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Research Article

A Numerical Method for Solving Fuzzy Differential Equations With Fractional Order

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Abstract

In this paper, we present a numerical method for solving fuzzy differential equation of fractional order under gH-fractional Caputo differentiability. The main idea of this method is to approximate the solution of fuzzy fractional differential equation (FFDE) by an implicit method as corrector and explicit method as predictor. This method is tested on numerical examples.

Keywords : Fuzzy fractional differential equations; Caputo generalized Hukuhara differentiability; Predictor-Corrector method.

1 Introduction

Fuzzy fractional differential equation (FFDE) is a generalization of fuzzy ordinary differential equation to arbitrary non-integer order. FFDE is used in mathematical modeling of several real world physical phenomena and various branch of science. The concept of FFDE was introduced by Agarwal, Lakshmikantham and Nieto [1]. Arshad and Lupulescu proved some results on the existence and uniqueness of solutions of Riemann-Liouville fuzzy fractional differential equations in [2, 3]. Allahviranloo et al in [14, 5] proposed the analytical methods for solving fuzzy fractional differential equations under Riemann-Liouville H-differentiability using Mittag- Leffler functions and Laplace transforms method. Mazandarani and Vahidian Kamyad [10], introduced the solution to Fuzzy Fractional Initial Value Problem under Caputo- type fuzzy fractional derivatives by a modified fractional Euler method. The numerical method for solving FFDE is an area not yet widely investigated, in this paper we introduce a Predictor-Corrector method for solving fuzzy fractional differential equation. In this approach, the FFDEs are expressed in terms of Caputo type under the Generalized Hukuhara differentiability.

The paper is organized as follows: In section 2, some basic definitions are brought. A proposed method for FFDE is introduced in section 3. A numerical example are presented in section 4 and finally conclusion is drawn.

2 Preliminaries

First notations which shall be used in this paper are introduced.

We denote by $\mathbb{R}_{\mathcal{F}}$, the set of fuzzy numbers, that is, normal, fuzzy convex, upper semi-continuous

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and compactly supported fuzzy sets which are defined over the real line.
For $0 < r \leq 1$, set $[u]^r = \Big\{t \in \mathbb{R}\Big| u(t) \geq r\Big\}$, and $[u]^0 = cl\Big\{t \in \mathbb{R}\Big| u(t) > 0\Big\}$. We represent $[u]^r = [u^-(r), u^+(r)]$, so if $u \in \mathbb{R}_{\mathcal{F}}$, the r-level set $[u]^r$ is a closed interval for all $r \in [0,\ 1]$. For arbitrary $u,\ v \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$, the addition and scalar multiplication are defined by $[u+v]^r = [u]^r + [v]^r$, $[ku]^r = k[u]^r$ respectively.

A triangular fuzzy number is defined as a fuzzy set in $\mathbb{R}_{\mathcal{F}}$, that is specified by an ordered triple $u = (a, b, c) \in \mathbb{R}^3$ with $a \leq b \leq c$ such that $u^-(r) = a + (b-a)r$ and $u^+(r) = c - (c-b)r$ are the endpoints of r-level sets for all $r \in [0, 1]$.

The Hausdorff distance between fuzzy numbers is given by $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}^+ \cup \{0\}$ as in [11]

$$D(u,\ v) = \sup_{r\in[0,\ 1]} \max\Big\{|u^-(r) - v^-(r)| , |u^+(r) - v^+(r)|\Big\}. \quad (2.1)$$

Consider $u, v, w, z \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, then the following properties are well-known for metric D,

1. $D(u \oplus w,\ v \oplus w) = D(u,\ v)$;
2. $D(\lambda u,\ \lambda v) = |\lambda| D(u,\ v)$;
3. $D(u \oplus v,\ w \oplus z) \leq D(u,\ w) + D(v,\ z)$;
4. $D(u \ominus v,\ w \ominus z) \leq D(u,\ w) + D(v,\ z)$, as long as $u \ominus v$ and $w \ominus z$ exist, where $u,\ v,\ w,\ z \in \mathbb{R}_{\mathcal{F}}$.

where, $\ominus$ is the Hukuhara difference(H-difference), it means that $w \ominus v = u$ if and only if $u \oplus v = w$.

Definition 2.1 *The generalized Hukuhara difference of two fuzzy numbers $u,\ v \in \mathbb{R}_{\mathcal{F}}$ is defined as follows*

$$u \ominus_{gH} v = w \Longleftrightarrow \begin{cases} (i).\ u = v + w; \\ or\ (ii).\ v = u + (-1)w. \end{cases}$$

The conditions for the existence of $u \ominus_{gH} v \in \mathbb{R}_{\mathcal{F}}$ are given in [7].

Definition 2.2 *Let $u, v \in R_F$. If there exists $w \in R_F$ such that*

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} (i) & u = v + w, \\ or\ (ii) & v = u + (-1)w, \end{cases}$$

Then w is called the generalized Hukuhara difference of u and v.

A function $f : [a, b] \to R_F$ so called fuzzy-valued function. The r-level representation of fuzzy valued function f is expressed by $[f]^r(t) = [f^-(t, r), f^+(t, r)]$, where $t \in [a, b]$, $r \in [0, 1]$.

Definition 2.3 *The generalized Hukuhara derivative of a fuzzy-valued function $f : (a, b) \to R_F$ at t_0 is defined as*

$$f'_{gH}(t_0) = \lim_{h\to 0} \frac{f(t_0 + h) \ominus_{gH} f(t_0)}{h}, \quad (2.2)$$

if $f'_{gH}(t_0) \in R_F$, we say that f is generalized Hukuhara differentiable (gH-differentiable) at t_0. Also we say that f is $[i - gH]$-differentiable at t_0 for $0 \leq r \leq 1$ if

$$[f'_{gH}]^r(t_0) = [(f^-)'(t_0, r), (f^+)'(t_0, r)], \quad (2.3)$$

and say f is $[ii - gH]$-differentiable at t_0 if

$$[f'_{gH}]^r(t_0) = [(f^+)'(t_0, r), (f^-)'(t_0, r)], \quad (2.4)$$

Definition 2.4 *We say that a point $t_0 \in (a, b)$, is a switching point for the differentiability of f, if in any neighborhood V of t_0 there exist points $t_1 < t_0 < t_2$ such that*
type (I): at t_1 (2.3) holds while (2.4) does not hold and at t_2 (2.4) holds and (2.3) does not hold, or type (II):at t_1 (2.4) holds while (2.3) does not hold and at t_2 (2.3) holds and (2.4) does not hold.

Definition 2.5 *A fuzzy-valued function $f : [a, b] \to R_F$ is said to be continuous at $t_0 \in [a, b]$ if for each $\epsilon > 0$ there is $\delta > 0$ such that $D(f(t), f(t_0)) < \epsilon$, whenever $t \in [a, b]$ and $|t - t_0| < \delta$. We say that f is fuzzy continuous on $[a, b]$ if f is continuous at each $t_0 \in [a, b]$ such that the continuity is one-sided at endpoints a, b.*

Definition 2.6 *([6]) Let $f : (a,\ b) \to \mathbb{R}_{\mathcal{F}}$. Consider $f(t)$ is gH-differentiable of order i, $i = 1, ..., n-1$ at t_0 with no switching point on $[a, b]$. We say that $f(t)$ is gH-differentiable of the n^{th}-order at t_0, if $(f)^{(n)}_{gH}(t_0) \in \mathbb{R}_{\mathcal{F}}$ exists such that*

$$(f)^{(n)}_{gH}(t_0) = \lim_{h\to 0} \frac{f^{(n-1)}(t_0 + h) \ominus_{gH} f^{(n-1)}(t_0)}{h}.$$

Definition 2.7 *Let $f : [a,b] \to R_F$, for each partition $P = \{t_0, t_1, \cdots, t_n\}$ of $[a,b]$ and for arbitrary $\xi_i \in [t_{i-1}, t_i]$, $1 \le i \le n$, suppose $R_p = \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})$, and $\triangle := \max |t_i - t_{i-1}|, 1 \le i \le n$. The definite Reimann integral of $f(t)$ over $[a,b]$ is*

$$\int_a^b f(t)dt = \lim_{\triangle \to 0} R_p,$$

provided that this limit exists in the metric D.

Note that if the fuzzy function $f(t)$ is continuous in the metric D, Lebesgue interval and Riemann integral yield the same value, and also for $0 \le r \le 1$,

$$[\int_a^b f(t)dt]^r = [\int_a^b f^-(t,r), \int_a^b f^+(t,r)],$$

In this paper $C_F[a,b]$ is the space of all continuous fuzzy-valued function on $[a,b]$. Also we denote the space of all Lebesgue integrable fuzzy-valued functions on the bounded interval $[a,b] \subset R$ by $L_F[a,b]$.

Definition 2.8 *[6]. Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$; the fuzzy Riemann-Liouville integral of fuzzy-valued function f is defined as follows:*

$$(J_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt$$

for $a \le x$, and $0 < \alpha \le 1$. For $\alpha = 1$, we set $J_a^1 = I$, the identity operator.

Definition 2.9 *Let $f_{gH}^{(m)} \in C_F[a\ ,\ b] \cap L_F[a\ ,\ b]$. The fuzzy gH-fractional Caputo differentiability of fuzzy-valued function f is defined as following:*

$$(\ _{gH}D_*^\alpha f)(x) = J_a^{m-\alpha}(f_{gH}^{(m)})(x) \qquad (2.5)$$

$$= \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{(m-\alpha-1)} (f_{gH}^{(m)})(t)\, dt$$

where $m-1 < \alpha \le m$, $m \in \mathbb{N}$, $x > a$.

In this paper, we only consider $gH-$differentiability of order $0 < \alpha \le 1$ for fuzzy-valued function f, so Eq.(2.5) can be written as the following form

$$(\ _{gH}D_*^\alpha f)(x) = J_a^{1-\alpha}(f'_{gH})(x)$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{(f'_{gH})(t)\mathrm{d}t}{(x-t)^\alpha}\ ,\ x > a$$

Theorem 2.1 *([6]) Let $f'_{gH} \in C_F[a\ ,\ b] \cap L_F[a\ ,\ b]$ where $f(x;r) = (f^-(x;r), f^+(x;r))$ for $0 \le r \le 1$, $x \in [a,b]$. Let $f^-(x;r)$ and $f^+(x;r)$ are Caputo differentiable functions then the function f is $gH-$differentiable. Furthermore*

$$(\ _{gH}D_*^\alpha f)(x;r) = [\min\{(D_*^\alpha f^-)(x;r), (D_*^\alpha f^+)(x;r)\}, \max\{(D_*^\alpha f^-)(x;r), (D_*^\alpha f^+)(x;r)\}],$$

Definition 2.10 *Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be $[gH]-$differentiable at $x_0 \in (a,b)$. We say that f is ${}^{CF}[(i)-gH]-$differentiable at x_0 for $0 \le r \le 1$, if*

(i) $(\ _{gH}D_*^\alpha f)(x_0;r) = [(D_*^\alpha f^-)(x_0;r), (D_*^\alpha f^+)(x_0;r)]$,

and that f is $[(ii)-gH]-$differentiable at x_0 if

(ii) $(\ _{gH}D_*^\alpha f)(x_0;r) = [(D_*^\alpha f^+)(x_0;r), (D_*^\alpha f^-)(x_0;r)]$,

where

$$(D_*^\alpha f^-)(x_0;r) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{(f^-)'(t)}{(x-t)^\alpha} dt,$$

$$(D_*^\alpha f^+)(x_0;r) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{(f^+)'(t)}{(x-t)^\alpha} dt,$$

3 Predictor-Corrector Method

Consider the following fuzzy fractional differential equation

$$\begin{cases} (\ _{gH}D_*^\alpha y)(t) = f(t, y(t)), & t \in [t_0, T], \\ y(t_0) = y_0 \in \mathbb{R}_{\mathcal{F}}, \end{cases} \qquad (3.6)$$

where $0 < \alpha \le 1$ is real number and the operator $_{gH}D_*^\alpha$ denote gH-fractional Caputo differentiability of fuzzy-valued function f of order α.
The following Peano-type theorem is given to derive solution of fuzzy fractional differential equations with order $0 < \alpha \le 1$ under Caputo's differentiability.

Theorem 3.1 *[13] Let $R_0 = [t_0, t_0 + p] \times \overline{B}(y_0, p), p, q > 0, y_0 \in \mathbb{R}_{\mathcal{F}}$, where $\overline{B}(y_0, p) = \{y \in \mathbb{R}_{\mathcal{F}} : d(y, y_0) \le q\}$ denote a closed ball in $\mathbb{R}_{\mathcal{F}}$ and let $f : R_0 \to \mathbb{R}_{\mathcal{F}}$ be a continuous function such that $d(0, f(t,y)) \le M$ for all $(t,y) \in R_0$ and f satisfies the Lipschitz condition*

Table 1: Error of Proposed method by Hausdorff distance at $t = 1$ in example 4.1

h	Error of y^-	Error y^+
0.1	1.3776	0.6888
0.01	0.3941	0.1970
0.001	0.1284	0.0642
0.002	0.0914	0.0457
0.005	0.0582	0.0291
0.0001	0.0413	0.0207
0.0005	0.0181	0.0091

Table 2: Error of Proposed method by Hausdorff distance at $t = 1$ in example 4.1

$h/\ r$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	0	0.4649	0.9298	1.3947	1.8596	2.3245	2.7894	3.2543	3.7192	4.1841	4.6491
0.01	0	0.4941	0.9882	1.4823	1.9763	2.4704	2.9645	3.4586	3.9527	4.446	4.9408
0.001	0	0.4988	0.9976	1.4964	1.9952	2.4940	2.9928	3.4916	3.9904	4.4892	4.9880
0.005	0	0.5000	0.9999	1.4999	1.9998	2.4998	2.9997	3.4997	3.9996	4.4996	4.9995
0.0001	0	0.5002	1.0005	1.5007	2.0009	2.5011	3.0014	3.5016	4.0018	4.5020	5.0023
0.0005	0	0.5006	1.0012	1.5018	2.0024	2.5030	3.0036	3.5042	4.0049	4.5054	5.0060

Table 3: The approximate solution to example (4.2)-y^+ at $t = 1$

$h/\ r$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	9.2981	8.8332	8.3683	7.9034	7.4385	6.9736	6.5087	6.0438	5.5789	5.1140	4.6491
0.01	9.8817	9.3876	8.8935	8.3994	7.9053	7.4113	6.9172	6.4231	5.9290	5.4349	4.9408
0.001	9.9760	9.4772	8.9784	8.4796	7.9808	7.4820	6.9832	6.4844	5.9856	5.4868	4.9880
0.005	9.9990	9.4991	8.9991	8.4992	7.9992	7.4993	6.9993	6.4994	5.9994	5.4995	4.9995
0.0001	10.0045	9.5043	9.0041	8.5038	8.0036	7.5034	7.0032	6.5029	6.0027	5.5025	5.0023
0.0005	10.0119	9.5113	9.0107	8.5101	8.0095	7.5089	7.0083	6.5077	6.0072	5.5066	5.0060

$$
\begin{aligned}
& d((t_1-s)^{\alpha-1}f(t,y),(t_2-s)^{\alpha-1}f(t,z)) \\
& \leq |(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}|.L.d(y,z),
\end{aligned}
$$

$\forall (t,y) \in R_0,\ \ t_1, t_2, s \in [t_0, t_0+p]$ and $d(y,z) \leq q$. If there exists $d > 0$ such that for $t \in [t_0, t_0+p]$ the sequence given by $\widehat{y}_0(t) = y_0$, $\widehat{y}_{n+1}(t) = y_0 \ominus \frac{(-1)}{\mu(\alpha)} \int_{t_0}^{t} (t-z)^{\alpha-1} f(z, y_n(z))dz$ is defined for any $n \in \mathbb{N}$. Then the fuzzy fractional differential equation (3.6) has two solution $y, \widehat{y} : [t_0, t_0+\tau] \to B(y_0, q)$ where $\tau = \min\{p, (\frac{q\mu(\alpha+1)}{M})^{\frac{1}{\alpha}}, (\frac{q\mu(\alpha+1)}{M_1})^{\frac{1}{\alpha}}, d\}$ and the successive iterations for $0 < \alpha \leq 1$,

$$
\begin{aligned}
& y(t_0) = y_0, \\
& y(t_{n+1}) = y(t_0) \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_{n+1}} (t_{n+1}-z)^{\alpha-1} f(z, y_n(z))dz,
\end{aligned}
$$

$$
\begin{aligned}
& \widehat{y}(t_0) = \widehat{y}_0, \\
& \widehat{y}(t_{n+1}) = \widehat{y}(t_0) \\
& \ominus \frac{-1}{\Gamma(\alpha)} \int_{t_0}^{t_{n+1}} (t_{n+1}-z)^{\alpha-1} f(z, \widehat{y}_n(z))dz,
\end{aligned}
$$

converge to these two solutions.

Here, we state the numerical method under ${}^{CF}[(i)-gH]-$differentiability.
Now, by theorem (3.1) is easy to verify that the problem (3.6) is solved when the following fuzzy Voltera integral

$$
\int_{t_0}^{t_{n+1}} (t_{n+1}-u)^{\alpha-1} f(u, y(u))du \qquad (3.7)
$$

is approximated.

$$\int_{t_0}^{t_{n+1}}(t_{n+1}-u)^{\alpha-1}f(u,y(u))du \simeq \int_{t_0}^{t_{n+1}}(t_{n+1}-u)^{\alpha-1}g(u)du, \tag{3.8}$$

where $g(u)$ is the fuzzy piecewise linear interpolation for fuzzy function f, whose nodes and knots are chosen at the t_j, $j=0,1,2,\cdots,n+1$. This means $g(u)=(g^-(u,r),g^+(u,r))$ is construct by

$$\begin{aligned} g^-(u;r) &= \ell_0(t)g^-(t_0;r) \\ &+ \sum_{j=1}^{n}\gamma_j(t)g^-(t_j;r)+\ell_{n+1}(t)g^-(t_{n+1};r), \end{aligned}$$

$$\begin{aligned} g^+(u;r) &= \ell_0(t)g^+(t_0;r) \\ &+ \sum_{j=1}^{n}\gamma_j(t)g^+(t_j;r)+\ell_{n+1}(t)g^+(t_{n+1};r), \end{aligned}$$

where the positive coefficients ℓ_0,ℓ_{n+1} and γ_j for $j=1,\cdots,n$ are as following:

$$\begin{aligned} \ell_0(t) &= \frac{(t_1-t)}{(t_1-t_0)}, t_0<t<t_1, \\ \ell_{n+1}(t) &= \frac{(t-t_n)}{(t_{n+1}-t_n)}, t_n<t<t_{n+1}, \\ \gamma_j(t) &= \begin{cases} \frac{t-t_{j-1}}{t_j-t_{j-1}} & t_{j-1}<t<t_j, \\ \frac{t_{j+1}-t}{t_{j+1}-t_j} & t_j<t<t_{j+1}, \end{cases} \end{aligned}$$

We can write the right-hand side of eq.(3.8) as

$$\begin{aligned} &\int_{t_0}^{t_{n+1}}(t_{n+1}-u)^{\alpha-1}g^-(u;r)du \\ =& \int_{t_0}^{t_{n+1}}(t_{n+1}-u)^{\alpha-1}\ell_0(u)g^-(t_0;r)du \\ +& \sum_{j=1}^{n}\int_{t_0}^{t_{n+1}}(t_{n+1}-u)^{\alpha-1}\gamma_j(u)g^-(t_j;r)du \\ +& \int_{t_0}^{t_{n+1}}(t_{n+1}-u)^{\alpha-1}\ell_{n+1}(u)g^-(t_{n+1};r)du \end{aligned}$$

and

$$\begin{aligned} &\int_{t_0}^{t_{n+1}}(t_{n+1}-u)^{\alpha-1}g^+(u;r)du \\ =& \int_{t_0}^{t_{n+1}}(t_{n+1}-u)^{\alpha-1}\ell_0(u)g^+(t_0;r)du \\ +& \sum_{j=1}^{n}\int_{t_0}^{t_{n+1}}(t_{n+1}-u)^{\alpha-1}\gamma_j(u)g^+(t_j;r)du \\ +& \int_{t_0}^{t_{n+1}}(t_{n+1}-u)^{\alpha-1}\ell_{n+1}(u)g^+(t_{n+1};r)du \end{aligned}$$

therefore by integration we have

$$\int_{t_0}^{t_{n+1}}(t_{n+1}-u)^{\alpha-1}g^-(u;r)du = \sum_{j=0}^{n+1}\omega_j g^-(t_j;r),$$

$$\int_{t_0}^{t_{n+1}}(t_{n+1}-u)^{\alpha-1}g^+(u;r)du = \sum_{j=0}^{n+1}\omega_j g^+(t_j;r),$$

whereas

$$\begin{aligned} \omega_0 &= \frac{h^\alpha}{\alpha(\alpha+1)}[n^{\alpha+1}+(n+1)^\alpha(\alpha-n)], \\ \omega_j &= \frac{h^\alpha}{\alpha(\alpha+1)}[(n-j)^{\alpha+1} \\ &\quad -2(n+1-j)^{\alpha+1} \\ &\quad +(n-j+2)^{\alpha+1}], \quad 1\le j\le n, \\ \omega_{n+1} &= \frac{h^\alpha}{\alpha(\alpha+1)}, \end{aligned}$$

in the case of equal space nodes $t_j=t_0+jh$, with some fixed h.

This gives us our implicit formula, which is:

$$\begin{aligned} &y^-(t_{n+1};r)=y^-(t_0;r) \\ &+\frac{1}{\Gamma(\alpha)}(\sum_{j=0}^{n}\omega_j\ f^-(t_j,y(t_j;r)) \\ &+\omega_{n+1}f^-(t_{n+1},y^P(t_{n+1};r))), \\ &y^+(t_{n+1};r)=y^+(t_0;r) \\ &+\frac{1}{\Gamma(\alpha)}(\sum_{j=0}^{n}\omega_j\ f^+(t_j,y(t_j;r)) \\ &+\omega_{n+1}f^+(t_{n+1},y^P(t_{n+1};r))), \end{aligned}$$

as corrector, we replace the integral on the right-hand side of eq. (3.8) by using a single interpolation point (the left endpoint of interval), i.e.

$$\int_{t_0}^{t_{n+1}} (t_{n+1}-u)^{\alpha-1} f(u;y(u))du = \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1}-u)^{\alpha-1} f(t_j,y(t_j))du,$$

$$\int_{t_0}^{t_{n+1}} (t_{n+1}-u)^{\alpha-1} f(u;y(u))du = \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} (t_{n+1}-u)^{\alpha-1} f(t_j,y(t_j))du,$$

where

$$\int_{t_j}^{t_{j+1}} (t_{n+1}-u)^{\alpha-1} du = q_j = \frac{h^{\alpha}}{\alpha}[(n+1-j)^{\alpha}-(n-j)^{\alpha}],$$

Therefore the predictor $(y^-)^P$, $(y^+)^P$ is determined by

$$(y^-)^P(t_{n+1},r) = y^-(t_0,r) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} q_j f^-(t_j,y(t_j,r)),$$

$$(y^+)^P(t_{n+1},r) = y^-(t_0,r) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} q_j f^+(t_j,y(t_j,r)),$$

Finally the fractional version of the Predictor-Corrector method is complete.
The Predictor-Corrector method under ${}^{CF}[(ii)-gH]-$ differentiability, is as follows.

$$(\widehat{y})^-(t_{n+1};r) = (\widehat{y})^-(t_0;r) \ominus \frac{-1}{\Gamma(\alpha)} (\sum_{j=0}^{n} \omega_j \ f^-(t_j,\widehat{y}(t_j;r)) + \omega_{n+1} f^-(t_{n+1},\widehat{y}^P(t_{n+1};r))),$$

$$(\widehat{y})^+(t_{n+1};r) = (\widehat{y})^-(t_0;r) \ominus \frac{-1}{\Gamma(\alpha)} (\sum_{j=0}^{n} \omega_j \ f^+(t_j,\widehat{y}(t_j;r)) + \omega_{n+1} f^+(t_{n+1},\widehat{y}^P(t_{n+1};r))),$$

as corrector and

$$((\widehat{y})^-)^P(t_{n+1},r) = (\widehat{y})^-(t_0,r)) \ominus \frac{-1}{\Gamma(\alpha)} \sum_{j=0}^{n} q_j f^-(t_j,\widehat{y}(t_j,r)),$$

$$((\widehat{y})^+)^P(t_{n+1},r) = (\widehat{y})^+(t_0,r)) \ominus \frac{-1}{\Gamma(\alpha)} \sum_{j=0}^{n} q_j f^+(t_j,y(t_j,r)),$$

as predictor.

4 Examples

We validate our theoretical results from the previous section by considering the following examples, which were solved by MathLab.

Example 4.1 *[6] Consider the following fuzzy Caputo differential equation*

$$\begin{cases} ({}_{gH}D_*^{0.5}y)(x) = y(x), & t \in [0,1], \\ y(0;r) = (r, 2-r), \end{cases} \quad (4.9)$$

$y(t)$ be ${}^{CF}[(i)-gH]-$differentiable, the solution of (4.9) is given by [6] as follows

$$y(t,r) = (r, 2-r)E_{0.5}(t^{0.5}) \quad (4.10)$$

we solve this example by predictor-corrector method and compare the solutions by real solution (4.10) in Table 1.

Example 4.2 *[6]Consider the following fuzzy Caputo differential equation*

$$\begin{cases} ({}_{gH}D_*^{0.5}y)(t) = y(t) \oplus c \odot t^2 & t \in [0,1], \\ y(0;r) = (r, 2-r), \end{cases} \quad (4.11)$$

In this example we consider $c = (r, 2-r)$. According to ${}_{gH}D_*^{0.5}$ and using Predictor-Corrector method, the approximate solution of (4.11) with $h = 0.1, 0.01, 0.001, 0.005, 0.0001, 0.005$ is shown in Tables 2 and 3.

5 Conclusion

In this paper a numerical method for solving fuzzy differential equation of fractional order under gH-fractional Caputo differentiability was proposed. We used fuzzy Lagrange interpolation for approximating unknown fuzzy solution of FFDE. The proposed method was predictor-corrector that was obtained by combining explicit and implicit methods.

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