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Research Article

Approximate Solution of the Stochastic Nonlinear Oscillator

T. Damercheli *†

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Abstract

In this paper, we consider the nonlinear equations with the additive white noise, which are commonly impossible to be solved by an analytical procedure. The Block-Pulse functions as basic functions are proposed to solve these equations. In order to investigate the validity of this method, we used the Adomian decomposition method to approximate the solution of the stochastic Duffing equations. The results reveal that the proposed method is very effective.

Keywords : Duffing's equation; Stochastic; Decomposition method; Block-Pulse functions.

1 Introduction

Stochastic systems play a prominent role in a range of application areas including biology, chemistry, epidemiology, mechanics, microelectronics, economics, finance and physics. Generally, mathematical modeling of such processes leads to nonlinear deterministic and stochastic systems [14, 15]. A realistic nonlinear stochastic differential equation is the mathematical model of damped-forced pendulum with noise, that can be described approximately by the second order nonlinear stochastic differential equations such as stochastic Duffing's equations [3]. These equations have many catastrophic, diverging and oscillating behaviors by several stable and unstable states due to the value of their coefficients. Therefore, it is impossible to find an analytical procedure to solve nonlinear equations such as Duffing's equations. Some authors reported analytical and perturbative or nonperturbative techniques to solve these equations [4, 20, 21, 23, 24, 25]. The Adomian decomposition method (ADM) is a solution method with a wide range of applications including the solution of linear and nonlinear algebraic, differential, integral and integro-differential equations or system of equations that acknowledge this method work extremely well for Duffing's equations [1, 2, 7, 8, 9, 10, 22, 26, 27]. In this method, the solution is considered as a rapidly converging, infinite series. The convergence of the method proved by Y. Cherrualt et al in [5, 6]. In recent years, the Block-Pulse functions (BPF) have been studied and applied extensively as a useful tool to solve linear and nonlinear equations [12]. Studies and applications show that these functions may have definite advantages for problems involving integrals and derivatives due to their clearness in expressions and their simplicity in formulations.

*Corresponding author. tdamercheli@gmail.com, Tel:+98(912)3599569.
†Department of Mathematics, Yadegar-e-Imam Khomeini (RAH) Share Rey Branch, Islamic Azad University, Tehran, Iran.

The main advantage of using BPF is that it reduces the problems to those of solving a system of algebraic equations. In this paper, we study the stochastic Duffing equation that of course cannot be solved explicitly. Therefore, it is important to find their approximate solutions by using some numerical methods. Recently, several authors used some numerical methods to solve stochastic equations [11, 13, 16, 17, 18, 19, 28]. The methods for solving stochastic equations are based on similar techniques used for solving deterministic equations, but generalized to provide support for stochastic equations. The BPF and the ADM are two examples of the methods which have been applied to solve a wide range of linear and nonlinear problems, both deterministic and stochastic. They are two powerful methods that consider the approximate solution of nonlinear problems as an infinite series converging to the exact solution.

In this paper, we use the BPF to approximate the solution of the stochastic nonlinear Duffing equation. To show the validity of the method, we apply the ADM to solve the stochastic equation. The comparison among numerical results shows that both methods give similar approximations and also the results are compared with the results of the deterministic equation. It is shown that these methods are all valid if the intensity of the noise is small.

The paper is organized as follows. In Section 2, we describe the basic properties of the BPF which are required for our subsequent development and Section 3 states the formulation of the stochastic nonlinear Duffing oscillator. We apply the BPF and the ADM to solve nonlinear stochastic Duffing's equations in section 4 and 5, respectively and in section 6, we report our numerical findings.

2 Preliminaries

2.1 Description of the BPF

Block-pulse functions, a set of orthogonal functions with piecewise constant values, is defined as follows [?, ?]

$$\phi_i(x) = \begin{cases} 1 & (i-1)h \leq x \leq ih, \\ 0 & otherwise, \end{cases} \tag{2.1}$$

with $x \in [0, T)$, $i = 1, 2, \ldots, m$ and $h = \frac{T}{m}$. The BPFs have some properties such as disjointness, orthogonality and completeness.

1. Disjointness

$$\phi_i(x)\phi_j(x) = \begin{cases} \phi_i(x) & i = j, \\ 0 & i \neq j, \end{cases} \tag{2.2}$$

2. Orthogonality

$$\int_0^T \phi_i(x)\phi_j(x)dx = \begin{cases} h & i = j, \\ 0 & i \neq j, \end{cases} \tag{2.3}$$

3. Completeness

For every $f \in L^2[0, T]$ when m approach to the infinity, Parsevals identity holds

$$\int_0^T f^2(x)dx = \sum_{i=1}^{\infty} f_i^2 \|\phi_i(x)\|^2, \tag{2.4}$$

where

$$f_i = \frac{1}{h}\int_0^T f(x)\phi_i(x)dx. \tag{2.5}$$

The set of Block-Pulse functions may be written as a m-vector $\phi(x)$,

$$\phi(x) = [\phi_1(x), \phi_2(x), \ldots, \phi_m(x)]^T. \tag{2.6}$$

From the above representation and disjointness property we have

$$\phi(x)\phi(x)^T = \begin{bmatrix} \phi_1(x) & 0 & \cdots & 0 \\ 0 & \phi_2(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_m(x) \end{bmatrix}, \tag{2.7}$$

$$\phi(x)^T\phi(x) = 1, \tag{2.8}$$

$$\phi(x)\phi(x)^T X = \tilde{X}\phi(x), \tag{2.9}$$

where X is an m-vector and $\tilde{X} = diag(X)$. Moreover, function $f(x) \in L^2[0, T]$ can be expanded by Block-Pulse functions as

$$f(x) = \sum_{i=1}^{m} f_i\phi_i(x) = F^T\phi(x) = \phi^T(x)F, \tag{2.10}$$

where F is an m-vector given by

$$F = [f_1, f_2, \dots, f_m]^T. \tag{2.11}$$

The integration of the vector $\phi(x)$ defined in Eq. (2.6) can be approximately obtained as

$$\int_0^x \phi(x)dx = P\phi(x), \tag{2.12}$$

where P is the operational matrix for integration and is given by

$$P = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{2.13}$$

3 Statement problem

We consider the Duffing equation under stochastic external excitations as follows

$$y''(x)+b_1y'(x)+b_2y(x)+b_3y^3(x) = f(x)+\gamma\xi(x), \tag{3.14}$$

The common form of the Duffing equations can be obtained from the dynamical equations of a physical pendulum, that is perturbed by additive noise. Also, it describes the model excited by inhomogeneities of the surrounding medium and external forces. The coefficient γ characterizes the noise intensity (for $\gamma = 0$ the motion is deterministic), b_1 is the measure of damping and b_3 is nonlinearities. Suppose that the pendulum starts its motion from $y = \alpha$ by the velocity $y' = \beta$ at the time $x = 0$. So, we have the initial conditions

$$y(0) = \alpha \qquad y'(0) = \beta. \tag{3.15}$$

In Eq. (3.14), $\xi(x) = dW(x)/dx$, $W(x)$ is a Wiener process which represents an intrinsic noise in the dynamical systems, and is often referred to as Gaussian white noise with zero mean and standard deviation $\sigma = 1$. In general, the Wiener process $W(x)$ is a continuous-time stochastic process with the following three properties [25]:

Property 1. For each x, the random variable $W(x)$ is normally distributed with mean 0 and variance x and also $W(0) = 0$.

Property 2. For each $x_1 < x_2$, the normal random variable $W(x_2)$-$W(x_1)$ is independent of the random variable $W(x_1)$, and in fact independent of all $W(x)$, $0 \le x \le x_1$.

Property 3. The Wiener process $W(x)$ can be represented by continuous paths where is not differentiable.

4 The BPF for Stochastic Duffing's equations

The BPF is based on converting the underlying differential equation into an integral equation through integration, approximating various signals involved in the equation by truncated orthogonal functions of BPF, and using the operational matrix of integration to eliminate the integral operations. We consider Eq. (3.14) as the form

$$y(x) = \alpha + \beta x + \int_0^x \int_0^x f(x)dxdx \tag{4.16}$$

$$+ \int_0^x \int_0^x \gamma\xi(x)dxdx - b_1 \int_0^x \int_0^x \frac{d}{dx}y(x)dxdx$$

$$- b_2 \int_0^x \int_0^x y(x)dxdx - b_3 \int_0^x \int_0^x y^3(x)dxdx,$$

By simplifying above equation, we can get

$$y(x) = z(x) + \int_0^x \int_0^x f(x)dxdx \tag{4.17}$$

$$+ \int_0^x \gamma W(x)dx - b_1 \int_0^x y(x)dx$$

$$- b_2 \int_0^x \int_0^x y(x)dxdx - b_3 \int_0^x \int_0^x y^3(x)dxdx,$$

where $z(x) = \alpha+(\beta+b_1)x$. To solve this equation by using BPF, we approximate functions $y(x)$, $f(x)$, $z(x)$, $W(x)$ and $y^3(x)$ by using Eq. (2.10) as follows

$$\begin{aligned} y^3(x) &= Y^T\tilde{Y}^2\phi(x), \\ W(x) &= W^T\phi(x), \\ y(x) &= Y^T\phi(x), \\ f(x) &= F^T\phi(x), \\ z(x) &= Z^T\phi(x). \end{aligned}$$

By substituting in Eq. (4.17), we obtain

$$Y^T\phi(x) = Z^T\phi(x) + \int_0^x\int_0^x F^T\phi(x)dxdx \quad (4.18)$$
$$+\gamma\int_0^x W^T\phi(x)dx - b_1\int_0^x Y^T\phi(x)dx$$
$$-b_3\int_0^x\int_0^x Y^T\tilde{Y}^2\phi(x)dxdx.$$

By using operational matrix of integration P given in Eq. (2.13), we get

$$Y^T\phi(x) = Z^T\phi(x) + F^T P^2\phi(x) + \gamma W^T P\phi(x)$$
$$-b_1 Y^T P\phi(x) - b_3 Y^T\tilde{Y}^2P^2\phi(x)$$

$$Y^T + b_1Y^TP + b_3Y^T\tilde{Y}^2P^2 = Z^T + F^TP^2 + \gamma W^TP. \quad (4.19)$$

Obviously Eq. (4.19) is a system of nonlinear algebraic equations that can be solved by nonlinear algorithms.

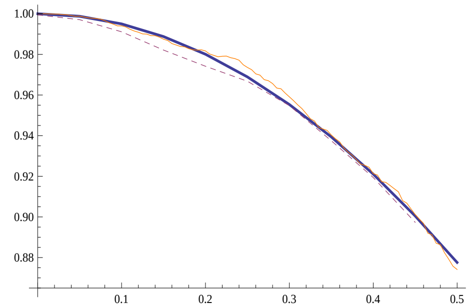

Figure 1: Trajectory deterministic (Thick),Trajectory stochastic: ADM (Orange), BPF (dashed)

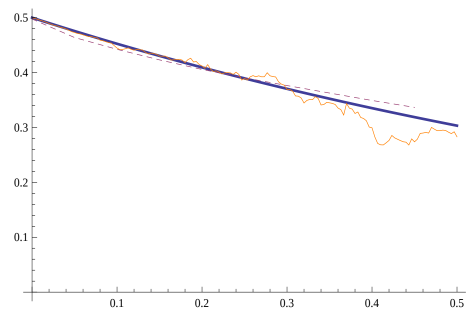

Figure 2: Trajectory deterministic (Orange),Trajectory stochastic: ADM (Thick), BPF (dashed)

5 The ADM for Stochastic Duffing's equation

In this section, we apply the ADM to solve Eq. (3.14) whit initial values. In this method, the solution is considered as the summation of an infinite series which converges to the exact solutions. To this end, we denote $\frac{d^2}{dx^2}$ by operator L, then operator L^{-1} is a two-fold integration from 0 to x. So, the Eq. (3.14) can be written

$$L(y(x)) = f(x)+\gamma\xi(x)-b_1y'(x)-b_2y(x)-b_3y^3(x) \quad (5.20)$$

which is converted to the following equation after applying L^{-1} and using Eq. (3.15),

$$y(x) = \alpha+\beta x+L^{-1}(f(x))+L^{-1}(\gamma\xi(x)) \quad (5.21)$$

$$-b_1L^{-1}(y'(x)) - b_2L^{-1}(y(x)) - b_3L^{-1}(y^3(x))$$

Now, to solve Eq. (5.21), consider

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \quad (5.22)$$

and

$$y^3(x) = \sum_{n=0}^{\infty} A_n(x), \quad (5.23)$$

where A_ns for $i = 0, 1, 2, \ldots$, are the Adomian polynomials [?]. By substituting Eq. (5.22) and Eq. (5.23) into Eq. (5.21), we have

$$\sum_{n=0}^{\infty} y_n(x) = \alpha+\beta x+\int_0^x\int_0^x f(x)dxdx+ \quad (5.24)$$

$$\int_0^x\int_0^x \gamma\xi(x)dxdx - b_1\int_0^x\int_0^x \frac{d}{dx}(\sum_{n=0}^{\infty} y_n(x)dxdx$$

$$-b_2\int_0^x\int_0^x\sum_{n=0}^{\infty} y_n(x)dxdx$$

$$-b_3\int_0^x\int_0^x\sum_{n=0}^{\infty} A_n(x)dxdx$$

Equating the terms, we get

$$y_0(x) = \alpha+\beta x+\int_0^x\int_0^x f(x)dxdx \quad (5.25)$$
$$+\int_0^x\int_0^x \gamma\xi(x)dxdx$$

where

$$\int_0^x \int_0^x \gamma\xi(x)dxdx = \int_0^x \int_0^x \gamma dW(x)dx \quad (5.26)$$

and the last integral is called Ito integral calculated by stochastic calculus and for $i = 1, 2, \ldots$, we have

$$y_i(x) = -b_1 \int_0^x \int_0^x \frac{d}{dx} y_{i-1}(x)dxdx \quad (5.27)$$

$$-b_2 \int_0^x \int_0^x y_{i-1}(x)dxdx - b_3 \int_0^x \int_0^x A_{i-1}(x)dxdx.$$

6 Examples

In this section, we determine the approximate solution of two examples from stochastic nonlinear Duffing's equations by the PBF and the ADM. The numerical results are presented by Figs 1 and 2. All computations are carried out using Mathematica 8.

Example 6.1 *Consider the nonlinear Duffing equation with additive Gaussian white noise in the external force as*

$$y''(x)+y'(x)+y(x)+y^3(x) = cos^3x - sinx + \xi(x), \quad (6.28)$$

where the initial conditions are $y(0) = 1$ and $y'(0) = 0$. Denoting $\frac{d^2}{dx^2}$ by L, we have L^{-1}, as a two-fold integration. Using the operator L, Eq. (6.28) becomes

$$L(y(x)) = cos^3x - sinx + \xi(x) - y'(x) - y(x) - y^3(x). \quad (6.29)$$

Applying the inverse operator L^{-1} on both sides of Eq. (6.29) and using the initial conditions, we have

$$y(x) = 1 + L^{-1}(cos^3x - sinx) + L^{-1}(\xi(x))$$

$$-L^{-1}(y'(x)) - L^{-1}(y(x)) - L^{-1}(y^3(x)). \quad (6.30)$$

To solve Eq. (6.30) by the BPF, we choose $m = 10$ and $T = 0.5$, therefore we obtain

$$y(x) = [y_1, y_2, \ldots, y_{10}][\phi_1(x), \phi_2(x), \ldots, \phi_{10}(x)]^T$$

$$= Y\phi(x),$$

$$z(x) = x + 1 =$$

$$[1.025, 1.075, 1.125, 1.175, 1.225, 1.275, 1.325,$$

$$1.375, 1.425, 1.475]\phi(x) = Z\phi(x)$$

$$f(x) = cos^3x - sinx =$$

$$[0.973756, 0.916362, 0.851818, 0.780506, 0.70293,$$

$$0.619705, 0.531549, 0.43926, 0.34373, 0.24589]\phi(x)$$

$$= F\phi(x)$$

$$W(x) =$$

$$[W(\frac{h}{2}), W(\frac{3h}{2}), W(\frac{5h}{2}), W(\frac{7h}{2}), W(\frac{9h}{2}), W(\frac{11h}{2}),$$

$$W(\frac{13h}{2}), W(\frac{15h}{2}), W(\frac{17h}{2}), W(\frac{19h}{2})]\phi(x)$$

$$= W\phi(x)$$

$$y^3(x) = [Y\phi(x)]^3 = Y\tilde{Y}^2\phi(x)$$

$$= [y_1^3, y_2^3, y_3^3, \ldots, y_{10}^3]\phi(x).$$

By substituting above equations in Eq. (6.30) and using operational matrix P, we can obtain the system of nonlinear algebraic equations as follows

$$Y[I+P+P^2] + Y\tilde{Y}^2P^2 = Z + FP^2 + WP. \quad (6.31)$$

To solve Eq. (6.30) by the ADM, consider the Eqs. (5.22) and (5.23), therefore we can get

$$\sum_{n=0}^{\infty} y_n(x) = 1 + \int_0^x \int_0^x (cos^3x - sinx)dxdx$$

$$+ \int_0^x \int_0^x \xi(x)dxdx - \int_0^x \int_0^x \sum_{n=0}^{\infty} y_n(x)dxdx$$

$$- \int_0^x \int_0^x \frac{d}{dx}(\sum_{n=0}^{\infty} y_n(x))dxdx$$

Table 1

x	$approximate by ADM$	$approximate by BPF$
0	1	0.999335
0.05	0.99847	0.996849
0.10	0.995485	0.992361
0.15	0.988915	0.985904
0.20	0.981633	0.973580
0.25	0.969365	0.965957
0.30	0.955885	0.952355
0.35	0.941122	0.936839
0.40	0.924878	0.920659
0.45	0.913456	0.915467

Table 2

x	$approximate by ADM$	$approximate by BPF$
0	0.5	0.497512
0.05	0.470779	0.463879
0.10	0.451952	0.442588
0.15	0.425432	0.423625
0.20	0.423725	0.404458
0.25	0.371617	0.384283
0.30	0.401555	0.373218
0.35	0.381442	0.362504
0.40	0.339443	0.358907
0.45	0.316615	0.334563

$$-\int_0^x \int_0^x \sum_{n=0}^{\infty} A_n(x)dxdx,$$

considering five terms of the Maclaurin series external force, $f(x)$, and equating the terms, we obtain

$$f(x) = cos^3x - sinx = 1 - x - \frac{3x^2}{2} + \frac{x^3}{6} + \frac{7x^4}{8}, \tag{6.32}$$

$$y_0(x) = 1 + \int_0^x \int_0^x (1 - x - \frac{3x^2}{2} + \frac{x^3}{6} + \frac{7x^4}{8})dxdx + \int_0^x \int_0^x dW(x)dx, \tag{6.33}$$

$$y_i(x) = -\int_0^x \int_0^x \frac{d}{dx} y_{i-1}(x)dxdx \tag{6.34}$$

$$-\int_0^x \int_0^x y_{i-1}(x)dxdx$$

$$-\int_0^x \int_0^x A_{i-1}(x)dxdx, \quad i = 1, 2, \ldots.$$

By calculating five terms of the series $y_i(x)$, $i = 1, 2, \ldots, 5$, an approximate solution for the Eq. (6.30) we obtain as follows

$$y_0(x) = 1 + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{8} + \frac{x^5}{120} + \frac{7x^6}{240} + W(x) \tag{6.35}$$

$$y_1(x) = -x^2 - \frac{x^3}{6} - \frac{x^4}{8} + \frac{7x^5}{120} - \frac{7x^6}{720} + \frac{x^7}{90} + \frac{x^8}{336} - \frac{x^9}{2880} + \cdots - 2x^2W(x) - \frac{1}{4}x^4W(x) + \frac{1}{20}x^5W(x) + \frac{3}{280}x^7W(x) + \frac{1}{192}x^8W(x) - \frac{1}{480}x^9W(x) - \frac{37x^{10}W(x)}{86400} + \cdots, \tag{6.36}$$

$\vdots to$ y(x)=1-x^2 $\frac{}{2+\frac{x^4}{24}-\frac{x^6}{720}-\frac{x^7}{5040}+\frac{x^8}{280}-}$

$$\frac{913x^9}{120960} - \cdots + W(x) - 2x^2W(x) + \frac{2}{3}x^3W(x) +$$

$$\frac{3}{4}x^4W(x) - \frac{19}{60}x^5W(x) - \frac{143}{360}x^6W(x) + \cdots.$$

The numerical results are shown in Table 1. The curves in Fig. 1 represent a trajectory of the approximate solution computed by the BPF and ADM.

Example 6.2 *Consider the nonlinear Duffing equation with additive Gaussian white noise in external force as form*

$$y''(x) + 2y'(x) + y(x) + 8y^3(x) = e^{-3x} + \xi(x), \tag{6.37}$$

where the initial conditions are $y(0) = \frac{1}{2}$ and $y'(0) = -\frac{1}{2}$. The numerical results are shown in Table 2. The curves in Fig. 2 represent a trajectory of the approximate solution computed by the BPF and ADM.

Integrating of both side of the Eq. (6.37) and using the initial conditions we have

$$y(x) = \frac{1}{2} + \frac{1}{2}x + \int_0^x\int_0^x e^{-3x}dxdx + \int_0^x\int_0^x \xi(x)dxdx$$
$$-2\int_0^x\int_0^x \frac{d}{dx}(y(x))dxdx - \int_0^x\int_0^x y(x)dxdx$$
$$-8\int_0^x\int_0^x y^3(x)dxdx. \tag{6.38}$$

To solve Eq. (6.38) by the BPF, we choose $m = 10$ and $T = 0.5$, by relation (4.16)-(4.19), we obtain the system of nonlinear algebraic equations as follows

$$Y[I + 2P + P^2] + 8Y\tilde{Y}^2P^2 = Z + FP^2 + WP. \tag{6.39}$$

To determine the approximate solution of Eq. (6.37) by the ADM, we consider five terms of the Maclaurin series external force, e^{-3x}, and calculate the $y_i(x)$, for $i = 0, 1, \ldots, 5$, therefore we obtain

$$y(x) = \frac{1}{2} + W(x) - \frac{x}{2} + \frac{x^2}{4} - \frac{7W(x)x^2}{2}$$
$$-6W(x)^2x^2 - 4W(x)^3x^2 - \frac{x^3}{12} + \frac{13W(x)x^3}{3}$$
$$+6W(x)^2x^3 + \frac{W(x)^3x^3}{3} + \frac{x^4}{48} - \frac{9W(x)x^4}{8} +$$
$$7W(x)^2x^4 + 20W(x)^3x^4 + 20W(x)^4x^4 + 8W(x)^5x^4$$
$$-\frac{3x^5}{80} - \frac{97W(x)x^5}{30} - \frac{93W(x)^2x^5}{5} + \cdots.$$

7 Conclusion

The stochastic nonlinear Duffing equation is one of the complex issues that cannot be solve explicitly. In the present paper, we used the BPF and the ADM to approximate solution of this equation. The main advantage of using BPF is that it reduce the problems to those of solving a system of algebraic equations and also the ADM has been known to be a powerful device for solving nonlinear equations, particularly Duffing's equations. Comparison of the numerical findings show that these methods have near approximates and also the numerical results are compared with the exact solution of the deterministic equation. Therefore, it can be concluded the presented method is valid to solve this kind of equations.

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Tayebeh Damercheli is Assistance Professor of Applied Mathematics at the Department of Mathematics, College of Science, Yadegar-e-Emam Khomeyni (RAH) Shahr-e-Rey Branch, Islamic Azad University. His main research interest is include numerical solution of deterministic and stochastic integral equations, differential equations and dynamic systems arise from optimal control problems.