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Solving Second-Order Fuzzy Cauchy-Euler Initial Value Problems Under Generalized Differentiability

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Abstract

In this paper, we study a class of second-order fuzzy initial value problems that are known as the Cauchy-Euler differential equations, in the crisp case. This work begins by studying the structure of solution function in the crisp case and providing a requirement space of the generalized differentiable functions. In sequel, the process of production and construction of the solution formula are discussed, in details. Finally, the obtained formulas are applied and illustrated by solving some examples.

Keywords : Differential equations; Fuzzy differential equations; Generalized differentiability; Cauchy-Euler equations.

1 Introduction

One of the interesting subjects in fuzzy mathematics and engineering sciences is to solve fuzzy differential equations ($FDEs$), which have been investigated by many researches. Up to now, many definitions are suggested to fuzzy derivative concept (for instance [14] and [27]). The generalized differentiability concept (G-differentiability) of fuzzy-valued functions, introduced by Bede et al. is a comprehensive and workable concept of fuzzy differentiability [8, 9, 11]. Some results of G-differentiability can be seen in [2, 10, 11, 12, 15, 16, 17, 29] and studies on $FDEs$ under G-differentiability can be summarized as follows: Bede et al. [11] obtained some of the solution formulas related to the classes of first-order linear $FDEs$ and then Khastan et al. [23] completed some results in [11], by obtaining the other formulas of solutions. Considering solution as a fuzzy set consisted of real functions was studied in [7] and [21]. Recently, Allahviranloo et al. [5], introduced a method based on length function properties and obtained all solutions related to the various forms of first-order linear $FDEs$. Some numerical methods to solve $FDEs$ have been introduced in [1, 4, 24, 25]. The existence and uniqueness problem of solution to a class second-order linear $FDEs$ under G-differentiability has been studied in [6]. An algorithm to calculate the solution of second-order fuzzy initial value problems was given in [3], which is based on determining sign of the upper and lower functions of the solution function and their derivatives. Gasilov et al.

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[22], considered a second-order fuzzy initial value problem as a set of crisp problems and introduced a solution method by using the properties of linear transformations.

The aim of this paper is to solve a class of $FDEs$ that are modeled as follows:

$$\begin{cases} (t-\gamma)^2 y''(t) + \alpha(t-\gamma)y'(t) + R(t) = \beta y(t), \\ t \in [a,b], \quad t \neq \gamma, \\ y(a) = y_0, \\ y'(a) = y_1, \end{cases} \tag{1.1}$$

where y_0 and y_1 are fuzzy numbers, α, β and γ are real numbers and $R(t)$ is a fuzzy polynomial of degree at most n about the point $t = \gamma$, i.e.,

$$R(t) = \sum_{i=0}^{n} (t-\gamma)^i u_i \tag{1.2}$$

with u_i for $i = 0, 1, ..., n$, given as fuzzy numbers. These equations are well known as Cauchy-Euler initial value problems, in the crisp case. We recently have studied on the first-order Cauchy-Euler initial value problems in [19]. The proposed approach in the present paper is based on the structure of solution in the crisp case of the problem. (to see the methods in the crisp case we proposed [13] and [18], for example). It is shown that under certain conditions the solution can be obtained as a fuzzy-valued function.

This paper is organized as follows. In Section 2, the basic concepts and essential results are reviewed which are needed to deal with $FDEs$. In Section 3, we first point out the structure of solution in the crisp case of equation (1.1) and next study the G-differentiability of some fuzzy-valued functions that are useful to solve equation (1.1). The manufacturing process of solutions and conditions of their existence are given, in Section 4. At the end of paper, some examples are solved to show that the obtained formulas gives us the requirement results.

2 Preliminaries

In this section, we present basic definitions, notations and some important results which are required in this paper.

Definition 2.1. *[20] A fuzzy number u is defined in the r-cut form as $[u]^r = [u^-(r), u^+(r)]$ for all $r \in [0,1]$, where*
(i) $u^-(r)$ is a left continuous, bounded and non-decreasing function in r,
(ii) $u^+(r)$ is a left continuous, bounded and non-increasing function in r,
(iii) $u^-(r) \leq u^+(r)$.

The set all fuzzy numbers defined on the real axis $\mathbb{R}$ is denoted by $\mathbb{R}_F$. For $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product λu are defined [30], by

$$\begin{aligned}[u+v]^r &= [u]^r + [v]^r \\ &= [u^-(r) + v^-(r), u^+(r) + v^+(r)],\end{aligned}$$

$$\begin{aligned}[\lambda u]^r &= \lambda [u]^r \\ &= [\min(\lambda u^-(r), \lambda u^+(r)), \\ &\quad \max(\lambda u^-(r), \lambda u^+(r))],\end{aligned}$$

for all $r \in [0,1]$.

Definition 2.2. *[9, 28] Let $u, v \in \mathbb{R}_F$. If there exists $w \in \mathbb{R}_F$ such that, $u = v + w$ then w is called the H-difference of u, v and it is denoted as $u \ominus v$.*

In this paper, the symbol " $\ominus$ " always stands for H-difference and it is noteworthy that $u \ominus v \neq u + (-1)v = u - v$.

The following properties of H-difference are well-known

(a) $\lambda(u \ominus v) = \lambda u \ominus \lambda v, \quad \forall \lambda \in \mathbb{R}, \ u, v \in \mathbb{R}_F$,

$$\begin{aligned}(b)\ (u+v) \ominus w &= (u \ominus w) + v \\ &= (v \ominus w) + u, \forall u, v, w \in \mathbb{R}_F,\end{aligned}$$

$$(c)\ (u+v) \ominus (w+z) = (u \ominus w) + (v \ominus z), \\ \forall u, v, w, z \in \mathbb{R}_F,$$

provided that all the above H-differences exist (see [5] and other properties in [26]).

Definition 2.3. *[30] The Hausdorff distance between two arbitrary fuzzy numbers u, $[u]^r = [u^-(r), u^+(r)]$ and v, $[v]^r = [v^-(r), v^+(r)]$ is defined as function $D : \mathbb{R}_F \times \mathbb{R}_F \to [0, +\infty)$ by*

$$D(u,v) = \sup_{0 \leq r \leq 1} d_r(u,v),$$

where

$$d_r(u,v) = \max\left\{|u^-(r) - v^-(r)|, |u^+(r) - v^+(r)|\right\}.$$

The following properties of the Hausdorff metric are well-known

$$(a)\ D(\lambda u, \lambda v) = |\lambda| D(u, v),\ \ \lambda \in \mathbb{R},$$

$$(b)\ D(u+w, v+w) = D(u, v),$$

$$(c)\ D(u+v, w+z) \leq D(u, w) + D(v, z),$$

for all $u, v, w, z \in \mathbb{R}_F$, and $(\mathbb{R}_F, D)$ is a complete metric space [30].

Definition 2.4. *[9] Let $f : (a, b) \to \mathbb{R}_F$. Fix $t_0 \in (a, b)$. It is said that f is G-differentiable at t_0, if the H-differences $f(t_0+h) \ominus f(t_0)$, $f(t_0) \ominus f(t_0-h)$ for all $h > 0$ or all $h < 0$, sufficiently close to 0 exist, and an element $f'(t_0) \in \mathbb{R}_F$ exists, such that either (i):*

$$\lim_{h\to 0^+} \frac{f(t_0+h) \ominus f(t_0)}{h} = \lim_{h\to 0^+} \frac{f(t_0) \ominus f(t_0-h)}{h} = f'(t_0),$$

or (ii):

$$\lim_{h\to 0^-} \frac{f(t_0+h) \ominus f(t_0)}{h} = \lim_{h\to 0^-} \frac{f(t_0) \ominus f(t_0-h)}{h} = f'(t_0),$$

or (iii):

$$\lim_{h\to 0^+} \frac{f(t_0+h) \ominus f(t_0)}{h} = \lim_{h\to 0^-} \frac{f(t_0+h) \ominus f(t_0)}{h} = f'(t_0),$$

or (iv):

$$\lim_{h\to 0^+} \frac{f(t_0) \ominus f(t_0-h)}{h} = \lim_{h\to 0^-} \frac{f(t_0) \ominus f(t_0-h)}{h} = f'(t_0).$$

Theorem 2.1. *[15] Let $f : (a, b) \to \mathbb{R}_F$ be fuzzy-valued function where $[f(t)]^r = [f^-(t, r), f^+(t, r)]$, for each $r \in [0, 1]$.*
(a) If f is (i)-differentiable at $t \in (a, b)$, then $f^-(t, r)$ and $f^+(t, r)$ are differentiable functions and

$$[f'(t)]^r = [{f^-}'(t, r), {f^+}'(t, r)].$$

(b) If f is (ii)-differentiable at $t \in (a, b)$, then $f^-(t, r)$ and $f^+(t, r)$ are differentiable functions and

$$[f'(t)]^r = [{f^+}'(t, r), {f^-}'(t, r)].$$

Definition 2.5. *[12] Assume that the function f is G-differentiable on interval (a, b). A point $t_0 \in (a, b)$ is called a switching point of f, if G-differentiability changes from type (i) to type (ii), or from type (ii) to type (i), in Definition 2.4.*

Chalco et al. [17], have been demonstrated if t_0 is a switching point, then f at t_0 is a differentiable function in the sense (iii) or (iv). Moreover, if f is differentiable on over (a, b) in the sense (iii) (or (iv)) then $f'(t) = \{c\}$, where $c \in \mathbb{R}$ is a real number.

3 Fuzzy Cauchy-Euler equations and the structure of solutions

Consider the second-order linear FDE which is appeared as below initial value problem

$$\begin{cases} y''(t) + P(t)y'(t) + R(t) = Q(t)y(t), \\ y(a) = y_0, \\ y'(a) = y_1, \end{cases} \tag{3.3}$$

where $t \in [a, b]$, $y_0, y_1 \in \mathbb{R}_F$ are fuzzy numbers, $P, Q : [a, b] \to \mathbb{R}$ are real functions and $R : [a, b] \to \mathbb{R}_F$ is a fuzzy-valued function. Recall that this equation is said to be homogeneous if $R(t) = 0$, $\forall t \in [a, b]$. Consider the homogeneous equation corresponding to equation (3.3). It is said that the value $t = \gamma$ is an ordinary point of the equation if both $P(t)$ and $Q(t)$ are defined at $t = \gamma$. In the event that $P(t)$ or $Q(t)$ (or both) are infinite at $t = \gamma$, then γ is called a singular point of that equation and in this case, in the light of the observations just made, we form

$$p(t) = (t-\gamma)P(t), \quad q(t) = (t-\gamma)^2 Q(t).$$

Then, under this situation, we will have the following equation

$$\begin{cases} (t-\gamma)^2 y''(t) + p(t)(t-\gamma)y'(t) = q(t)y(t), \\ \quad t \in [a, b], \quad t \neq \gamma, \\ y(a) = y_0, \\ y'(a) = y_1, \end{cases} \tag{3.4}$$

where if both $p(t)$ and $q(t)$ remain finite as $t \to \gamma$, then we have a regular singular point at $t = \gamma$, that means these functions have convergent Taylor series about $t=\gamma$, i.e.

$$p(t)=p_0+p_1(t-\gamma)+p_2(t-\gamma)^2+\cdots,$$

$$q(t)=q_0+q_1(t-\gamma)+q_2(t-\gamma)^2+\cdots,$$

with p_n and q_n constants, and $q_0\neq 0$ so that $(t-\gamma)$ is not a common factor of the coefficients in equation (3.4).

The following equations are the simplest class of fuzzy differential equations in the form (3.4) that have only one regular singular point

$$\begin{cases} (t-\gamma)^2y''(t)+\alpha(t-\gamma)y'(t)=\beta y(t), \\ \quad t\in I=[a,b], \quad t\neq\gamma \\ y(a)=y_0, \\ y'(a)=y_1, \end{cases} \tag{3.5}$$

where $y_0,y_1\in\mathbb{R}_F$ and $\alpha,\beta\in\mathbb{R}$, and $\beta\neq 0$. The equations (3.5) are the homogeneous equations corresponding to equations (1.1) which are known as the Cauchy-Euler differential equations, in the crisp case.

In what follows we try to find the structure of functions that may be the solution to problem (1.1). For this end, we first point out the structure of solution in the crisp case of equation. Because of the behavior of solution in the uncertainty conditions should reflects the behavior of solution in the crisp case of the equation, namely 1-cut equation of (1.1). So, let us first consider the fuzzy Cauchy-Euler equations (1.1) and (3.5), in the crisp case. In this case, the solution is represented as $y(t)=y_p(t)+y_h(t)$, where y_p (particular solution) satisfies the inhomogeneous equation (1.1) and not necessarily satisfying the initial conditions, and y_h (homogeneous solution) is solution to equation (3.5). The function y_h will be searched as a power function $y_h(t)=(t-\gamma)^k$, where the unknown number k must be found such that y_h is valid as a solution to the homogeneous equation (3.5), and the function y_p will be usually searched based on the structure of function $R(t)$. Therefore, under this process of finding the solution, the criterion of function $R(t)$ must be given. For instance, let $R(t)$ be a real polynomial about the point $t=\gamma$, then the solution function will be obtained in the following set

$$F_\gamma(I)=\Big\{f:I\to\mathbb{R}\,\Big|\; f(t)=\sum_{i=0}^{n}a_i|t-\gamma|^{\alpha_i},\ \alpha_i\in\mathbb{R},\ a_i\in\mathbb{R}\Big\}$$

Since the functions from $F_\gamma(I)$ do not differentiable at point $t=\gamma$, so, in the crisp case, the study of Cauchy-Euler equations (1.1) and (3.5) with $R(t)$ given as a real polynomial will be limited to one of two areas $t<\gamma$ or $t>\gamma$.

Now, in order to study the fuzzy Cauchy-Euler differential equations (1.1) with $R(t)$ given as the fuzzy polynomial (1.2), and the arbitrary real number γ, we produce the following set

$$FF_\gamma(I)=\Big\{f:I\to\mathbb{R}_F\,\Big|\; f(t)=\sum_{i=0}^{n}|t-\gamma|^{\alpha_i}u_i,\ \alpha_i\in\mathbb{R},\ u_i\in\mathbb{R}_F\Big\}.$$

It is clear that $F_\gamma(I)\subset FF_\gamma(I)$ and in fact $FF_\gamma(I)$ is a extension of $F_\gamma(I)$ into fuzzy sets. Moreover, to apply the G-differentiability concept we present the following general result

Theorem 3.1. *Let $j\in\{i,ii\}$ be fixed. Suppose that y_p is (j)-differentiable solution of equation*

$$y''(t)+P(t)y'(t)+R(t)=Q(t)y(t), \quad t\in[a,b],$$

with P and Q given as real-valued functions and $R(t)$ given as fuzzy-valued function, and suppose that y_h is a (j)-differentiable solution of the homogeneous equation corresponding to (3.3), then $y=y_p+y_h$ is (j)-differentiable solution of equation (3.3).

Proof. It is straightforward. □

Let $\gamma\in\mathbb{R}$ be arbitrary but fixed. We denote by $D_F^1(I,\gamma)$, the set of functions $f:I\to\mathbb{R}_F$ such that

(a) f and f' are (i)-differentiable ((ii)-differentiable) for $t>\gamma$ and are (ii)-differentiable (respectively, (i)-differentiable) for $t<\gamma$.

(b) γ is a switching point for f.

We investigate the classes of fuzzy-valued functions which are included in $D_F^1(\mathbb{R},\gamma)$.

Example 3.1. *Take $\gamma\in\mathbb{R}$ as fixed and consider the family of functions $f_r(t)=|t-\gamma|^r u\in FF_\gamma(\mathbb{R})$, where $u\in\mathbb{R}_F$ and $r\in\{0\}\cup(1,+\infty)$. It is clear that $f_0\in D_F^1(\mathbb{R},\gamma)$. By Theorem 3.1 in [19], function f_r is (ii)-differentiable, if $t<\gamma$ and is (i)-differentiable, if $t>\gamma$ and also*

$$f_r'(t)=r|t-\gamma|^{r-1}v, \quad t\neq\gamma, \quad r\in(1,+\infty) \tag{3.6}$$

where $v = \begin{cases} u; & t > \gamma \\ -u; & t < \gamma \end{cases}$. *Take* $r > 1$ *fixed and put* $g_r(t) = r|t-\gamma|^{r-1}$. *We use of Theorem 5 in [11] for determining G-differentiability of* f'_r. *For* $t > \gamma$, *we get*

$$g_r(t)g'_r(t) = r^2(r-1)(t-\gamma)^{2r-3} > 0$$

and for $t < \gamma$, *we get*

$$g_r(t)g'_r(t) = -r^2(r-1)(\gamma-t)^{2r-3} < 0.$$

Therefore, according to cases (a) and (b) of Theorem 5 in [11], it follows that the function f'_r *is (ii)-differentiable, if* $t < \gamma$ *and is (i)-differentiable, if* $t > \gamma$. *In addition, since* $r > 1$, *we obtain*

$$\lim_{h\to 0^+} \frac{f_r(\gamma+h)\ominus f_r(\gamma)}{h} = \lim_{h\to 0^+} \frac{h^r u \ominus 0}{h} = 0,$$

and

$$\lim_{h\to 0^-} \frac{f_r(\gamma+h)\ominus f_r(\gamma)}{h} = \lim_{h\to 0^-} \frac{(-h)^r u \ominus 0}{h} = 0.$$

Therefore, f_r *is (iii)-differentiable at point* $t = \gamma$ *and consequently,* $f_r \in D^1_F(\mathbb{R},\gamma)$.
Furthermore, by Theorem 2 in [11], it can be resulted that for $r \in \{0\} \cup (1, +\infty)$,

$$f''_r(t) = r(r-1)|t-\gamma|^{r-2}u, \quad t \neq \gamma. \tag{3.7}$$

Remark 3.1. *It is noteworthy that if* $f \in D^1_F(\mathbb{R},\gamma)$ *then* γ *is not necessarily a switching point for the function* f'. *For example, consider the special function* $f_{\frac{3}{2}}(t)$ *described by Example 3.1. Then, by (3.6),* $f'_{\frac{3}{2}}(t) = \frac{3}{2}|t-\gamma|^{\frac{1}{2}}v$ *which easily gives us*

$$\lim_{h\to 0^+} \frac{1}{h}\left(f'_{\frac{3}{2}}(\gamma+h)\ominus f'_{\frac{3}{2}}(\gamma)\right) = \frac{3}{2}\left(\lim_{h\to 0^+} \frac{h^{\frac{1}{2}}}{h}v\right) = +\infty,$$

and

$$\lim_{h\to 0^-} \frac{1}{h}\left(f'_{\frac{3}{2}}(\gamma+h)\ominus f'_{\frac{3}{2}}(\gamma)\right) = \frac{3}{2}\left(\lim_{h\to 0^-} \frac{(-h)^{\frac{1}{2}}}{h}v\right) = -\infty.$$

Consequently, γ *is not a switching point for the function* $f'_{\frac{3}{2}}$.

Example 3.2. *Consider the fuzzy polynomial* $R(t) = \sum_{i=0}^n (t-\gamma)^i u_i$ *where* $t \in \mathbb{R}$, u_i *is a fuzzy number for* $i \in \{0,1,...,n\}$ *and* γ *is a real constant. It is clear that* $R \in FF_\gamma(\mathbb{R})$. *By Remark 3.3 in [19],* R *is (i)-differentiable for* $t > \gamma$ *and (ii)-differentiable for* $t < \gamma$ *and further*

$$R'(t) = \sum_{i=1}^n i(t-\gamma)^{i-1}u_i.$$

If we set $R_i(t) = (t-\gamma)^i u_i$ *then, similar to Example 3.1, it can be resulted that* R_i *is (iii)-differentiable at* $t = \gamma$ *and also, the differentiability type of function* R'_i *is same as* R_i. *Then,* $R_i \in D^1_F(\mathbb{R},\gamma)$ *for* $i \in \{0,1,...,n\}$, *which gives us* $R \in D^1_F(\mathbb{R},\gamma)$, *by Lemma 4 in [11]. Moreover, we get*

$$R''(t) = \sum_{i=2}^n i(i-1)(t-\gamma)^{i-2}u_i.$$

We now give the process of obtaining solution to the fuzzy Cauchy-Euler differential equations (1.1), with $R(t)$ given as (1.2), under the G-differentiability concept. Since $R(t)$ is a fuzzy polynomial about the point $t = \gamma$, then $R \in D^1_F(\mathbb{R},\gamma)$, by Example 3.1. We thus focus on the set $D^1_F(\mathbb{R},\gamma)$ to search the solution of the equation. First, consider equation (1.1) in the following popular form

$$(t-\gamma)^2 y''(t) + \alpha(t-\gamma)y'(t) + R(t) = \beta y(t), t \neq \gamma. \tag{3.8}$$

In order to establish a balance on both sides of equality (3.8), we assume that the following fuzzy polynomial satisfies the equation (3.8)

$$y_p(t) = \sum_{i=0}^n (t-\gamma)^i v_i,$$

where the fuzzy numbers v_i, $[v_i]^\alpha = [v_i^-(\alpha), v_i^+(\alpha)]$, $\forall\alpha \in [0,1]$ are unknown. Then, we should be have

$$(t-\gamma)^2 y''_p(t) + \alpha(t-\gamma)y'_p(t) + R(t) = \beta y_p(t).$$

According to the results given in Example 3.2, we get

$$\sum_{i=0}^n i(i-1)(t-\gamma)^i v_i + \alpha \sum_{i=0}^n i(t-\gamma)^i v_i + \sum_{i=0}^n (t-\gamma)^i u_i = \beta \sum_{i=0}^n (t-\gamma)^i v_i,$$

which easily results in

$$m(m-1)v_m + m\alpha v_m + u_m = \beta v_m, \tag{3.9}$$

for $m = 0, 1, ..., n$. If $\beta = 0$ then the system (3.9) has not solution in an unreal fuzzy environment. We now study the system (3.9) for various cases of signs α and β, in the following.

Case (a). $\alpha \geq 0$ and $\beta > 0$.
In this case, the equalities (3.9) can be written as follows:

$$(m(m-1) + m\alpha)\, v_m + u_m = \beta v_m,$$

i.e.

$$\beta v_m \ominus m(m+\alpha-1)v_m = u_m, \tag{3.10}$$

for $m = 0, 1, ..., n$. Hence for each $\alpha \geq 0$ the final sequence $\{m(m+\alpha-1)\}_{m=0}^n$ is an increasing sequence in m, included as nonnegative real numbers then all H-differences (3.10) exist if and only if $\beta \geq n(n+\alpha-1)$. Under this situation, we get

$$(\beta - m(m+\alpha-1))v_m = u_m, \quad m = 0, 1, ..., n.$$

If $\beta = n(n+\alpha-1)$, then the left side of the equation corresponding to $m = n$ of the last system is equal to zero while $u_n \neq 0$, because R is a fuzzy polynomial of degree n. Thus, $\beta > n(n+\alpha-1)$, and we infer

$$v_m = \frac{1}{\beta - m(m+\alpha-1)} u_m, \tag{3.11}$$

for $m = 0, 1, ..., n$.

Case (b). $\alpha \geq 0$ and $\beta < 0$.
Let us consider each one of equations (3.10) in the r-cut form:

$$[\beta v_m \ominus m(m+\alpha-1)v_m]^r = [u_m]^r, \quad \forall r \in [0,1].$$

It easily follows that

$$\begin{cases} \beta v_m^+(r) - m(m+\alpha-1)v_m^-(r) = u_m^-(r) \\ \beta v_m^-(r) - m(m+\alpha-1)v_m^+(r) = u_m^+(r) \end{cases}$$

that results in

$$\begin{cases} a_m v_m^-(r) = m(m+\alpha-1)u_m^-(r) + \beta u_m^+(r) \\ a_m v_m^+(r) = \beta u_m^-(r) + m(m+\alpha-1)u_m^+(r) \end{cases} \tag{3.12}$$

where $a_m = \beta^2 - m^2(m+\alpha-1)^2$. By monotonicity the functions $u_m^-(r)$ and $u_m^+(r)$, and since $\alpha \geq 0$ and $\beta < 0$, we realize the functions $v_m^-(r)$ and $v_m^+(r)$, satisfying the system (3.12), are valid as lower and upper functions of a fuzzy number if and only if $a_m > 0$, for $m = 0, 1, \cdots, n$. But, the conditions $a_m > 0$ are equivalent to conditions $\beta + m(m+\alpha-1) < 0$, because of

$$a_m = (\beta - m(m+\alpha-1))(\beta + m(m+\alpha-1))$$

and $\beta - m(m+\alpha-1) < 0$. Therefore, by assuming that $\beta < -n(n+\alpha-1)$, we obtain from systems (3.12), the following:

$$\begin{aligned} v_m &= \frac{\beta}{\beta^2 - m^2(m+\alpha-1)^2} u_m \\ &\quad + \frac{m(m+\alpha-1)}{\beta^2 - m^2(m+\alpha-1)^2} u_m, \end{aligned} \tag{3.13}$$

for $m = 0, 1, \cdots, n$.

Case (c). $\alpha \leq 0$ and $\beta > 0$.
Similar to previous case we obtain by (3.9) the following system

$$\begin{cases} m(m-1)v_m^-(r) + m\alpha v_m^+(r) + u_m^-(r) = \beta v_m^-(r) \\ m(m-1)v_m^+(r) + m\alpha v_m^-(r) + u_m^+(r) = \beta v_m^+(r) \end{cases}$$

that results in

$$\begin{cases} b_m v_m^-(r) = (\beta - m(m-1))u_m^-(r) + m\alpha u_m^+(r) \\ b_m v_m^+(r) = m\alpha u_m^-(r) + (\beta - m(m-1))u_m^+(r) \end{cases} \tag{3.14}$$

where $b_m = (\beta - m(m-1))^2 - m^2\alpha^2$. We show that if u_m is an unreal fuzzy number then the system (3.14) has solution as unreal fuzzy number v_m, if and only if $b_m > 0$. By subtracting two sides equations in system (3.14), we get

$$\begin{aligned} &b_m(v_m^+(r) - v_m^-(r)) \\ &= (\beta - m(m-1) - m\alpha)(u_m^+(r) - u_m^-(r)). \end{aligned} \tag{3.15}$$

Suppose that v_m is an unreal fuzzy number and $b_m < 0$. In this case, the equality (3.15) implies $\beta - m(m-1) - m\alpha < 0$. Since

$$b_m = (\beta - m(m-1) - m\alpha)(\beta - m(m-1) + m\alpha) \tag{3.16}$$

then, we should be have

$$\beta - m(m-1) + m\alpha > 0,$$

which implies $\beta - m(m-1) > 0$. It follows that, the monotony type of functions located on two sides of each of equations of system (3.14) is not the same which is a contradiction. Conversely, suppose that $b_m > 0$. According to system (3.14), it is sufficient to show that $\beta - m(m-1) > 0$. Suppose that there exists $m \in \{0, 1, \cdots, n\}$ such that $\beta - m(m-1) < 0$, then $\beta - m(m-1) + m\alpha < 0$, which follows that $\beta - m(m-1) - m\alpha < 0$, by (3.16) and that $b_m > 0$. Since u_m is an unreal fuzzy number then the equality (3.15) implies $v_m^+(r) < v_m^-(r)$ and, thus v_m can not be a fuzzy number.

According to (3.16), the conditions $b_m > 0$ for $m = 0, 1, \cdots, n$ are equivalent to conditions $\beta > m(m - \alpha - 1)$, for $m = 0, 1, \cdots, n$. Consequently, by assuming that $\beta > n(n - \alpha - 1)$, we obtain

$$v_m = \frac{m\alpha}{(\beta - m(m-1))^2 - m^2\alpha^2} u_m + \frac{\beta - m(m-1)}{(\beta - m(m-1))^2 - m^2\alpha^2} u_m, \tag{3.17}$$

for $m = 0, 1, \cdots, n$.

Case (d). $\alpha \leq 0$ and $\beta < 0$.

Similar to the previous cases, we obtain

$$\begin{cases} c_m v_m^-(r) = m(m-1)u_m^-(r) + (\beta - m\alpha)u_m^+(r) \\ c_m v_m^+(r) = (\beta - m\alpha)u_m^-(r) + m(m-1)u_m^+(r) \end{cases} \tag{3.18}$$

where $c_m = (\beta - m\alpha)^2 - m^2(m-1)^2$. Similar to case (c), it can be shown that the system (3.18) has solution as unreal fuzzy number v_m, if and only if $c_m > 0$, which are equivalent to conditions $\beta < -m(m - \alpha - 1)$, for $m = 0, 1, \cdots, n$. Therefore, by assuming that $\beta < -n(n - \alpha - 1)$, we obtain

$$v_m = \frac{m(m-1)}{(\beta - m\alpha)^2 - m^2(m-1)^2} u_m + \frac{\beta - m\alpha}{(\beta - m\alpha)^2 - m^2(m-1)^2} u_m, \tag{3.19}$$

for $m = 0, 1, \cdots, n$. By aggregating the results of cases (a)-(d), it can be said that if $|\beta| > n(n + |\alpha| - 1)$ then the equation (3.8) has a solution as

$$y_p(t) = \sum_{i=0}^{n} (t-\gamma)^i v_i, \tag{3.20}$$

which is uniquely obtained from set $FF_\gamma(\mathbb{R})$, using one of the formulas (3.11), (3.13), (3.17) or (3.19).

Now, we consider the equation (3.8) on the interval $I = [a, b]$, with fuzzy initial conditions $y(a) = y_0$ and $y'(a) = y_1$. For the sake of instituting the initial conditions, we add a complementary function $y_h(t)$ and represent the solution function as

$$y(t) = y_p(t) + y_h(t).$$

Since $y_p(t) \in FF_\gamma(\mathbb{R})$ is a unique solution to the equation (3.8), then if $y_h(t) \in FF_\gamma(\mathbb{R})$ then it should be satisfied in the following homogeneous equation

$$(t-\gamma)^2 y_h''(t) + \alpha(t-\gamma)y_h'(t) = \beta y_h(t). \tag{3.21}$$

Hence the G-differentiability type y_h should be same as y_p and hence y_h should be satisfied the equation (3.21) so, we suggest and search the following representation as a solution of (3.21)

$$y_h(t) = |t-\gamma|^s v, \quad v \in \mathbb{R}_F, \quad s > 0.$$

According to the formulas (3.6) and (3.7), we have

$$y_h'(t) = \begin{cases} s(t-\gamma)^{s-1}v; & t > \gamma \\ -s(\gamma-t)^{s-1}v; & t < \gamma \end{cases} \tag{3.22}$$

and

$$y_h''(t) = s(s-1)|t-\gamma|^{s-2}v, \quad t \neq \gamma. \tag{3.23}$$

By substituting (3.22) and (3.23) into (3.21), it is easy to deduce that

$$s(s-1)|t-\gamma|^s v + \alpha s|t-\gamma|^s v = \beta|t-\gamma|^s v.$$

Since $t \neq \gamma$, the last equation results in

$$s(s-1)v + \alpha s v = \beta v,$$

By assuming $s > 1$, the last equation leads to one of the following equations:

$$\begin{cases} s(s-1) + \alpha s = \beta, \\ \text{if } \alpha \geq 0, \quad \beta > 0, \\ s(s-1) + \alpha s = -\beta, \\ \text{if } \alpha \geq 0, \quad \beta < 0, \quad v = -v, \\ -s(s-1) = \beta - \alpha s, \\ \text{if } \alpha \leq 0, \quad \beta < \alpha s, \quad v = -v, \\ \alpha s = \beta - s(s-1), \\ \text{if } \alpha \leq 0, \quad \beta > s(s-1), \quad v = -v. \end{cases}$$

By assuming that

$$|\beta|> max\{s|\alpha|,\ s(s-1)\},$$

so, each one of the last equalities can be written as follows:

$$s(s-1)+|\alpha|s=|\beta|$$

that gives

$$s_1=\frac{1}{2}\left(\sqrt{4|\beta|+(|\alpha|-1)^2}+1-|\alpha|\right)$$

and

$$s_2=\frac{1}{2}\left(-\sqrt{4|\beta|+(|\alpha|-1)^2}+1-|\alpha|\right).$$

It is easy to see that $s_2<1$. Since $|\beta|>|\alpha|$ then $s_1>1$ and we thus put $s=s_1$. Since $s>1$, then, according to results from Example 3.1, the G-differentiability type of y_h is similar to the G-differentiability type of y_p, on each given interval $I=(a,b)$ and also, y_h has a switching point at $t=\gamma$. Now, we check that y_h is not one of terms contained in y_p. Since $|\beta|>n(n+|\alpha|-1)$, we obtain

$$\begin{aligned} s &= \frac{1}{2}\left(\sqrt{4|\beta|+(|\alpha|-1)^2}+1-|\alpha|\right)\\ &> \frac{1}{2}\Big(\sqrt{4n(n+|\alpha|-1)+(|\alpha|-1)^2}\\ &\quad +1-|\alpha|\Big)\\ &= \frac{1}{2}\left(|2n+|\alpha|-1|+1-|\alpha|\right)\geq n. \end{aligned}$$

Consequently, the solution function is obtained as follows:

$$y(t)=\sum_{i=0}^{n}(t-\gamma)^i v_i+|t-\gamma|^s v.$$

Finally, under initial condition $y(a)=y_0$, it can be obtained that $v=w_1$, where

$$w_1=|a-\gamma|^{-s}\left\{y_0\ominus\left(\sum_{i=0}^{n}(a-\gamma)^i v_i\right)\right\}, \tag{3.24}$$

and $\gamma\neq a$. Under the initial condition $y'(a)=y_1$, we get $v=w_2$, where

$$w_2=\frac{|a-\gamma|^{2-s}}{s(a-\gamma)}\left\{y_1\ominus\left(\sum_{i=1}^{n}i(a-\gamma)^{i-1}v_i\right)\right\}, \tag{3.25}$$

and $\gamma\neq a$. Of course, provided that the above mentioned H-differences exist.

Consequently, we have proved the following result.

Theorem 3.2. *Let $\gamma\in\mathbb{R}$ be as fixed and consider the fuzzy differential equation*

$$\begin{cases} (t-\gamma)^2y''(t)+\alpha(t-\gamma)y'(t)\\ +\sum_{i=0}^{n}(t-\gamma)^i u_i=\beta y(t),\\ y(a)=y_0\in\mathbb{R}_F,\quad and/or\\ y'(a)=y_1\in\mathbb{R}_F \end{cases} \tag{3.26}$$

where $t\in[a,b]$, $t\neq\gamma$,, $u_i\in\mathbb{R}_F$, for $i=0,1,...,n$ and α, $\beta\in\mathbb{R}$, such that

$$|\beta|> max\{n(n+|\alpha|-1),s|\alpha|,s(s-1)\}$$

and

$$s=\frac{1}{2}\left(\sqrt{4|\beta|+(|\alpha|-1)^2}+1-|\alpha|\right). \tag{3.27}$$

Then, the expression

$$y(t)=\sum_{i=0}^{n}(t-\gamma)^i v_i+|t-\gamma|^s v, \tag{3.28}$$

is solution of the equation such that for cases (a) $\alpha\geq 0$ and $\beta>0$, (b) $\alpha\geq 0$ and $\beta<0$, (c) $\alpha\leq 0$ and $\beta>0$, and (d) $\alpha\leq 0$ and $\beta<0$, the numbers v_i are respectively, described by (3.11), (3.13), (3.17) and (3.19). The fuzzy number v is described by $v=w_1$ given as (3.25), if $y(a)=y_0$ and it is described by $v=w_2$ given as (3.26), if $y'(a)=y_1$ and it must be satisfied $v=-v$ for cases (b), (c) and (d). Furthermore, if $\gamma\leq a$ then $y(t)$ is a (i)-differentiable solution on (a,b), and if $\gamma\geq b$ then $y(t)$ is a (ii)-differentiable solution on (a,b), and if $\gamma\in(a,b)$ then $y(t)$ has a switching point at $t=\gamma$.

The following property is easy to obtain.

Proposition 3.1. *Suppose that the conditions of Theorem 3.2 hold on the equation (3.26). Then, the function $y(t)$ from (3.28), satisfies both initial conditions $y(a)=y_0$ and $y'(a)=y_1$, if and only if both the H-differences (3.24) and (3.25) exist, $w_1=w_2$ and further, for each one of cases (b), (c) and (d) of Theorem 3.2, we have $w_1=-w_2$.*

Remark 3.2. *According to Theorem 3.2, if the H-differences (3.24) or (3.25) exist, then the found solution function is satisfied at least one of the initial conditions $y(a) = y_0$ or $y'(a) = y_1$. This can be an advantage. In fact, in the crisp case, the popular solution to a second-order differential equation consists of two arbitrary constants, so two conditions (usually, determined as initial or boundary values) are required to obtain a particular solution of it equation. Determining two initial values need to more cognition of the behavior of the solution function at the first point of interval under study. This could be a strong constraint especially when the initial values are uncertain.*

4 Examples

In order to the practical application and observing the behavior of solution function, we solve some examples.

Example 4.1. *Consider fuzzy differential equation*

$$\begin{cases} (t-1)^2y''(t) + (t-1)y'(t) \\ +u_0 + (t-1)u_1 = \frac{9}{4}y(t), \ t \geq 0, \ t \neq 1, \\ y(0) = y_0, \end{cases} \tag{4.29}$$

where $[u_0]^r = [\frac{1}{2}r, 1-\frac{1}{2}r]$, $[u_1]^r = [\frac{1}{4}r, \frac{1}{2}-\frac{1}{4}r]$ *and* $[y_0]^r = [-1+r, 1-r]$, *for* $r \in [0,1]$.

Here $n = 1$, $\alpha = 1$, *and* $\beta = \frac{9}{4}$. *So, the conditions of Theorem 3.2 are clearly hold. Also, the H-difference (3.24) exists, because for each* $r \in [0,1]$, *we obtain*

$$\begin{aligned} [w_1]^r &= \Big[y_0 \ominus \Big(\sum_{i=0}^{1} \frac{(-1)^i}{\beta - i^2} u_i\Big)\Big]^r \\ &= \Big[y_0 \ominus \Big(\frac{4}{9}u_0 - \frac{4}{5}u_1\Big)\Big]^r \\ &= \Big[\frac{-3}{5} + \frac{26}{45}r, \frac{5}{9} - \frac{26}{45}r\Big]. \end{aligned}$$

Therefore, by using Theorem 3.2, we obtain the solution expression as follows:

$$y(t) = \frac{4}{9}u_0 + \frac{4}{5}(t-1)u_1 + |t-1|^{\frac{3}{2}}w_1,$$

which represents two criteria as solution functions, for equation (4.29), y_1 *that is (i)-differentiable for* $t > 1$, *with the following r-cut form*

$$[y_1(t)]^r = \Big[\frac{2}{9}r + \frac{1}{5}r(t-1) + (\frac{-3}{5} + \frac{26}{45}r)(t-1)^{\frac{3}{2}}, \\ \frac{4}{9} - \frac{2}{9}r + (\frac{2}{5} - \frac{1}{5}r)(t-1) + (\frac{5}{9} - \frac{26}{45}r)(t-1)^{\frac{3}{2}}\Big]$$

and function y_2 *that is (ii)-differentiable for* $0 \leq t < 1$, *with the following r-cuts form*

$$[y_2(t)]^r = \Big[\frac{2}{9}r + (\frac{2}{5} - \frac{1}{5}r)(t-1) + (\frac{-3}{5} + \frac{26}{45}r)(1-t)^{\frac{3}{2}}, \\ \frac{4}{9} - \frac{2}{9}r + \frac{1}{5}r(t-1) + (\frac{5}{9} - \frac{26}{45}r)(1-t)^{\frac{3}{2}}\Big].$$

Moreover, by Theorem 3.2, $y(t)$ *has a switching point at* $t = 1$.

The graphical representation of the solution function $y(t)$, *for* $t \in [0,2]$ *and for three r-cuts* $r = 0$, $r = 0.5$ *and* $r = 1$, *can be seen in Figure 1.*

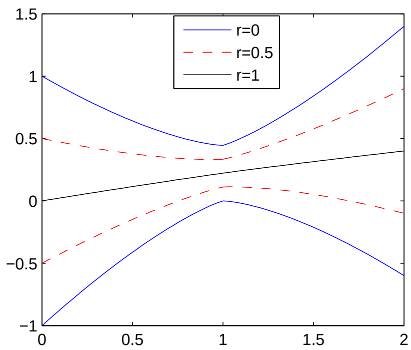

Figure 1. The solution of example 4.1.

Example 4.2. *Consider fuzzy differential equation*

$$\begin{cases} (t-2)^2y''(t) - 2(t-2)y'(t) + \\ u_0 + (t-2)u_1 + (t-2)^2u_2 = 12y(t), \\ y'(1) = y_1, \quad t \geq 1, \quad t \neq 2. \end{cases} \tag{4.30}$$

Where

$$[u_0]^r = [\frac{1}{2} + \frac{1}{2}r, \frac{3}{2} - \frac{1}{2}r],$$

$$[u_1]^r = [\frac{3}{2} + \frac{1}{2}r, \frac{5}{2} - \frac{1}{2}r],$$

$$[u_2]^r = [2+r, 4-r],$$

and

$$[y_1]^r = [-1 + \frac{5}{7}r, \frac{3}{7} - \frac{5}{7}r],$$

for all $r \in [0,1]$.

Here $n = 2$, $\alpha = -2$ *and* $\beta = 12$ *which give* $s = 3$. *The H-difference (3.25) exists, because by substituting (3.19) into (3.25), we have*

$$[w_2]^r = -\frac{1}{3}\Big[y_1 \ominus \Big(\sum_{i=1}^{2} i(-1)^{i-1}\big(\frac{-2i}{(12-i(i-1))^2-4i^2}u_i + \frac{12-i(i-1)}{(12-i(i-1))^2-4i^2}u_i\big)\Big)\Big]^r = \frac{139}{1260}[-1+r, 1-r].$$

We thus obtain the solution representation as follows:

$$\begin{aligned} y(t) &= v_0 + (t-2)v_1 + (t-2)^2 v_2 + |t-2|^3 w_2 \\ &= \frac{1}{12}u_0 + (t-2)(-\frac{1}{70}u_1 + \frac{3}{35}u_1) \\ &+ (t-2)^2(-\frac{1}{21}u_2 + \frac{5}{42}u_2) + |t-2|^3 w_2. \end{aligned}$$

This function provides two functions

$$\begin{aligned} [y_1(t)]^r = \Big[&\frac{1}{24} + \frac{1}{24}r + (\frac{13}{140} + \frac{1}{20}r)(t-2) \\ &+ (\frac{1}{21} + \frac{7}{42}r)(t-2)^2 \\ &+ \frac{139}{1260}(-1+r)(t-2)^3, \\ &\frac{1}{8} - \frac{1}{24}r + (\frac{27}{140} - \frac{1}{20}r)(t-2) \\ &+ (\frac{8}{21} - \frac{7}{42}r)(t-2)^2 \\ &+ \frac{139}{1260}(1-r)(t-2)^3,\Big] \end{aligned}$$

which is (i)-differentiable for $t > 2$, *and*

$$\begin{aligned} [y_2(t)]^r &= \Big[\frac{1}{24} + \frac{1}{24}r + (\frac{27}{140} - \frac{1}{20}r)(t-2) \\ &+ (\frac{1}{21} + \frac{7}{42}r)(t-2)^2 + \frac{139}{1260}(1-r)(t-2)^3, \\ &\frac{1}{8} - \frac{1}{24}r + (\frac{13}{140} + \frac{1}{20}r)(t-2) \\ &+ (\frac{8}{21} - \frac{7}{42}r)(t-2)^2 + \frac{139}{1260}(-1+r)(t-2) \end{aligned}$$

which is (ii)-differentiable for $1 < t < 2$.

Also, $t = 2$ *is switching point for* $y(t)$, *by Theorem 3.2.*

The graphical representation of solution function $y(t)$ *in interval* $[1,3]$, *for three* r*-cuts,* $r = 0$, $r = 0.5$ *and* $r = 1$, *can be seen in Figure 2.*

For more explanation of solution, let us consider the equation (4.30) in the crisp case, i.e.

$$\begin{cases} (t-2)^2 y''(t) - 2(t-2)y'(t) + [u_0]^1 + \\ (t-2)[u_1]^1 + (t-2)^2[u_2]^1 = 12y(t), \\ y'(1) = [y_1]^1, \quad t \geq 1, \quad t \neq 2. \end{cases}$$

that is

$$\begin{cases} (t-2)^2 y''(t) - 2(t-2)y'(t) + 1 + \\ 2(t-2) + 3(t-2)^2 = 12y(t), \\ y'(1) = -\frac{2}{7}, \quad t \geq 1, \quad t \neq 2. \end{cases}$$

For this initial value problem, it is clear that the function $\bar{y}(t)$, *defined as*

$$\bar{y}(t) = [y_2(t)]^1 = \frac{1}{12} + \frac{1}{7}(t-2) + \frac{3}{14}(t-2)^2,$$

is solution defined on $t \geq 1$. *However, according to classical method initial value problem*

$$\begin{cases} (t-2)^2 y''(t) - 2(t-2)y'(t) + 1 + \\ 2(t-2) + 3(t-2)^2 = 12y(t), \\ y(1) = c \\ y'(1) = -\frac{2}{7}, \quad t \geq 1, \quad t \neq 2 \end{cases}$$

with c *as an arbitrary constant is solved for* $1 \leq t < 2$ *and it has solution as follows:*

$$\begin{aligned} y(t) = \bar{y}(t) + &\\ \frac{1}{2}(c - \frac{13}{84})\Big(&(1 - \frac{3}{\sqrt{57}})(2-t)^{\frac{3+\sqrt{57}}{2}} \\ &+ (1 + \frac{3}{\sqrt{57}})(2-t)^{\frac{3-\sqrt{57}}{2}}\Big). \end{aligned}$$

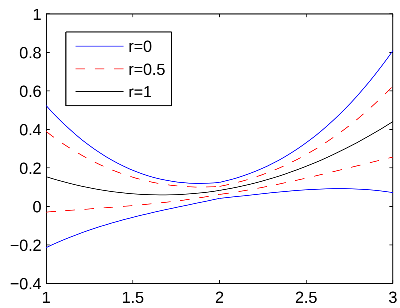

Figure 2. The solution of example 4.2.

Example 4.3. *Let us consider a FDE in different form as follows:*

$$\begin{cases} (t-\frac{1}{2})^2 y''(t) + (t-\frac{1}{2})y'(t) + e^{-t}a = 3y(t), \\ y(0) = y_0, \end{cases} \tag{4.31}$$

where $0 \leq t < \frac{1}{2}$, *and*

$$[a]^r = [\frac{1}{2}r, 1 - \frac{1}{2}r],$$

$$[y_0]^r = [-\frac{1}{2} + \frac{13}{8}r, \frac{11}{4} - \frac{13}{8}r],$$

for all $r \in [0,1]$.

Hence $0 \leq t < \frac{1}{2}$, *then a linear approximation can be a suitable alternative for factor* $e^{-t}a$. *Therefore, we write*

$$\begin{aligned} e^{-t}a \cong (1-t)a &= \Big(\frac{1}{2} - (t - \frac{1}{2})\Big)a \\ &= \frac{1}{2}a - (t - \frac{1}{2})a. \end{aligned}$$

We solve the following approximation equation instead of equation (4.31)

$$\begin{cases} (t-\frac{1}{2})^2 y''(t) + (t-\frac{1}{2})y'(t) + \\ u_0 + (t-\frac{1}{2})u_1 = 3y(t), \\ y(0) = y_0, \quad 0 \leq t < \frac{1}{2}, \end{cases} \tag{4.32}$$

where $u_0 = \frac{1}{2}a$ *and* $u_1 = -a$.

Here $n = 1$, $\alpha = 1$ *and* $\beta = 3$, *which hold the conditions of Theorem 3.2 and we get* $s = \sqrt{3}$, *by (3.27). Also, the H-difference (3.24) exist, because*

$$\begin{aligned} [w_1]^r &= 2^{\sqrt{3}}\Big[y_0 \ominus \Big(\frac{1}{3}u_0 - \frac{1}{4}u_1\Big)\Big]^r \\ &= 2^{\sqrt{3}}\Big[-\frac{1}{2} + \frac{17}{12}r, \frac{7}{3} - \frac{17}{12}r\Big], \ \forall r \in [0,1]. \end{aligned}$$

Consequently, we obtain (ii)-differentiability solution of (4.32), as follows:

$$y(t) = \frac{1}{3}u_0 - \frac{1}{2}(t - \frac{1}{2})u_1 + (\frac{1}{2} - t)^{\sqrt{3}}v_1$$

which has the r-cut form

$$\begin{aligned} &[y(t)]^r = \\ &\Big[\frac{1}{12}r - \frac{1}{4}r(t - \frac{1}{2}) + 2^{\sqrt{3}}(-\frac{1}{2} + \frac{17}{12}r)(\frac{1}{2} - t)^{\sqrt{3}}, \\ &\frac{1}{6} - \frac{1}{12}r + (-\frac{1}{2} + \frac{1}{4}r)(t - \frac{1}{2}) \\ &+ 2^{\sqrt{3}}(\frac{7}{3} - \frac{17}{12}r)(\frac{1}{2} - t)^{\sqrt{3}}\Big]. \end{aligned}$$

The graphical representation of solution, for $r = 0$, $r = 0.5$ *and* $r = 1$ *of r-cuts is given in Figure 3.*

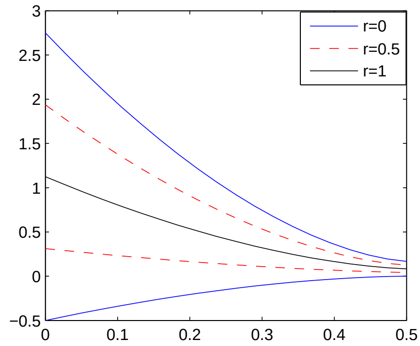

Figure 3. The solution of example 4.3.

Example 4.4. *Consider fuzzy differential equation*

$$\begin{cases} (t-1)^2 y''(t) + (t-1)y'(t) + u_0 = 4y(t), \\ y(0) = y_0, \\ y'(0) = y_1, \end{cases} \tag{4.33}$$

where $t \geq 0$, $t \neq 1$, $[u_0]^r = [\frac{1}{2} + \frac{1}{2}r, \frac{3}{2} - \frac{1}{2}r]$, $[y_0]^r = [-1+r, 1-r]$ *and* $[y_1]^r = [-\frac{5}{4} + \frac{7}{4}r, \frac{9}{4} - \frac{7}{4}r]$, *for* $r \in [0,1]$.

Here $n = 0$, $\alpha = 1$, *and* $\beta = 4$. *So, the conditions of Theorem 3.2 are clearly, hold and we have* $s = 2$. *Also, we get by (3.24) and (3.25) the following*

$$[w_1]^r = \Big[y_0 \ominus \frac{1}{4}u_0\Big]^r = \Big[-\frac{9}{8} + \frac{7}{8}r, \frac{5}{8} - \frac{7}{8}r\Big],$$

and

$$[w_2]^r = -\frac{1}{2}[y_1]^r = [w_1]^r.$$

Then $v = w_1 = w_2$ *and by Theorem 3.2, we obtain for* $r \in [0,1]$,

$$\begin{aligned} [y(t)]^r &= \Big[\frac{1}{4}u_0 + (t-1)^2 v\Big]^r \\ &= \Big[\frac{1}{8} + \frac{1}{8}r + \Big(-\frac{9}{8} + \frac{7}{8}r\Big)(t-1)^2, \\ &\quad \frac{3}{8} - \frac{1}{8}r + \Big(\frac{5}{8} - \frac{7}{8}r\Big)(t-1)^2\Big], \end{aligned}$$

According to Theorem 3.2, the function $y(t)$ *is (ii)-differentiable for* $0 < t < 1$, *(i)-differentiable for* $t > 1$ *and (iii)-differentiable at* $t = 1$. *This function is shown in Figure 4, for* $t \in [0,2]$ *and for* $r = 0$, $r = 0.5$ *and* $r = 1$, *from r-cuts.*

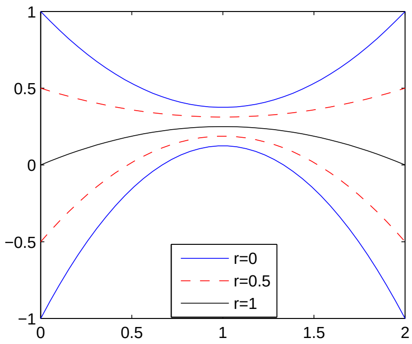

Figure 4. The solution of example 4.4.

5 Conclusion and future research

In this work, we have obtained the representation of explicitly solution function and the conditions of it's existence to a class of second-order fuzzy differential equations, under generalized differentiability. We saw that the solution function can be obtained by studying the generalize differentiable fuzzy-valued functions without turning the problem into a system of ordinary differential equations.

For future research, researchers can study the other forms of the problem and obtain solutions under other concepts of differentiability such as inclusion derivative and Zadeh's extension principle. Also, the fuzzy Cauchy-Euler equation can be studied as a boundary value problem.

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