\ (\alpha_{io}^*, \nu_{io}^*, \gamma_{io}^*)_{L-R} & if\ i \neq t. \end{cases}$$

Considering (3.32), the following inequality is obtained:

$$\epsilon \leq \frac{\kappa_{to}^2 - \kappa_{to}^1}{2} \implies \kappa_{to}^1 \leq \kappa_{to}^2 - 2\epsilon.$$

Because $\theta_t^* = 1$, therefore

$$\sum_{j=1}^n \hat{\lambda}_j x_{tj} - (\sum_{j=1}^n \hat{\lambda}_j \nu_{tj}) \int_0^1 L^{-1}(\alpha)d\alpha + (\sum_{j=1}^n \hat{\lambda}_j \gamma_{tj}) \int_0^1 R^{-1}(\alpha)d\alpha \leq 2\alpha_{to}^* - \nu_{to}^* \int_0^1 L^{-1}(\alpha)d\alpha + \gamma_{to}^* \int_0^1 R^{-1}(\alpha)d\alpha - 2\epsilon = \theta_t^* 2((\alpha_{to}^* - \epsilon) - \nu_{to}^* \int_0^1 L^{-1}(\alpha)d\alpha + \gamma_{to}^* \int_0^1 R^{-1}(\alpha)d\alpha).$$

In other words,

$$\sum_{j=1}^{n} \hat{\lambda}_j \tilde{x}_{tj} \preccurlyeq \theta_t^* \hat{\tilde{\alpha}}_{to}. \qquad (3.33)$$

By (3.32), we have

$$\epsilon \le \frac{\rho_{to}^1 - \rho_{to}^2}{2} \implies \rho_{to}^2 \le \rho_{to}^1 - 2\epsilon,$$

then

$$2x_{to} - \nu_{to} \int_0^1 L^{-1}(\alpha)d\alpha + \gamma_{to} \int_0^1 R^{-1}(\alpha)d\alpha$$

$$\le 2\alpha_{to}^* - \nu_{to}^* \int_0^1 L^{-1}(\alpha)d\alpha$$

$$+\gamma_{to}^* \int_0^1 R^{-1}(\alpha)d\alpha - 2\epsilon = 2(\alpha_{to}^* - \epsilon)$$

$$-\nu_{to}^* \int_0^1 L^{-1}(\alpha)d\alpha + \gamma_{to}^* \int_0^1 R^{-1}(\alpha)d\alpha.$$

In other words,

$$\tilde{x}_{to} \preccurlyeq \hat{\tilde{\alpha}}_{to}. \qquad (3.34)$$

By (3.19) and (3.34) ($\tilde{x}_{io} \preccurlyeq \tilde{\alpha}_{io}^* \approx \hat{\tilde{\alpha}}_{io}$ for $i \neq t$), it is clear that

$$\tilde{x}_{io} \preccurlyeq \hat{\tilde{\alpha}}_{io}, \qquad i = 1, \ldots, m. \qquad (3.35)$$

In addition,

$$\hat{\tilde{\alpha}}_o \in \tilde{\Lambda}, \qquad (3.36)$$

because $\tilde{x}_o \preccurlyeq \tilde{\alpha}_o^* \in \tilde{\Lambda}$, by (3.35), and

$$d(\tilde{\alpha}_{io}^*, \hat{\tilde{\alpha}}_{io}) = \begin{cases} 2\epsilon & if \;\; i = t, \\ 0 & if \;\; i \neq t, \end{cases}$$

imply that $\tilde{x}_o \preccurlyeq \hat{\tilde{\alpha}}_o \in \tilde{\Lambda}$.

By Eqs. (3.23), (3.24), and (3.33) we have

$$\sum_{j=1}^{n} \hat{\lambda}_j \tilde{x}_{ij} \preccurlyeq \theta_i^{+*} \tilde{\alpha}_{io}^*$$

$$\preccurlyeq \theta_i^* \tilde{\alpha}_{io}^* \approx \theta_i^* \hat{\tilde{\alpha}}_{io}^*, \;\; \forall i, \;\; i \neq t, \qquad (3.37)$$

$$\sum_{j=1}^{n} \hat{\lambda}_j \tilde{x}_{tj} \preccurlyeq \theta_t^* \hat{\tilde{\alpha}}_{to}, \qquad (3.38)$$

$$\sum_{j=1}^{n} \hat{\lambda}_j \tilde{y}_{rj} \succcurlyeq \varphi_r^{+*} \tilde{\beta}_{ro}^*$$

$$\succcurlyeq \varphi_r^* \tilde{\beta}_{ro}^* \approx \varphi_r^* \hat{\tilde{\beta}}_{ro}^*, \;\; \forall r, \qquad (3.39)$$

Since $\hat{\lambda} \in \Omega$, because of (3.35)-(3.39) and $\tilde{y}_o \preccurlyeq \tilde{\beta}_o^* \approx \hat{\tilde{\beta}}_o \in \tilde{\Gamma}$, $(\hat{\lambda}, \hat{\tilde{\alpha}}_o, \hat{\tilde{\beta}}_o)$ is a feasible solution to problem (3.16), in which

$$d(\tilde{\alpha}_{io}^*, \hat{\tilde{\alpha}}_{io}) = \begin{cases} 2\epsilon & if \;\; i \in t, \\ 0 & if \;\; i \neq t, \end{cases}$$

$$d(\hat{\tilde{\beta}}_{ro}, \tilde{\beta}_{ro}^*) = 0, \qquad r = 1, \ldots, s.$$

Therefore,

$$\hat{\tilde{\alpha}}_o \preccurlyeq \; \tilde{\alpha}_o^*, \hat{\tilde{\alpha}}_o \not\approx \tilde{\alpha}_o^*,$$

$$\hat{\tilde{\beta}}_o \approx \tilde{\beta}_o^*.$$

This contradicts the assumption that $(\lambda^*, \tilde{\alpha}_o^*, \tilde{\beta}_o^*)$ is a Pareto solution to problem (3.16), and the proof of case (a) is completed.

Case (b) If $\frac{1}{s}\sum_{r=1}^{s} \varphi_r^{+*} > 1$, then there exists at least one t, $1 \le t \le m$, such that $\varphi_t^{+*} > 1$. Since $\tilde{\beta}_{to}^*$ is a positive fuzzy number, for all $\alpha \in [0, 1]$ the following inequality is obtained:

$$\beta_{to}^* - \eta_{to}^* \int_0^1 L^{-1}(\alpha)d\alpha > 0,$$

$$\beta_{to}^* + \mu_{to}^* \int_0^1 R^{-1}(\alpha)d\alpha > 0.$$

Therefore,

$$2\beta_{to}^* - \eta_{to}^* \int_0^1 L^{-1}(\alpha)d\alpha$$

$$+\mu_{to}^* \int_0^1 R^{-1}(\alpha)d\alpha > 0. \qquad (3.40)$$

By (3.24) we have

$$\varphi_t^{+*}(2\beta_{to}^* - \eta_{to}^* \int_0^1 L^{-1}(\alpha)d\alpha$$

$$+\mu_{to}^* \int_0^1 R^{-1}(\alpha)d\alpha) \le 2\sum_{j=1}^{n} \hat{\lambda}_j y_{tj}$$

$$-(\sum_{j=1}^{n} \hat{\lambda}_j \eta_{tj}) \int_0^1 L^{-1}(\alpha)d\alpha$$

$$+(\sum_{j=1}^{n} \hat{\lambda}_j \mu_{tj}) \int_0^1 R^{-1}(\alpha)d\alpha. \qquad (3.41)$$

By (3.40), (3.41), and $\varphi_t^{+*} > 1$, we get

$$2\beta_{to}^* - \eta_{to}^* \int_0^1 L^{-1}(\alpha)d\alpha + \mu_{to}^* \int_0^1 R^{-1}(\alpha)d\alpha) < 2\sum_{j=1}^{n} \hat{\lambda}_j y_{tj} - (\sum_{j=1}^{n} \hat{\lambda}_j \eta_{tj}) \int_0^1 L^{-1}(\alpha)d\alpha + (\sum_{j=1}^{n} \hat{\lambda}_j \mu_{tj}) \int_0^1 R^{-1}(\alpha)d\alpha. \quad (3.42)$$

In other words,

$$\sum_{j=1}^{n} \hat{\lambda}_j \tilde{y}_{tj} \succ \tilde{\beta}_{to}^*. \quad (3.43)$$

If

$$\kappa_{to}^1 = 2\sum_{j=1}^{n} \hat{\lambda}_j y_{tj} - (\sum_{j=1}^{n} \hat{\lambda}_j \eta_{tj}) \int_0^1 L^{-1}(\alpha)d\alpha + (\sum_{j=1}^{n} \hat{\lambda}_j \mu_{tj}) \int_0^1 R^{-1}(\alpha)d\alpha,$$

$$\kappa_{to}^2 = 2\beta_{to}^* - \eta_{to}^* \int_0^1 L^{-1}(\alpha)d\alpha + \mu_{to}^* \int_0^1 R^{-1}(\alpha)d\alpha,$$

then there exists a $\epsilon > 0$ that satisfies

$$\epsilon \le \frac{\kappa_{to}^1 - \kappa_{to}^2}{2}. \quad (3.44)$$

Now, define $\hat{\tilde{\alpha}}_o \approx \tilde{\alpha}_o^*$ and

$$\hat{\tilde{\beta}}_{ro} \approx \begin{cases} (\beta_{ro}^* + 2\epsilon, \eta_{ro}^*, \mu_{ro}^*)_{L-R} \ if \ r = t, \\ (\beta_{ro}^*, \eta_{ro}^*, \mu_{ro}^*)_{L-R} \ \ if \ r \neq t. \end{cases}$$

Considering (3.44), the following inequality is obtained:

$$\kappa_{to}^2 + 2\epsilon \le \kappa_{to}^1.$$

Since $\varphi_t^* = 1$, then

$$2\sum_{j=1}^{n} \hat{\lambda}_j y_{tj} - (\sum_{j=1}^{n} \hat{\lambda}_j \eta_{tj}) \int_0^1 L^{-1}(\alpha)d\alpha + (\sum_{j=1}^{n} \hat{\lambda}_j \mu_{tj}) \int_0^1 R^{-1}(\alpha)d\alpha \ge 2\beta_{to}^* - \eta_{to}^* \int_0^1 L^{-1}(\alpha)d\alpha + \mu_{to}^* \int_0^1 R^{-1}(\alpha)d\alpha + 2\epsilon = \varphi_t^*(2(\beta_{to}^* + \epsilon) - \eta_{to}^* \int_0^1 L^{-1}(\alpha)d\alpha + \mu_{to}^* \int_0^1 R^{-1}(\alpha)d\alpha).$$

In other words,

$$\sum_{j=1}^{n} \hat{\lambda}_j \tilde{y}_{tj} \succcurlyeq \varphi_t^* \hat{\tilde{\beta}}_{to}. \quad (3.45)$$

On the other hand, because $\tilde{\beta}_{to}^* \succcurlyeq \tilde{y}_{to}$ we get

$$2y_{to} - \eta_{to} \int_0^1 L^{-1}(\alpha)d\alpha + \mu_{to} \int_0^1 R^{-1}(\alpha)d\alpha$$

$$\le 2\beta_{to}^* - \eta_{to}^* \int_0^1 L^{-1}(\alpha)d\alpha + \mu_{to}^* \int_0^1 R^{-1}(\alpha)d\alpha < 2(\beta_{to}^* + \epsilon) - \eta_{to}^* \int_0^1 L^{-1}(\alpha)d\alpha + \mu_{to}^* \int_0^1 R^{-1}(\alpha)d\alpha.$$

In other words,

$$\tilde{y}_{to} \prec \hat{\tilde{\beta}}_{to}. \quad (3.46)$$

By (3.20) and (3.46) ($\tilde{y}_{ro} \preccurlyeq \tilde{\beta}_{ro}^* \approx \hat{\tilde{\beta}}_{ro}$ for $r \neq t$), it is clear that

$$\tilde{y}_{ro} \preccurlyeq \hat{\tilde{\beta}}_{ro}, \quad r = 1, \ldots, s. \quad (3.47)$$

By Eqs. (3.23), (3.24), and (3.45) we have

$$\sum_{j=1}^{n} \hat{\lambda}_j \tilde{x}_{ij} \preccurlyeq \theta_i^{+*} \tilde{\alpha}_{io}^* \preccurlyeq \theta_i^* \tilde{\alpha}_{io}^* \approx \theta_i^* \hat{\tilde{\alpha}}_{io}^*, \quad \forall i, \quad (3.48)$$

$$\sum_{j=1}^{n} \hat{\lambda}_j \tilde{y}_{tj} \succcurlyeq \varphi_t^* \hat{\tilde{\beta}}_{to}, \quad (3.49)$$

$$\sum_{j=1}^{n} \hat{\lambda}_j \tilde{y}_{rj} \succcurlyeq \varphi_r^{+*} \tilde{\beta}_{ro}^* \succcurlyeq \varphi_r^* \tilde{\beta}_{ro}^* \approx \varphi_r^* \hat{\tilde{\beta}}_{ro}^*, \quad \forall r, \ r \neq t. \quad (3.50)$$

Since $\hat{\lambda} \in \Omega$, because of (3.47)-(3.50) and $\tilde{x}_o \preccurlyeq \tilde{\alpha}_o^* \approx \hat{\tilde{\alpha}}_o \in \tilde{\Lambda}$, $(\hat{\lambda}, \hat{\tilde{\alpha}}_o, \hat{\tilde{\beta}}_o)$ is a feasible solution to problem (3.16), in which

$$d(\hat{\tilde{\beta}}_{ro}, \tilde{\beta}_{ro}^*) = \begin{cases} 2\epsilon & if \;\; r = t, \\ 0 & if \;\; r \neq t, \end{cases}$$

$$d(\hat{\tilde{\alpha}}_{io}, \tilde{\alpha}_{io}^*) = 0, \qquad i = 1, \ldots, m.$$

Therefore,

$$\hat{\tilde{\beta}}_o \succcurlyeq \tilde{\beta}_o^*, \hat{\tilde{\beta}}_o \not\approx \tilde{\beta}_o^* \qquad \text{and} \qquad \hat{\tilde{\alpha}}_o \approx \tilde{\alpha}_o^*.$$

This contradicts the assumption that $(\lambda^*, \tilde{\alpha}_o^*, \tilde{\beta}_o^*)$ is a Pareto solution to problem (3.16), and the proof of case (b) is completed.

So, both cases lead to contradiction. Therefore, $\rho_o^{+*} = \rho_o^*$, and the proof is completed. □

Remark 3.1 *It is easy to see that Theorem 3.1 will remain valid if one replaces the objective function of fuzzy MOLP (3.16) with* "$\min(\tilde{\alpha}_{1o}, \ldots, \tilde{\alpha}_{mo})$".

To illustrate the using of the methodology that extended, the following numerical example with real data is considered to show the accuracy of the proposed method.

Example 3.1 *The data from the work of by Ghobadi and Jahangiri (Research work done at Khomeinishahr Azad University, Isfahan, Iran 2010), could be used to demonstrate the application of the method proposed in this paper. There are 14 educational departments in this district; D_1, D_2,...,D_{14}. Each educational department(DMU) uses two inputs to produce two outputs. The considered inputs and outputs, denoted by $\tilde{x}^1$; $\tilde{x}^2$ and $\tilde{y}^1$; $\tilde{y}^1$, are as follows, respectively:*
$\tilde{x}^1$: Facilities,
$\tilde{x}^2$: Amount of the attention paid to the department by the university; and
$\tilde{y}^1$: Satisfaction of the students,
$\tilde{y}^2$: Satisfaction of the professors and staff.

Suppose that $L(x) = R(x) = 1 - x$ and all the inputs and outputs are symmetric triangular fuzzy number. The data of input, output, and the ER-measure for all DMUs are shown in the table 1. It can be seen that D_3 is an ER-efficient DMU. Suppose that the decision maker determined the variations rate of increase input-output levels for D_3 as follows:

$$(9.7, 1.0) \preccurlyeq \tilde{x}^1 \preccurlyeq (11.0, 1.0),$$
$$(8.5, 1.0) \preccurlyeq \tilde{x}^2 \preccurlyeq (10.5, 1.0),$$
$$(12.4, 1.0) \preccurlyeq \tilde{y}^1 \preccurlyeq (16.7, 1.0),$$
$$(16.0, 1.0) \preccurlyeq \tilde{y}^2 \preccurlyeq (19.5, 1.0).$$

In order to propose pattern to the decision maker to increase input-output levels for this DMU, provided that the efficiency score remains unchanged, fuzzy MOLP (3.16) is considered and using the weight-sum method [8] the following Pareto solutions are generated:

$$(\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \tilde{\beta}_1^*, \tilde{\beta}_2^*) = ((9.7, 1.0), (9.9, 1.0), (14.4, 1.0), (19.5, 1.0)),$$
$$(\tilde{\bar{\alpha}}_1^*, \tilde{\bar{\alpha}}_2^*, \tilde{\bar{\beta}}_1^*, \tilde{\bar{\beta}}_2^*) = ((9.7, 1.0), (10.5, 1.0), (16.5, 1.0), (19.5, 1.0)).$$

According to Theorem 3.1, it is obvious that:

$$eff((\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \tilde{\beta}_1^*, \tilde{\beta}_2^*)) = eff((\tilde{\bar{\alpha}}_1^*, \tilde{\bar{\alpha}}_2^*, \tilde{\bar{\beta}}_1^*, \tilde{\bar{\beta}}_2^*))$$
$$= eff(\tilde{x}_3^1, \tilde{x}_3^2, \tilde{y}_3^1, \tilde{y}_3^2).$$

Therefore, the above Pareto solutions offer the two patterns to the decision maker to take better decision in order to extend D_3. That is to say that the decision maker can take necessary actions by choosing a suitable strategy for spreading D_3. Note that, according to Theorem 3.1 the ER-measure for the two patterns are equal one.□
The rest of the paper in this section will focus on converse version of Theorem 3.1. In other words, the following theorem is converse version of Theorem 3.1. Notice that, unlike Theorem 3.1, Theorem 3.2 holds in general case without assuming the efficiency DMU_o.

Theorem 3.2 *Suppose that $(\lambda^*, \theta^*, \varphi^*)$ is an optimal solution to problem (3.13). Let $(\overline{\lambda}, \overline{\tilde{\alpha}}_o = (\overline{\alpha}_o, \overline{\nu}_o, \overline{\gamma}_o)_{L-R}, \overline{\tilde{\beta}}_o = (\overline{\beta}_o, \overline{\eta}_o, \overline{\mu}_o)_{L-R})$ be a feasible solution to the problem (3.16). If*

$$eff(\overline{\tilde{\alpha}}_o, \overline{\tilde{\beta}}_o) = eff(\tilde{x}_o, \tilde{y}_o),$$

then $(\overline{\lambda}, \overline{\tilde{\alpha}}_o, \overline{\tilde{\beta}}_o)$ must be a Pareto solution to problem (3.16).

Table 1: The data and ER-measure under VRS assumption.

	Inputs		Outputs		
D_1	(3.9,1.0)	(7.8,1.0)	(11.8,1.0)	(14.0,1.0)	0.90
D_2	(5.4,1.0)	(7.8,1.0)	(12.2,1.0)	(10.0,1.0)	0.70
D_3	(9.7,1.0)	(8.5,1.0)	(12.4,1.0)	(16.0,1.0)	1.00
D_4	(5.5,1.0)	(7.1,1.0)	(11.3,1.0)	(14.4,1.0)	1.00
D_5	(3.6,1.0)	(9.1,1.0)	(12.5,1.0)	(13.1,1.0)	0.85
D_6	(4.2,1.0)	(6.7,1.0)	(9.6,1.0)	(14.5,1.0)	1.00
D_7	(5.1,1.0)	(7.8,1.0)	(12.3,1.0)	(14.5,1.0)	0.99
D_8	(3.9,1.0)	(8.4,1.0)	(13.2,1.0)	(10.4,1.0)	1.00
D_9	(5.6,1.0)	(8.3,1.0)	(12.3,1.0)	(12.7,1.0)	0.69
D_{10}	(2.7,1.0)	(8.1,1.0)	(12.3,1.0)	(14.1,1.0)	1.00
D_{11}	(2.0,1.0)	(8.8,1.0)	(11.6,1.0)	(12.8,1.0)	1.00
D_{12}	(5.5,1.0)	(7.5,1.0)	(12.5,1.0)	(14.4,1.0)	1.00
D_{13}	(5.6,1.0)	(9.6,1.0)	(13.0,1.0)	(14.4,1.0)	1.00
D_{14}	(10.0,1.0)	(10.0,1.0)	(12.2,1.0)	(14.4,1.0)	0.60

Proof. If $(\overline{\lambda}, \overline{\tilde{\alpha}}_o, \overline{\tilde{\beta}}_o)$ is not a Pareto solution to problem (3.16), then there exists another feasible solution of problem (3.16), $(\hat{\lambda}, \hat{\tilde{\alpha}}_o = (\hat{\alpha}_o, \hat{\nu}_o, \hat{\gamma}_o)_{L-R}, \hat{\tilde{\beta}}_o = (\hat{\beta}_o, \hat{\eta}_o, \hat{\mu}_o)_{L-R})$, such that $\hat{\tilde{\alpha}}_{io} \preccurlyeq \overline{\tilde{\alpha}}_{io}$ and $\hat{\tilde{\beta}}_{ro} \succcurlyeq \overline{\tilde{\beta}}_{ro}$ for all i, r and there exist at least one i in which $\hat{\tilde{\alpha}}_{io} \prec \overline{\tilde{\alpha}}_{io}$ or at least one r such that $\hat{\tilde{\beta}}_{ro} \succ \overline{\tilde{\beta}}_{ro}$. Without loss of generality, we assume that $\hat{\tilde{\alpha}}_{to} \prec \overline{\tilde{\alpha}}_{to}$. Therefore

$$2\overline{\alpha}_{to} - \overline{\nu}_{to}\int_0^1 L^{-1}(\alpha)d\alpha + \overline{\gamma}_{to}\int_0^1 R^{-1}(\alpha)d\alpha > 2\hat{\alpha}_{to} - \hat{\nu}_{to}\int_0^1 L^{-1}(\alpha)d\alpha + \hat{\gamma}_{to}\int_0^1 R^{-1}(\alpha)d\alpha.$$

Because $0 < \theta_t^* \leq 1$, we have

$$\theta_t^*(2\overline{\alpha}_{to} - \overline{\nu}_{to}\int_0^1 L^{-1}(\alpha)d\alpha + \overline{\gamma}_{to}\int_0^1 R^{-1}(\alpha)d\alpha) > \theta_t^*(2\hat{\alpha}_{to} - \hat{\nu}_{to}\int_0^1 L^{-1}(\alpha)d\alpha + \hat{\gamma}_{to}\int_0^1 R^{-1}(\alpha)d\alpha). \quad (3.51)$$

Feasibility of $(\hat{\lambda}, \hat{\tilde{\alpha}}_o, \hat{\tilde{\beta}}_o)$ for fuzzy MOLP (3.16), implies

$$\sum_{j=1}^{n} \hat{\lambda}_j \tilde{x}_{ij} \preccurlyeq \theta_i^* \hat{\tilde{\alpha}}_{io}, \ i = 1, \ldots, m, \quad (3.52)$$

$$\sum_{j=1}^{n} \hat{\lambda}_j \tilde{y}_{rj} \succcurlyeq \varphi_r^* \hat{\tilde{\beta}}_{ro}, \ r = 1, \ldots, s, \quad (3.53)$$

$$\hat{\lambda} \in \Omega. \quad (3.54)$$

Considering $i = t$ and by (3.52), the following inequality is obtained:

$$2\sum_{j=1}^{n} \hat{\lambda}_j x_{tj} - (\sum_{j=1}^{n} \hat{\lambda}_j \nu_{tj})\int_0^1 L^{-1}(\alpha)d\alpha + (\sum_{j=1}^{n} \hat{\lambda}_j \gamma_{tj})\int_0^1 R^{-1}(\alpha)d\alpha \leq \theta_t^*(2\hat{\alpha}_{to} - \hat{\eta}_{to}\int_0^1 L^{-1}(\alpha)d\alpha + \hat{\mu}_{to}\int_0^1 R^{-1}(\alpha)d\alpha). \quad (3.55)$$

Inequalities (3.51) and (3.55) imply that

$$2\sum_{j=1}^{n} \hat{\lambda}_j x_{tj} - (\sum_{j=1}^{n} \hat{\lambda}_j \nu_{tj})\int_0^1 L^{-1}(\alpha)d\alpha + (\sum_{j=1}^{n} \hat{\lambda}_j \gamma_{tj})\int_0^1 R^{-1}(\alpha)d\alpha < \theta_t^*(2\overline{\alpha}_{to} - \overline{\nu}_{to}\int_0^1 L^{-1}(\alpha)d\alpha + \overline{\gamma}_{to}\int_0^1 R^{-1}(\alpha)d\alpha). \quad (3.56)$$

In other words

$$\sum_{j=1}^{n} \hat{\lambda}_j \tilde{x}_{tj} \prec \theta_t^* \overline{\tilde{\alpha}}_{to}. \quad (3.57)$$

By inequality (3.56), there exists positive scalar $0 < k_t < 1$ such that

$$2\sum_{j=1}^{n}\hat{\lambda}_j x_{tj} - (\sum_{j=1}^{n}\hat{\lambda}_j \nu_{tj})\int_0^1 L^{-1}(\alpha)d\alpha + (\sum_{j=1}^{n}\hat{\lambda}_j \gamma_{tj})\int_0^1 R^{-1}(\alpha)d\alpha \leq \theta_t^* k_t(2\overline{\alpha}_{to} - \overline{\nu}_{to}\int_0^1 L^{-1}(\alpha)d\alpha + \overline{\gamma}_{to}\int_0^1 R^{-1}(\alpha)d\alpha). \quad (3.58)$$

In other words

$$\sum_{j=1}^{n}\hat{\lambda}_j \tilde{x}_{tj} \preccurlyeq \theta_t^* k_t \bar{\tilde{\alpha}}_{to}. \quad (3.59)$$

By Eqs. (3.52), (3.53), (3.59), $\hat{\tilde{\alpha}}_{io} \preccurlyeq \bar{\tilde{\alpha}}_{io}$ and $\hat{\tilde{\beta}}_{ro} \succcurlyeq \bar{\tilde{\beta}}_{ro}$ for all i, r, we have

$$\sum_{j=1}^{n}\hat{\lambda}_j \tilde{x}_{ij} \preccurlyeq \theta_i^* \hat{\tilde{\alpha}}_{io} \preccurlyeq \theta_i^* \bar{\tilde{\alpha}}_{io}, \quad i = 1, \ldots, m, i \neq t, \quad (3.60)$$

$$\sum_{j=1}^{n}\hat{\lambda}_j \tilde{x}_{tj} \preccurlyeq \theta_t^* k_t \bar{\tilde{\alpha}}_{to}, \quad (3.61)$$

$$\sum_{j=1}^{n}\hat{\lambda}_j \tilde{y}_{rj} \succcurlyeq \varphi_r^* \hat{\tilde{\beta}}_{ro} \succcurlyeq \varphi_r^* \bar{\tilde{\beta}}_{ro}, \quad \forall r, \quad (3.62)$$

For each $i = 1, \ldots, n$ and $r = 1, \ldots, s$, defining

$$\bar{\theta}_i = \begin{cases} \theta_i^* & if \quad i \neq t, \\ k_i\theta_i^* & if \quad i = t, \end{cases}$$

$$\bar{\varphi}_r = \varphi_r^*.$$

According to Eqs. (3.54), and (3.60)-(3.62), ($\lambda = \hat{\lambda}$, $\lambda_{n+1} = 0, \bar{\theta} = (\bar{\theta}_1, \ldots, \bar{\theta}_m), \bar{\varphi} = (\bar{\varphi}_1, \ldots, \bar{\varphi}_s)$) is a feasible solution to problem (3.15)(considering $\tilde{\alpha}_o^* = \bar{\tilde{\alpha}}_o$ and $\tilde{\beta}_o^* = \bar{\tilde{\beta}}_o$ in problem (3.15)). The value of the objective function of fuzzy LP (3.15) at this feasible point is equal to $\frac{1}{m}(\sum_{i=1, i\neq t}^{m}\bar{\theta}_i + \bar{\theta}_t)/\frac{1}{s}\sum_{r=1}^{s}\bar{\varphi}_r$. Therefore,

$$eff(\bar{\tilde{\alpha}}_o, \bar{\tilde{\beta}}_o) = \rho_o^{+*} \leq \frac{\frac{1}{m}(\sum_{i\neq t}\bar{\theta}_i + \bar{\theta}_t)}{\frac{1}{s}\sum_{r=1}^{s}\bar{\varphi}_r} < \frac{\frac{1}{m}(\sum_{i=1}^{m}\theta_i^*)}{\frac{1}{s}\sum_{r=1}^{s}\varphi_r^*} = eff(\tilde{x}_o, \tilde{y}_o).$$

This contradicts the assumption and completes the proof.□

4 Conclusion

In this paper, we have provided a technique to treat the fuzzy data in the problem of simultaneous estimation of input-output levels in the framework of inverse DEA. In this technique, the fuzzy DEA model is transferred to a mathematical programming model, using the weighted signed distance of fuzzy data. The sufficient conditions, in the problem of simultaneous estimation of input-output levels in the presence of fuzzy data, were provided not only for the ER-efficient DMUs, but also for the given necessary conditions in general case without assuming the efficiency of DMU. It should be noted that there are various approaches/methods in the literature of DEA for solving fuzzy DEA models. Some approaches are as follows:

1) Some of the proposed approaches transform a fuzzy DEA model to a category of ordinary DEA models, see e.g. [25]. It is not an appropriate computational point of view because in this method, to approximate the membership function of the efficiency measure, multiple LP problems must be solved.

2) Some of the proposed methods develop an ordinary DEA model to a fuzzy DEA model, using some ranking methods. This method changes the number of the constraints of the model, see e.g. [29]; therefore, this approach increases the computational complexity.

3) Some of the proposed methods develop a fuzzy DEA model in which to evaluate a DMU, a nonlinear multiple-objective mathematical programming problem needs to be solved, see e.g. [23]. This method is not appropriate from a computational point of view.

4) A number of suggested methods develop a possibility approach to treat the fuzzy DEA models, see e.g. [28]. This approach is almost useless with regard to any RTS assumption of product technology.

Moreover, Soleimani-damaneh et al. [33] reviewed some of the existing fuzzy DEA models and showed that many of the current models/methods are suffering from theoretical and computational problems. However, the approach used in the present study does not have the above mentioned pitfalls. In this method, unlike other proposed methods, at first the fuzzy DEA model is transferred to a mathematical programming model, while the number of constraints and the computational complexity will not increase. Also, this technique can be applied under any RTS assumption of technology. The results obtained are theoretically significant since they provide some of theoretical extensions of inverse DEA theory in the presence of fuzzy data, which can expand the application area of inverse DEA. Future works can be based on ratio-based (polynomial-time) approaches which are able to solve the proposed models from a computational view. In addition, analyzing other helpful ranking functions which are available in fuzzy mathematics to denazify the provided problem can be experimented theoretically.

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