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Estimate Output of a Production Unit in Production Possibility Set with Fuzzy Inference Mechanism

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Abstract

In this paper, we consider the production possibility set with n production units such that the following four principles that governs: inclusion observations, conceivability, immensity and convexity. Our goal is to estimate the output of a same and new production unit with existing production possibility and amount of input is specified. So, initially we find the interval changes of each inputs and outputs and then partitioning each interval with fuzzy numbers. We use each of the possible to produce with specific inputs and outputs the fuzzy if-then rules and estimate the amount of outputs with mamdany method in fuzzy inference mechanism. In the end, we present an example of implementation.

Keywords: Production possibility set, Fuzzy number, Fuzzy inference.

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1. Introduction

Data Envelopment Analysis (DEA) was originally proposed by Charnes et al. [7] as a method for evaluating the relative efficiency of Decision Making Units (DMUs) performing essentially the same task. Units use similar multiple inputs to produce similar multiple outputs. DEA deals with the evaluation of the performance of DMU performing a transformation process of several inputs to several outputs. There are several ways to evaluate the units such as CCR [7], BCC [4] and etc. Production units are always looking for the best output. Zadeh in [12], introduced the concept of fuzzy number. The concept of fuzzy numbers and arithmetic operations with fuzzy numbers were first introduce and investigated by Chang and Zadeh [6], and others. Applications of fuzzy numbers for indicating uncertain and vague information in decision making, linguistic controllers, expert systems, data mining, pattern recognition, etc. were stated in recently published articles [1,2,11]. Systems of fuzzy IF-THEN rules are widely used in applications of fuzzy set theory, for example in fuzzy control, identification of dynamic systems, prediction of dynamic systems, decision-making, etc. In this paper, we consider the fuzzy IF-THEN rules method for problem "Estimate output of a production unit in production possibility set with specific input such that in comparison with similar units is efficient". Some researchers worked on the estimate output such as in [5], R.D. Banker present the estimating most productive scale size using data envelopment analysis. In this work, in section 2 we present some basic definitions of DEA, fuzzy numbers and its property and then inference mechanism. In section 3, we state the the new method for estimate output of a production unit with using fuzzy inference mechanism with specific input. In section 4, we present one example for illustration of method. Conclusion is drawn in section 5.

2. Background

We provide the contents of two topics, data envelopment analysis (DEA) and fuzzy mechanism.

2.1. Data envelopment analysis

DEA utilizes a technique of mathematical programming for evaluation of n DMUs. Suppose there are n DMUs: $DMU_1, DMU_2, \dots, DMU_n$. Let the input and output data for DMU_j be $X_j = (x_{1j}, \dots, x_{mj})$ and $Y_j = (y_{1j}, \dots, y_{sj})$, respectively.[3]

The set of feasible activities is called the Production Possibility Set (PPS) and is denoted by P with the properties: inclusion observations, conceivability, immensity and convexity, where [8]

$$P = \left\{ (X,Y) \middle| X^t \geq \sum_{j=1}^{n} \lambda_j X_j^t, Y^t \leq \sum_{j=1}^{n} \lambda_j Y_j^t \right\} \quad (1)$$

CCR model [7] for evaluate relative efficiency of DMU_k is as follows,

$$\begin{aligned} \min \quad & \theta \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{ik} \qquad i = 1, \dots, m \qquad (2) \end{aligned}$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \leq \theta y_{rk} \qquad r = 1, \dots, s$$
$$\lambda_j \geq 0, \qquad j = 1, \dots, n$$

Unit DMU_k is CCR efficient if the optimal value of the objective function in model (2) equals one $(\theta^* = 1)$.

2.2 Fuzzy mechanism

There are many definitions for fuzzy numbers. We denote set of all fuzzy numbers by $F(R)$ and define as follows.[9]

Definition 1. For fuzzy number $A \in F(R)$, we show the membership function by $A(x)$ which is given by

$$A(x) = \begin{cases} 0 & x \leq a_1 \\ l_A(x) & a_1 \leq x \leq a_2 \\ 1 & a_2 \leq x \leq a_3 \\ r_A(x) & a_3 \leq x \leq a_4 \\ 0 & a_4 \leq x \end{cases} \qquad (3)$$

Where $\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \mathrm{a}_4 \in \mathrm{R}$ and $l_A(x)$ is non-decreasing and $r_A(x)$ is non-increasing and $l_A(a_1) = 0, l_A(a_2) = 1, r_A(a_3) = 1$ and $r_A(a_4) = 0$. For any $\alpha \in (0,1]$; $\alpha -$cut of fuzzy numbe A is a crisp interval as

$$A_\alpha = \{x \in R: A(x) \geq \alpha\} = [A_l(\alpha), A_r(\alpha)] \quad (4)$$

If $A, B \in F(R)$ and $\lambda \in R$ then $(\mathrm{A} + \mathrm{B})_\alpha = A_\alpha + \mathrm{B}_\alpha$ and $\mathrm{A} = \mathrm{B}$, if $A_l(\alpha) = B_l(\alpha)$ and $A_r(\alpha) = B_r(\alpha)$ almost everywhere, $\alpha \in [0,1]$.

$$(\lambda A)_\alpha = \begin{cases} [\lambda A_l(\alpha), \lambda A_r(\alpha)] & \lambda \geq 0 \\ [\lambda A_l(\alpha), \lambda A_r(\alpha)] & \lambda < 0 \end{cases} \quad (5)$$

Trapezoidal fuzzy numbers are special cases of fuzzy numbers which are denoted by $F^T(R)$ and for $T \in F^T(R)$ its membership function is as follows:

$$T(x) = \begin{cases} 0 & x \leq a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & a_1 \leq x \leq a_2 \\ 1 & a_2 \leq x \leq a_3 \\ \dfrac{a_4 - x}{a_4 - a_3} & a_3 \leq x \leq a_4 \\ 0 & a_4 \leq x \end{cases} \quad (6)$$

Where $\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \mathrm{a}_4 \in \mathrm{R}$ and $\alpha -$level of fuzzy number T is as:

$$T_\alpha = [T_l(\alpha), T_r(\alpha)] = [a_1 + (a_2 - a_1)\alpha , a_4 - (a_4 - a_3)\alpha] \qquad (7)$$

Triangular fuzzy numbers are also special cases of trapezoidal fuzzy numbers with $a_2 = a_3$.

2.2.1 Inference mechanism

When we model a knowledge system, it is often represented by the form of fuzzy rule base. The fuzzy rule base consists of fuzzy if-then rules. The rule base has the form of a MIMO (multiple input multiple output) system.[10]

2.2.2 Mamdani inference mechanism

We assume that we have k fuzzy control rules of the form

$Rule1: If\ x_1 in\ A_{11}, \cdots, x_m in\ A_{m1}\ Then\ y_1 in\ B_{11}, \cdots, y_s in\ B_{s1}$

$\vdots$

$Rule\ k: If\ x_1 in\ A_{1k}, \cdots, x_m in\ A_{mk}\ Then\ y_1 in\ B_{1k}, \cdots, y_s in\ B_{sk}$

$fact: If\ x_1 = \overline{x_1}, \cdots, x_m = \overline{x_m}$

$consequence: \ y_1 in\ B_1, \cdots, y_s in\ B_s \qquad (8)$

The firing levels of the rules, denoted by $\alpha_j : j = 1, \dots, k$ are computed by

$$\alpha_j = A_{1j}(x_1) \wedge \dots \wedge A_{mj}(x_m); \quad j = 1, \dots, k \tag{9}$$

The individual rule outputs are obtained by

$$\acute{C}_{rj}(w_r) = B_{rj}(w_r) \wedge \alpha_j; \ j = 1, \dots, k, \quad r = 1, \dots, s \tag{10}$$

$$C_r(w_r) = \acute{C}_{r1}(w_r) \vee \dots \vee \acute{C}_{rk}(w_r); \quad r = 1, \dots, s \tag{11}$$

Finally, to obtain a deterministic control action, we employ any defuzzification strategy [10].

2.2.3 Middle-of-Maxima defuzzification

If C is not discrete then defuzzified value of a fuzzy set C is defined as [10]

$$Z_0 = \frac{\int_G z dz}{\int_G dz} \tag{12}$$

where G denotes the set of maximizing element of C.

3 Estimate output of a production units with fuzzy Inference mechanism

Suppose there are n $DMUs : DMU_1, DMU_2, \dots, DMU_n$. Let the input and output data for DMU_j be $X_j = (x_{1j}, \dots, x_{mj})$ and $Y_j = (y_{1j}, \dots, y_{sj})$, respectively.

We want to create a new similar production unit with the specified input $X = (x_1, \dots, x_m)$ were new input should be in the convex hull of the units input observed. For this purpose, first we identify efficient units by existing methods such as CCR or BCC, etc. Let, k $DMUs$ be effective. Then, we arranged increasing order inputs and outputs of effecient units. Without loss of generality argument suppose that $x_{i1} \leq x_{i2} \leq \dots \leq x_{ik}$ for i−th input and $y_{r1} \leq y_{r2} \leq \dots \leq y_{rk}$ for r−th output.

Now we partition the interval $[x_{i1}, x_{ik}]$ for $i = 1, \dots, m$ with triangular fuzzy numbers are as follows:

$$A_{i1}(x) = \begin{cases} \frac{x_{i2} - x}{x_{i2} - x_{i1}} & x_{i1} \leq x \leq x_{i2} \\ 0 & otherwise \end{cases} \tag{13}$$

$$A_{ij}(x) = \begin{cases} \frac{x - x_{i(j-1)}}{x_{ij} - x_{i(j-1)}} & x_{i(j-1)} \leq x \leq x_{ij} \\ \frac{x_{i(j+1)} - x}{x_{i(j+1)} - x_{ij}} & x_{ij} \leq x \leq x_{i(j+1)} ; j = 2, \dots, k-1 \\ 0 & otherwise \end{cases} \tag{14}$$

$$A_{ij}(x) = \begin{cases} \frac{x - x_{i(k-1)}}{x_{ik} - x_{i(k-1)}} & x_{i(k-1)} \leq x \leq x_{ik} \\ 0 & otherwise \end{cases} \tag{15}$$

And also, we partition the interval $[y_{r1}, y_{rk}]$ for $r = 1, \dots, s$ with triangular fuzzy numbers are as follows:

$$B_{r1}(y) = \begin{cases} \frac{y_{r2} - y}{y_{r2} - y_{r1}} & y_{r1} \leq y \leq y_{r2} \\ 0 & otherwise \end{cases} \tag{16}$$

$$B_{rj}(y) = \begin{cases} \frac{y - y_{r(j-1)}}{y_{rj} - y_{r(j-1)}} & y_{r(j-1)} \leq y \leq y_{rj} \\ \frac{y_{r(j+1)} - y}{y_{r(j+1)} - y_{rj}} & y_{rj} \leq y \leq y_{r(j+1)}; j = 2, \dots, k-1 \\ 0 & otherwise \end{cases} \tag{17}$$

$$B_{rj}(y) = \begin{cases} \frac{y - y_{r(k-1)}}{y_{rk} - y_{r(k-1)}} & y_{r(k-1)} \leq y \leq y_{rk} \\ 0 & otherwise \end{cases} \tag{18}$$

We consider each efficient unit observed as a fuzzy control rules as follows for $j = 1,\dots,k$:

$$Rule1: If\ x_1 in\ A_{11},\cdots,x_m in\ A_{m1}\ Then\ y_1 in\ B_{11},\cdots,\ y_s in\ B_{s1}$$

⋮

$$Rule\ k: If\ x_1 in\ A_{1k},\cdots,x_m in\ A_{mk}\ Then\ y_1 in\ B_{1k},\cdots,\ y_s in\ B_{sk} \quad (19)$$

Where A_{ij} corresponding to $i-$th input of $j-$th unit. And also, B_{rj} corresponding to $r-$th output of $j-$th unit.

Now suppose the input the new unit is as: $x_1 = \overline{x1},\cdots,x_m = \overline{x_m}$, we obtain with Mamdanis max min method and Middle-of-Maxima defuzzification method the outputs of $y_1 = \overline{y_1},\cdots,y_s = \overline{y_s}$.

Proposition 1. In provided inference method, interpolation is maintains. In other words, if input is: $x_1 = x_{1t},\cdots,x_m = x_{mt}$, then with Mamdanis max min method and Middle-of-Maxima defuzzification method the outputs is as $y_1 = y_{1t},\cdots,y_s = y_{st}$ for $t = 1,\dots,k$.

Proof. With rules (19) and fuzzy partition of input interval and fuzzy partition of output interval, Mamdanis max min method is as following

$$\alpha_j = A_{1j}(x_1) \wedge \dots \wedge A_{mj}(x_{mj}); \quad j = 1,\dots,k \quad (20)$$

The individual rule outputs are obtained by

$$\acute{C}_{rj}(w_r) = B_{rj}(w_r) \wedge \alpha_j; \quad j = 1,\dots,k, r = 1,\dots,s \quad (21)$$

$$C_r(w_r) = \acute{C}_{r1}(w_r) \vee \dots \vee \acute{C}_{rk}(w_r); \quad r = 1,\dots,s \quad (22)$$

Finally, to obtain a deterministic control action the outputs, we employ Middle-of-Maxima defuzzification method.

Now if $x_1 = x_{1t},\cdots,x_m = x_{mt}$, for $t = 1,\dots,k$.

With Eq. (20)

$$\alpha_j = \begin{cases} 1 & j = t \\ 0 & j \neq t \end{cases} \quad (23)$$

With Eq. (21) for $r = 1,\dots,s$

$$\acute{C}_{rj}(w_r) = \begin{cases} B_{rt}(y_{rt}) & j = t \\ 0 & j \neq t \end{cases} \quad (24)$$

And with Eq. (22) for $r = 1,\dots,s$

$$C_r(w_r) = \acute{C}_{r1}(w_r) \vee \dots \vee \acute{C}_{rk}(w_r) = B_{rt}(y_{rt}); \quad r = 1,\dots,s \quad (25)$$

Now, with Middle-of-Maxima defuzzification method $y_r = y_{rt}$ for $r = 1,\dots,m$.

3.1 Examples

In this section, we consider 19 $DMUs$ with two inputs and two outputs.(Table 1). First, we identify efficient units by CCR model where has come on table 2 (6 DMUs are efficient). To implement the example we use MATLAB software. We partition the interval $[0, 216]$ (range of input 1) and $[0, 203]$ (range of input 2) with triangular fuzzy numbers regarding to Eqs (13)-(15) are as figures 1 and 2. And also, we partition the interval $[0, 5368]$ (range of output 1) and $[0, 639]$ (range of output 2) with triangular fuzzy numbers regarding to Eqs $(16) - (18)$ are as figures 3 and 4.

Table 1. 19 DMUs with two inputs and two outputs.

DMU	input 1	input 2	output 1	output 21
1	81	87.6	5191	205
2	85	12.8	3629	0
3	56.7	55.2	3302	0
4	91	78.8	3379	8
5	216	72	5368	639
6	58	25.6	1674	0
7	112.2	8.8	2350	0
8	293.2	52	6315	414
9	186.6	0	2865	0
10	143.4	105.2	7689	66
11	108.7	127	2165	266
12	105.7	134.4	3963	315
13	235	236.8	6643	236
14	146.3	124	4611	128
15	57	203	4869	540
16	118.7	48.2	3313	16
17	58	47.4	1853	230
18	146	50.8	4578	217
19	0	91.3	0	508

Table 2. DMUs are CCR efficient.

DMU	input 1	input 2	output 1	output 21
1	81	87.6	5191	205
2	85	12.8	3629	0
5	216	72	5368	639
9	186.6	0	2865	0
15	57	203	4869	540
19	0	91.3	0	508

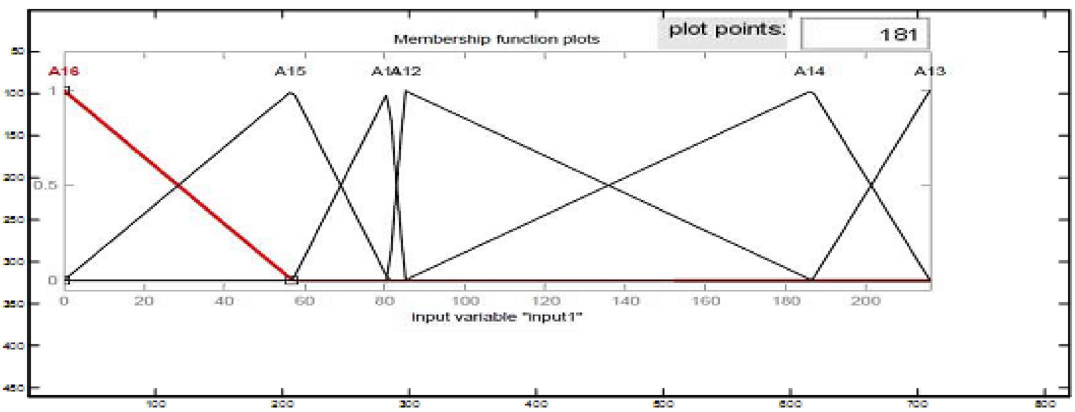

Figure 1: Partition of interval $[0, 216]$ for input 1 with triangular fuzzy numbers.

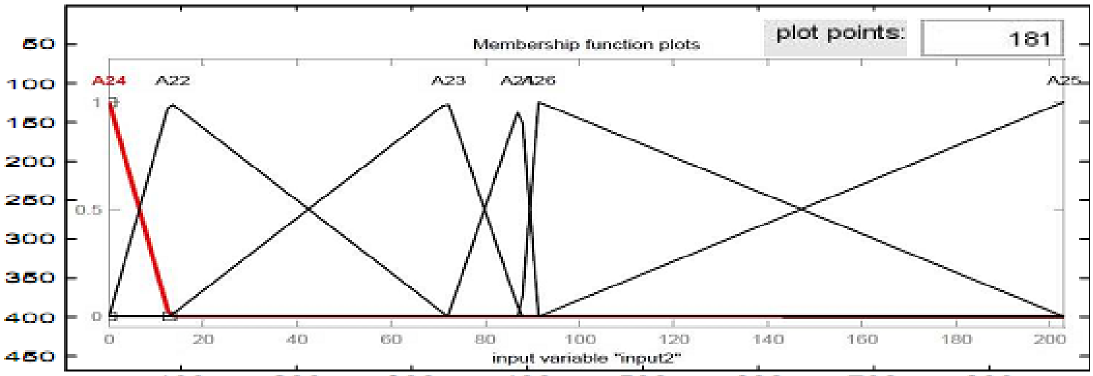

Figure 2: Partition of interval [0, 203] for input 2 with triangular fuzzy numbers.

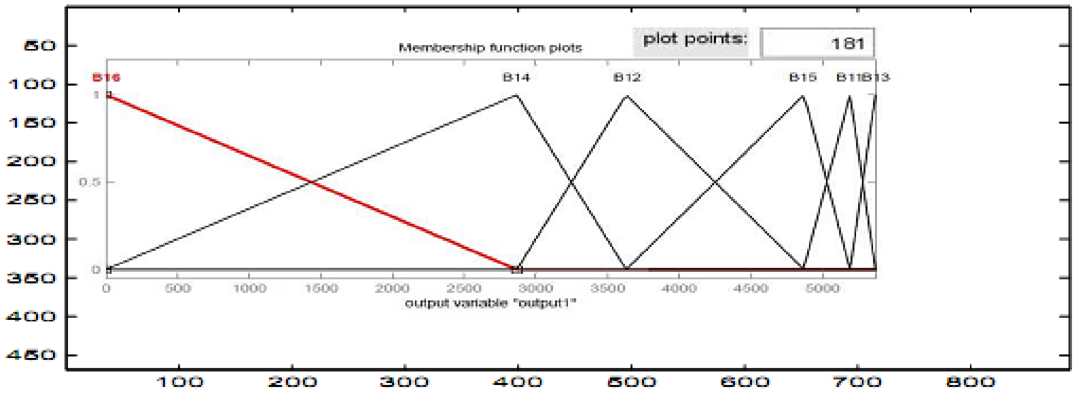

Figure 3: Partition of interval [0, 5368] for output 1 with triangular fuzzy numbers.

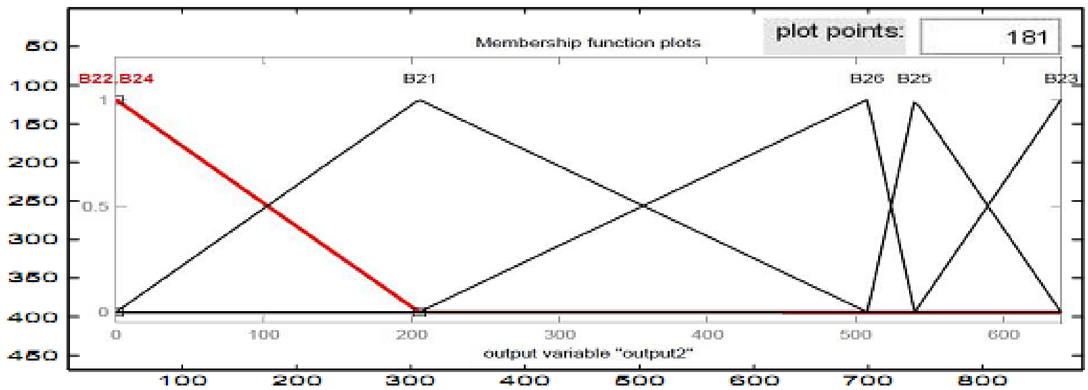

Figure 4: Partition of interval [0, 639] for output 2 with triangular fuzzy numbers.

We consider each efficient unit observed as a fuzzy control rule as figure 5 with partitions of inputs and outputs.

For example, Approximation output to several units with specific inputs is shown in Table 3.

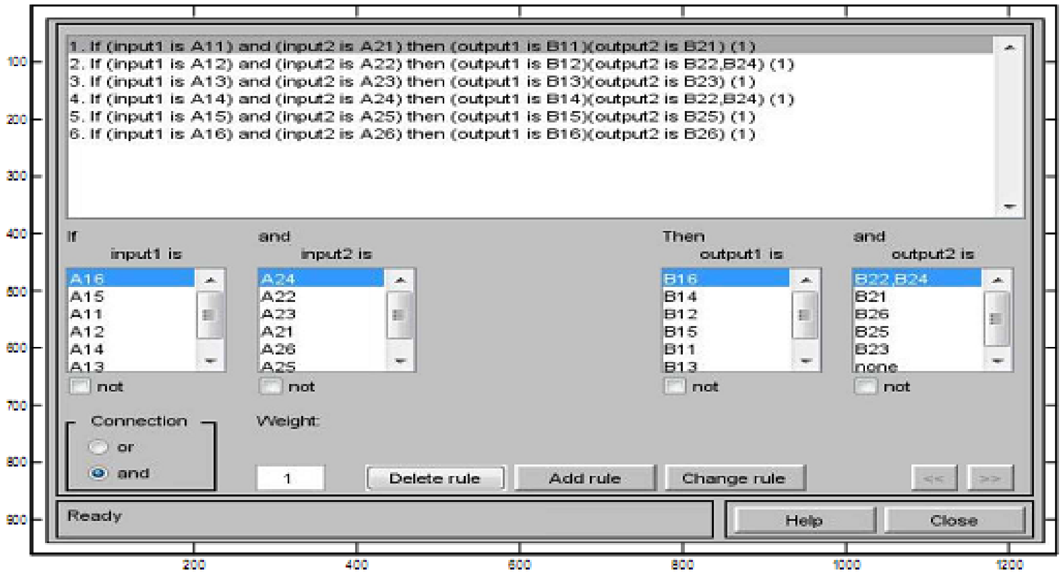

Figure 5: Fuzzy control rules

Table 3. Approximation output with known input.

Input 1	Input 2	Outnput 1	Outnput 2
x_1=90	x_2=20	$y_1 \approx 3.65e+003$	$y_2 \approx 9.58$
x_1=200	x_2=80	$y_1 \approx 5.34e+003$	$y_2 \approx 613$
x_1=118	x_2=42.8	$y_1 \approx 3.76e+003$	$y_2 \approx 51.1$
x_1=69	x_2=180	$y_1 \approx 4.62e+003$	$y_2 \approx 559$

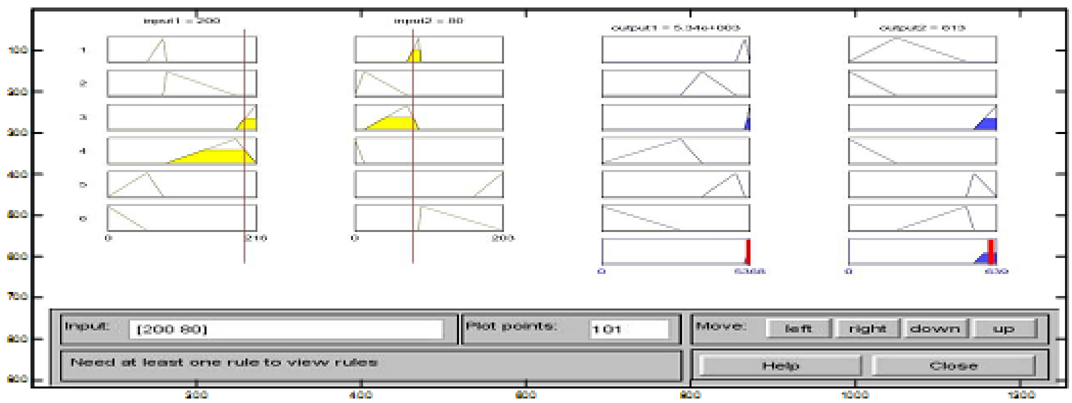

Figure 6: Approximation outputs with $x_1 = 200$ and $x_2 = 80$.

4 Conclusions

In this paper, we consider the production possibility set with n production units such that the following four principles that govern: inclusion observations, conceivability, immensity and convexity. We proposed a method for estimate the output of a same and new production unit with existing production possibility and amount of input is specified. We in Proposition 1. Prove that in this inference method, interpolation is maintains. In the end, we present an example of implementation.

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