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Efficiency of DMUs in Presence of New Inputs and Outputs in DEA

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Abstract

Examining the impacts of data modification is considered as sensitivity analysis. A lot of studies have considered the data modification of inputs and outputs in DEA. The issues which has not heretofore been considered in DEA sensitivity analysis is modification in the number of inputs and (or) outputs and determining the impacts of this modification in the status of efficiency of DMUs. This paper is going to present systems that show the impacts of adding one or multiple inputs or outputs on the status of efficiency of DMUs and furthermore a model is presented for recognizing the minimum number of inputs and (or) outputs from among specified inputs and outputs which can be added whereas an inefficient DMU will become efficient. Finally the presented systems and model have been utilized for a set of real data and the results have been reported.

***Keywords**:* Data envelopment analysis; Sensitivity analysis; efficiency.

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1. Introduction

Data envelopment analysis (DEA) is a nonparametric approach for evaluating the relative efficiency of DMUs with multiple inputs and multiple outputs. The basic models of DEA (Charnes et al, 1978; Banker et al, 1984 and Charnes et al ,1985) are used for evaluating of relative efficiency in similar economical systems. DEA has been extended in different areas these days. For example consider sensitivity analysis. Sensitivity analysis of DEA models which is based on the linear programming are both theoretically and practically important. The first DEA sensitivity analysis paper by (Charnes et al ,1985) determined change in a single output. later many studies have been conducted in changing some ofthe inputs and (or) outputs simultaneously by (Seiford et al ,1998; Zhu ,2001; Cooper et al, 2001;G.R.Jahanshahloo et al,2004; Jahanshahloo et al,2005a; Jahanshahloo et al,2005b) and etc.Heretofore one of the important issues which has considered in DEA sensitivity analysis is modification (increasing or decreasing) in the value of the inputs and (or) out puts. In this paper is going to investigate the impact of increasing the number of the inputs and (or) out puts on the status of efficiency in DMUs.

The present study has been organized as follows: First some basic DEA models and related concepts have been reviewed. Thereafter the number of inputs and (or) outputs has been modified and the impact of this modification (adding of one or multiple inputs and outputs) has been presented through some systems show the status of efficiency or inefficiency in DMUs and a model is presented for recognizing the minimum number of inputs and (or) outputs from among specified inputs and outputs which can be added whereas an inefficient DMU will become efficient. Then a set of DMUs have been presented. By changing (adding) the number of inputs and (or) outputs, the presented systems and a model in former section have been utilized for this set of DMUs and the results have been reported. Finally the results have been synthesized and conclude.

2. preliminary

Suppose n DMUs are evaluated, each of them consumes m inputs to produce s outputs. Suppose. $X_j = (x_{1j}, x_{2j}, \dots, x_{mj})^T$ and $Y_j = (y_{1j}, y_{2j}, \dots, y_{sj})^T$ are as the inputs and outputs of DMU_j for j=1,…,n. For the first time (Charnes et al, 1978) laid the foundation of DEA through introducing the CCR model. The multiplier form of this model is as follows:

$$
\begin{aligned}
\text{Max} \quad & \sum_{r=1}^{s} u_r y_{ro} \\
\text{S.t.} \quad & \sum_{i=1}^{m} v_i x_{io} = 1 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \\
& \qquad j = 1, \dots, n \qquad\qquad (1) \\
& u_r \ge 0 \quad r = 1, \dots, s \\
& v_i \ge 0 \quad i = 1, \dots, m
\end{aligned}
$$

That O is the index of the evaluated DMU and U= $(u_1, u_2, \dots, u_s) \in R^s$ and V= $(v_1, v_2, \dots, v_m) \in R^m$

Definition1. DMU_o is called CCR efficient if and only if the following conditions are acknowledged:

1) $\sum_{r=1}^{s} u_r^* y_{ro} = 1$

2) At least in one optimal solution of this model $u_r > 0$ *for each* $r = 1,2,\dots,s$ and $v_i > 0$ for each i=1,2,…,m.

If both conditions are acknowledged DMU_o is called strong efficient and if just first condition is acknowledged DMU_o is called weak efficient.

Definition2. DMU_o is called CCR inefficient if and only if $\sum_{r=1}^{s} u_r^* y_{ro} < 1$.

3. Adding of one or multiple inputs and (or) outputs

Suppose DMU_o with m inputs and s outputs have been evaluated efficient. The optimal value of objective function in model (1) will not be worse through the addition of one or multiple inputs and (or) outputs because adding input or output is equivalent to adding a new variable in model (1) ,so it is still preserved its efficiency. Then the inefficient DMUs are considered. In this section some systems has been presented for adding of one or multiple inputs and (or) outputs which by utilizing them it can be recognize the impact of these modification. Furthermore a model is presented for recognizing the minimum number of inputs and (or) outputs from among specified inputs and outputs which can be added whereas an inefficient DMU will become efficient.

3.1 Adding of one input or output

Suppose DMU_o is an inefficient DMU. Now it's going to present a system to find out whether DMU_o is preserved its inefficiency or it has been changed to efficient with adding of one input ((m+1)th input). For this reason consider the following system.

$$\begin{cases} \sum_{r=1}^{s} u_r\, y_{ro} = 1 \\ \sum_{i=1}^{m} v_i\, x_{io} + t x_{(m+1)o} = 1 \\ \sum_{r=1}^{s} u_r\, y_{rj} - \sum_{i=1}^{m} v_i\, x_{ij} - t x_{(m+1)j} \le 0 \\ \quad u_r \ge 0 \qquad j = 1,\dots,n \\ \quad v_i \ge 0 \qquad r = 1,\dots,s \\ \quad t \ge 0 \qquad i = 1,\dots,m \end{cases} \tag{2}$$

In above system the new input DMU_j for $j = 1,\dots,n$ with $x_{(m+1)j}$ and its corresponding weight is illustrated by t. The first constraint is the condition of efficiency for DMU_o and the other constraints are the constrains of model (1) in presences of the new input.

Theorem 1. Suppose that DMU_o has been evaluated inefficient with m inputs and s outputs.

a) If system (2) is feasible then $t > 0$ and with adding input *(m+1)*th, DMU_o will become efficient.

b) If system (2) is infeasible then with adding input *(m+1)*th, DMU_o will become inefficient.

Proof a) Suppose that system (2) is feasible and t=0. Therefore model (1) has a feasible solution with value of objective function equals one. This means DMU_o has been efficient in absence of the new input that is in contradiction with inefficiency assumption of DMU_o. Now it should be proved that DMU_o becomes efficient in presence of the new input. Consider the feasible solution of system (2) that is feasible and also optimal for model (3) with value of objective function equals one.

$$\begin{aligned} \text{Max} \quad & \sum_{r=1}^{s} u_r y_{ro} \\ \text{S.t.} \quad & \sum_{i=1}^{m} v_i x_{io} + t x_{(m+1)o} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - t x_{(m+1)j} \leq 0 \\ & v_i \geq 0 \qquad j = 1, \dots, n \\ & u_r \geq 0 \qquad i = 1, \dots, m \\ & t \geq 0 \qquad r = 1, \dots, s \end{aligned} \tag{3}$$

This means DMU_o becomes efficient in presence of the new input.

b) Suppose that system (2) is infeasible and DMU_o has become efficient in presence of the new input .Therefore in optimal solution of model (3), the value of objective function equals one. This solution is feasible for system (2) that is in contradiction with the infeasibility assumption of it. ■

It is worthwhile note this point that if system (2) has alternative optimal solutions, decision maker can choose any one of these solutions. System (2) can be generalized for adding on output as follows:

$$\begin{cases} \sum_{r=1}^{s} u_r y_{ro} + w y_{(s+1)o} = 1 \\ \sum_{i=1}^{m} v_i x_{io} = 1 \\ \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + w y_{(s+1)j} \leq 0 \\ \qquad u_r \geq 0 \qquad j = 1, \dots, n \\ \qquad v_i \geq 0 \qquad r = 1, \dots, s \\ \qquad w \geq 0 \qquad i = 1, \dots, m \end{cases} \tag{4}$$

3. 2 Adding of multiple inputs and (or) outputs

Now the impact of adding multiple inputs and (or) outputs on inefficient DMU_o is considered. Suppose inputs (m+1),…,(m+k) and outputs (s+1),…,(s+h) are added to find out whether DMU_o is still preserved its inefficiency or it will become efficient.

The following system is presented through the generalization of system (2) and (4):

$$\begin{aligned} & \sum_{r=1}^{s} u_r y_{ro} + \sum_{q=1}^{h} w_q y_{(s+q)o} = 1 \qquad (5) \\ & \sum_{r=1}^{s} u_r y_{rj} + \sum_{q=1}^{h} w_q y_{(s+q)j} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{p=1}^{k} t_p x_{(m+p)j} \leq 0 \\ & \qquad j = 1, \dots, n \\ & \sum_{i=1}^{m} v_i x_{io} + \sum_{p=1}^{k} t_p x_{(m+p)o} = 1 \\ & v_i \geq 0 \qquad i = 1, \dots, m \\ & u_r \geq 0 \qquad r = 1, \dots, s \\ & t_p \geq 0 \qquad p = 1, \dots, k \\ & w_q \geq 0 \qquad q = 1, \dots, h \end{aligned}$$

Where $x_{(m+p)j}$ and $y_{(s+q)j}$ are respectively the p(th) input and q(th) output of DMU_j for $p = 1 \dots k$ and $q = 1 \dots h$. Furthermore t_p and w_q are respectively related weights to p(th) input and q(th) output.

Theorem 2. Suppose that DMU_o has been evaluated inefficient with m inputs and s outputs.

a) If system (5) is feasible then at least one element of $\{t_1, \dots, t_k, w_1, \dots, w_h\}$ is positive. Furthermore by adding these k inputs and h outputs DMU_o will become efficient.

b) If system (5) is infeasible then by adding these k inputs and h outputs, DMU_o will still preserve its inefficiency.

Proof is similar to theorem 1. ■

4. Finding out the minimum number of added inputs and (or) outputs

Suppose that inefficient DMU_o has become efficient by adding all of these k inputs and h outputs. However it may be not necessary to add all these inputs and outputs for becoming DMU_o efficient. Now the question that will be raised is finding the minimum number of inputs and (or) outputs from among k inputs and h outputs which can be added whereas inefficient DMU_o becomes efficient. For this reason the following model can be presented.

$$z^* = Min \sum_{p=1}^{k} d_p + \sum_{r=1}^{h} d'_q$$

$$\text{S.t.} \quad \sum_{r=1}^{s} u_r y_{ro} + \sum_{q=1}^{h} w_q \, y_{(s+q)o} = 1$$

$$\sum_{r=1}^{s} u_r y_{rj} + \sum_{q=1}^{h} w_q y_{(s+q)j} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{p=1}^{k} t_p x_{(m+p)j} \le 0 \quad j = 1, \dots, n \qquad (6)$$

$$\sum_{i=1}^{m} v_i x_{io} + \sum_{p=1}^{k} t_p x_{(m+p)o} = 1$$

$$0 \le t_p \le M d_p \qquad p = 1, \dots, k$$
$$0 \le w_q \le M d'_q \qquad q = 1, \dots, h$$
$$d_p \in \{0,1\} \qquad p = 1, \dots, k$$
$$d'_q \in \{0,1\} \qquad q = 1, \dots, h$$
$$v_i \ge 0 \qquad i = 1, \dots, m$$
$$u_r \ge 0 \qquad r = 1, \dots, s$$
$$t_p \ge 0 \qquad p = 1, \dots, k$$
$$w_q \ge 0 \qquad q = 1, \dots, h$$

Theorem 3. If DMU_o becomes efficient with adding all of these p inputs and h outputs then Model (6) is feasible.

Proof Since DMU_o has become efficient then the following model has a feasible solution with value of objective function equals one.

$$z^* = Min \sum_{r=1}^{s} u_r y_{ro} + \sum_{q=1}^{h} w_q \, y_{(s+q)o}$$

$$\text{S.t.} \quad \sum_{i=1}^{m} v_i x_{io} + \sum_{p=1}^{k} t_p x_{(m+p)o} = 1$$

$$\sum_{r=1}^{s} u_r y_{rj} + \sum_{q=1}^{h} w_q y_{(s+q)j} - \sum_{i=1}^{m} v_i x_{ij} - \sum_{p=1}^{k} t_p x_{(m+p)j} \le 0$$

$$j = 1, \dots, n \qquad (7)$$
$$v_i \ge 0 \qquad i = 1, \dots, m$$
$$u_r \ge 0 \qquad r = 1, \dots, s$$
$$t_p \ge 0 \qquad p = 1, \dots, k$$
$$w_q \ge 0 \qquad q = 1, \dots, h$$

The optimal solution of the above model with $d_p = 1$ for $p = 1, \dots, k$ and $d'_q = 1$ for $q = 1, \dots, h$ is a feasible solution for model (6). ■

Theorem 4 . The minimum number of inputs and (or) outputs among these k inputs and h outputs which should be added whereas inefficient DMU_o will become efficient, equals z^*.

Proof With regard to minimization of model (6) it's evident that the binary variables d_p^* and d'^*_q for $p = 1, \dots, k$ and $q = 1, \dots, h$ are preferred to choose zero value. So by regarding to the constraints $0 \le t_p \le M d_p$ and $0 \le w_q \le M d'_q$, if $d_p^* = 0$ ($d'^*_q = 0$) then $t_p^* = 0$ ($w_q^* = 0$). It means $(m+p)$th input ($(s+q)$th output) cannot be added. On the other hand if $d_p^* = 1$ ($d'^*_q = 1$) then $t_p^* > 0$($w_q^* > 0$). It means the corresponding input (output) should be added. So $\sum_{p=1}^{k} d_p + \sum_{r=1}^{h} d'_q$ indicates the number of added inputs and or outputs. With regard to this issue and feasibility of model (6), it can be concluded that z^* equals the minimum number of inputs and (or) outputs among these k inputs and h outputs which should beadded in a way that the inefficient DMU_o will become efficient.■

5. Empirical example

Now the presented systems and model in this paper are used for the data of tables 1and 2 related to twenty DMUs with three inputs and three outputs. These data are real and extracted from (Alder et al, 2002).The original data and the data that are supposed to be add are given respectively in table 1 and table 2. Evaluating the presented DMUs in table 1 through CCR model has been revealed that only DMU_{15} is efficient.

Table 1. Data of input1 and output1of DMUs (Alder et al , 2002)

DMU	(I)input1	(O)output1
1	0.95	0.19
2	0.796	0.227
3	0.798	0.228
4	0.865	0.193
5	0.815	0.233
6	0.842	0.207
7	0.719	0.182
8	0.785	0.125
9	0.476	0.08
10	0.678	0.082
11	0.711	0.212
12	0.811	0.123
13	0.659	0.176
14	0.976	0.144
15	0.685	1
16	0.613	0.115
17	1	0.09
18	0.634	0.059
19	0.372	0.039
20	0.583	0.11

Table2. Data of input2,3 and output2,3 of DMUs (Alder et al, 2002)

DMU	(I) input2	(I) input3	(O) output 2	(O) output 3
1	0.7	0.155	0.521	0.293
2	0.6	1	0.627	0.462
3	0.75	0.513	0.97	0.261
4	0.55	0.21	0.632	1
5	0.85	0.268	0.722	0.246
6	0.65	0.5	0.603	0.569
7	0.6	0.35	0.9	0.716
8	0.75	0.12	0.234	0.298
9	0.6	0.135	0.364	0.244
10	0.55	0.51	0.184	0.049
11	1	0.305	0.318	0.403
12	0.65	0.255	0.923	0.628
13	0.85	0.34	0.645	0.261
14	0.8	0.54	0.514	0.243
15	0.95	0.45	0.262	0.098
16	0.9	0.525	0.402	0.464
17	0.6	0.205	1	0.161
18	0.65	0.235	0.349	0.068
19	0.7	0.238	0.19	0.111
20	0.55	0.5	0.615	0.764

Now it’s going to present the results of adding the inputs and outputs in table 2 on the status of efficiency of DMUs in table 3. The results of table 3 are obtained through using the software DEA-Solver.

Now systems (2), (4) and (5) for adding one or multiple inputs and or outputs are solved by using the software lingo. The related results for $DMU_o = DMU_1, DMU_4, DMU_7, DMU_{12},$ $, DMU_{17} and\ DMU_{20}$ are presented in tables 4, 5, 6, 7 and 8.

Table 3. Results of extracted from DEA-Solver

DMU_o	input1 output1	input1,2 Output3	input1 output1,2	input1,3 output1,2	input1,3 output1,3	input1,2,3 output1,2,3
DMU_1	0.1370	0.3317	0.4829	1	0.7738	1
DMU_2	0.1953	0.5677	0.6925	0.6925	0.5677	0.8334
DMU_3	0.1957	0.3973	0.9911	0.9911	0.4046	0.9911
DMU_4	0.1528	0.9173	0.6216	0.8719	1	1
DMU_5	0.1958	0.3804	0.7613	0.8974	0.4958	0.8974
DMU_6	0.1684	0.6076	0.6229	0.6229	0.6274	0.7484
DMU_7	0.1734	0.8278	1	1	0.8697	1
DMU_8	0.1091	0.3544	0.2878	0.7161	0.7677	0.7979
DMU_9	0.1151	0.4495	0.6177	0.7597	0.4863	0.7875
DMU_{10}	0.0828	0.1236	0.2500	0.2500	0.1236	0.2897
DMU_{11}	0.2042	0.5666	0.4617	0.5055	0.5974	0.6045
DMU_{12}	0.1039	0.6158	0.9092	1	0.6642	1
DMU_{13}	0.1829	0.4322	0.8166	0.8166	0.4492	0.8166
DMU_{14}	0.1011	0.2591	0.4413	0.4413	0.2681	0.4693
DMU_{15}	1	1	1	1	1	1
DMU_{16}	0.1286	0.6263	0.5515	0.5515	0.6263	0.6390
DMU_{17}	0.0617	0.1642	0.7989	1	0.2907	1
DMU_{18}	0.0637	0.1299	0.4398	0.4727	0.1462	0.4727
DMU_{19}	0.0718	0.2660	0.4088	0.4088	0.2729	0.4088
DMU_{20}	0.1292	1	0.8427	0.8427	1	1

Table 4.solving system (4) for adding output2

DMU_o	v	u	w_1
DMU_1		Infeasible	
DMU_4		Infeasible	
DMU_7	1.390821	0	1.111111
DMU_{12}		Infeasible	
DMU_{17}		Infeasible	
DMU_{20}		Infeasible	

Table 5.solving system (4) for adding output3

DMU_o	v	u	w_1
DMU_1		Infeasible	
DMU_4		Infeasible	
DMU_7		Infeasible	
DMU_{12}		Infeasible	
DMU_{17}		Infeasible	
DMU_{20}	1.715266	0	1.308901

Table 6. solving system(5) for adding input3 and output2

DMU_o	v	u	t_2	w_1
DMU_1	0.149183	2.311984	5.537265	1.076244
DMU_4		Infeasible		
DMU_7	1.390821	0	0	1.111111
DMU_{12}	0.803122	0	1.367327	1.083424
DMU_{17}	0	0	4.878049	1
DMU_{20}		Infeasible		

Table7. solving system(5) for adding input3 and outpu3

DMU_o	v	u	t_2	w_1
DMU_1		Infeasible		
DMU_4	1.095108	0	0.2511046	1
DMU_7		Infeasible		
DMU_{12}		Infeasible		
DMU_{17}		Infeasible		
DMU_{20}	1.433387	0	0.3286709	1.308901

Now by using model (6) is determined the minimum number of inputs and or outputs that should be added in a way that the inefficient units become efficient. For this purpose model (6) is solved through using the software Lingo and the related results are presented in table 9

In table 9, $z^* = 1$ for $DMU_o = DMU_7$ and DMU_{20} . This means the minimum number of inputs and outputs among $\{\text{input2}, \text{input3}, \text{output2}, \text{output3}\}$ that should be added in a way that DMU_7 and DMU_{20} become efficient equals 1. The added inputs and (or) outputs for DMU_7 and DMU_{20} are respectively outpt2 and output3. But $z^* = 2$ for $DMU_o = DMU_1, DMU_4, DMU_{12}$ and DMU_{17}. The added inputs and (or) outputs for DMU_1, DMU_{12} and DMU_{17} are input3 and output2 whereas the added inputs and (or) outputs for DMU_4 are input3 and output3. These results are consistent with table3.

Table 8. Solving system (5) for adding input2, input3, output2 and output3

DMU_o	v	u	t_1	t_2	w_1	w_2
DMU_1	0.2374179	2.119988	0	4.996471	1.040665	0.1877675
DMU_4	1.095108	0	0	0.2511046	0	1
DMU_7	0.0963206	0	1.113514	0.7503923	0.8693287	0.3039165
DMU_{12}	0.2900087	0	0	2.999227	0.8494963	0.3438136
DMU_{17}	0	0.990099	1.666667	0	0.9108911	0
DMU_{20}	1.433387	0	0	0.3286709	0	1.308901

Table 9. Results of solving model (6)

	DMU_1	DMU_4	DMU_7	DMU_{12}	DMU_{17}	DMU_{20}
z^*	2	2	1	2	2	1

6. Conclusion

In this paper is presented a system for showing that whether inefficient DMU_o is still preserved its inefficiency or it will become efficient through adding a given input or output. Next this system has been generalized for adding given multiple inputs and (or) outputs. Afterwards a model is presented that can be obtained the minimum number of inputs and outputs among the given inputs and outputs which should be added whereas inefficient DMU_o will become efficient. Finally the mentioned systems and model have been utilized in a set of DMUs and the results have been presented.

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