Available online at http://ijdea.srbiau.ac.ir

Int. J. Data Envelopment Analysis (ISSN 2345-458X)

Vol.2, No.4, Year 2014 Article ID IJDEA-00245,7 pages

Research Article

International Journal of Data Envelopment Analysis

Science and Research Branch (IAU)

DEA Models with Interval Scale Inputs and Outputs

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Received 18 August 2014, Revised 25 November 2014, Accepted 21 December 2014

Abstract

This paper proposes an alternative approach for efficiency analysis when a set of DMUs uses interval scale variables in the productive process. To test the influence of these variables, we present a general approach of deriving DEA models to deal with the variables. We investigate a number of performance measures with unrestricted-in-sign interval and/or interval scale variables.

Keywords: data envelopment analysis; interval scale variables; ratio scale variables.

1. Introduction

Data envelopment analysis (DEA) is a non-parametric method for evaluating the relative efficiency of decision-making units (DMUs) on the basis of multiple inputs and outputs. The original DEA models use (positive) ratio scale input and output variables but these models do not apply to variables in which interval scale data can appear. With the widespread use of interval scale variables, such as profit or the increase/decrease in bank accounts emphasis has led to the development of alternative models aiming at assessing efficiency in presence of ratio and interval scale data in DEA models .However, interval scale inputs/outputs cannot be used widely in DEA models. The problem with interval scale variables arises from the fact that rations of measurement on such a scale are meaningless. Consequently, a DEA model can be used to handle interval scale variables in which the ratio of virtual effect inputs and/or outputs and observed inputs and/or outputs does not have any role in the calculations. This paper deal with the problem of interval scale data in DEA, then investigate a number of different performance measures. The measures are modified in such a way that they also are able to handle variables consisting

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of ratio scale for some and interval scale for other sample DMUs. This paper unfolds as follows. In section 2, we summarize three of the initial DEA models: the CCR model (Charnes et al., 1978), the BCC model (Banker et al., 1984), the additive model (Charnes et al., 1978). In sections 3 and 4, we examine a modified slacks-based measure model (Tone, 2001) and weighted-additive model of DEA proposed by Pastor and Ruiz (2005), respectively. Conclusions are summarized in section 5.

2. The initial DEA models

Suppose we have *n* peer observed decision-making units (DMUs) where every $DMU_j (j = 1,2, \dots, n)$ produce multiple outputs $y_{rj}, (r = 1, \dots, s)$ by utilizing multiple inputs $x_{ij}, (i = 1, \dots, m)$. The input and output vectors of DMU_j are denoted by $\boldsymbol{x}_j$ and $\boldsymbol{y}_j$, respectively, and we assume $\boldsymbol{x}_j$and $\boldsymbol{y}_j$ are semi positive, i.e., $(\boldsymbol{x}_j, \boldsymbol{y}_j) \geq (\boldsymbol{0}, \boldsymbol{0}) \epsilon \mathbb{R}^{m+s}, \boldsymbol{x}_j \neq \boldsymbol{0},\ \boldsymbol{y}_j \neq \boldsymbol{0},\ j = 1, \dots, n$. We use by $(\boldsymbol{x}_j,\ \boldsymbol{y}_j)$ to descript DMU_j, and specially use $(\boldsymbol{x}_o,\ \boldsymbol{y}_o)$ $(o \epsilon \{1,2, \dots, n\})$ as the DMU under evaluation. Throughout this paper, vectors will be denoted by bold letters. Let

$\forall i \left(i \epsilon \{ 1, \dots, m\} \Rightarrow \exists j \big(j \epsilon \{1, \dots, n\}\ \&\ x_{ij} > 0 \big) \right)$,

$\forall r \left(r \epsilon \{ 1, \dots, s\} \Rightarrow \exists j \big(j \epsilon \{1, \dots, n\}\ \&\ y_{rj} > 0 \big) \right)$

The efficiency of each DMU_o can be evaluated by the envelopment form of the input-oriented CCR model by solving the following linear program,

$$\begin{array}{ll} \min & \theta - \varepsilon \left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \right) \\ & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io}\ ; \quad i = 1, \dots, m, \\ \text{s.t.} & \sum_{j=1}^{n} \lambda_j y_{ij} - s_r^+ = y_{ro}; \quad r = 1, \dots, s, \\ & \lambda_j \geq 0,\ s_i^- \geq 0,\ s_r^+ \geq 0;\ j = 1, \dots, n,\ i = 1, \dots\ ,\ m,\ r = 1, \dots, s, \end{array} \tag{1}$$

where θ is a scalar, and $\varepsilon > 0$ is the non-Archimedean element. In model (1) the ratio of virtual effect inputs and observed inputs (positive data) play a central role in the calculations, thus input data must be measured on ratio scale (Mohammadpour et al., 2015). However, in model (1) the ratio of virtual effect outputs and observed outputs hasn't any role in the calculations; therefore, the model can be applied with interval scale output data. But, since an assumption of CRS is not possible in technologies where negative data can exist (see Portela et al., 2004), thus the model (1) cannot be used with the negative interval scale output data. If some inputs are interval scale data, a modified ratio scale input-oriented CCR model is proposed as follows:

$$
\begin{array}{ll}
\min & \theta-\varepsilon\left(\sum_{\mathrm{i}=1}^{\mathrm{m}} s_i^{-}+\sum_{\mathrm{r}=1}^{\mathrm{s}} s_r^{+}\right) \\
\mathrm{s.t.} & \sum_{j=1}^{n} \lambda_j \boldsymbol{x}_{ij}+s_i^{-}=\theta \boldsymbol{x}_{io}; \quad i \epsilon R_{inp}, \\
& \sum_{j=1}^{n} \lambda_j \boldsymbol{x}_{ij}+s_i^{-}=\boldsymbol{x}_{io}; \quad i \epsilon I_{inp}, \\
& \sum_{j=1}^{n} \lambda_j \boldsymbol{y}_{rj}-s_r^{+}=\boldsymbol{y}_{ro}; \quad r \epsilon R_{out} \cup I_{out}, \\
& \theta \geq 0, i=1, \ldots,\left|R_{inp}\right|,\ \lambda_j \geq 0, \forall j;\ s_i^{-} \geq 0, \forall i;\ s_r^{+} \geq 0, \forall r.
\end{array} \tag{2}
$$

where

$$
\begin{aligned}
R_{inp} &= \{i: i\epsilon\{1,\ldots,m\}\ \&\ \forall j(j\epsilon\{1,\ldots,n\} \Rightarrow\ x_{ij} \text{ is a ratio scale variable})\}\\
I_{inp} &= \{i: i\epsilon\{1,\ldots,m\}\ \&\ \forall j(j\epsilon\{1,\ldots,n\} \Rightarrow\ x_{ij} \text{ is an interval scale variable})\}\\
R_{out} &= \{r: r\epsilon\{1,\ldots,s\}\ \&\ \forall j(j\epsilon\{1,\ldots,n\} \Rightarrow y_{rj} \text{ is a ratio scale variable})\}\\
I_{out} &= \{r: r\epsilon\{1,\ldots,s\}\ \&\ \forall j(j\epsilon\{1,\ldots,n\} \Rightarrow y_{rj} \text{ is an interval scale variable})\},
\end{aligned} \tag{3}
$$

and for simplification in using the defined sets let $R_{inp}=\{1,\ldots,|R_{inp}|\}, R_{out}=\{1,\ldots,|R_{out}|\}$, where symbol of $|.|$ is the cardinality of sets. Since DMU_j ($j = 1,\ldots,n$) are peer, then

$R_{inp} \cup I_{inp}=\{1,\ldots,m\},\ R_{out} \cup I_{out}=\{1,\ldots,s\}$.

Model (2), at first evaluates the radial efficiencies of ratio scale input data $\theta_i^*(i\epsilon R_{inp})$, then it takes account of the interval scale input excesses and output shortfalls that are represent by non-zero slacks. Thus model (2) detects all inefficiencies of inputs and outputs, if any. Therefore, model (2) indicates $\mathrm{DMU_o}$ with the interval scale input and/or output data is CCR-efficient. Also, assuming that $((\lambda_1^*,\ldots,\lambda_n^*),\theta^*,(s_1^{-*},\ldots,s_m^{-*}),(s_1^{+*},\ldots,s_s^{+*}))$ be an optimal solution of model (2), the production possibility $(\sum_{j=1}^{n}\lambda_j^*\boldsymbol{x}_j, \sum_{j=1}^{n}\lambda_j^*\boldsymbol{y}_j)$ is CCR-efficient. Models (1) and (2) are not translation invariant with respect to either outputs or inputs; however these models with convexity constraint $\sum_{j=1}^{n}\lambda_j=1$ are translation invariant with respect to interval scale outputs. Therefore, model (2) with convexity constraint $\sum_{j=1}^{n}\lambda_j=1$ can be used with non-negative interval scale inputs and negative interval scale outputs. Also, model (2) with constraint $\sum_{j=1}^{n}\lambda_j=1$ both indicates $\mathrm{DMU_o}$ with interval input / output data is BCC-efficient and provide a performance measure.

In model (2) we assume that $\mathrm{R_{inp}} \neq \emptyset$. Now, if $\mathrm{R_{inp}} = \emptyset$, then the following model is obtained:

$$
\begin{array}{ll}
\max & \sum_{i=1}^{m} s_i^{-}+\sum_{r=1}^{s} s_r^{+} \\
\mathrm{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij}+s_i^{-}=x_{io}; \quad i=1,\ldots,m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}+s_r^{+}=y_{ro}; \quad r=1,\ldots,s, \\
& \lambda_j \geq 0, s_i^{-} \geq 0, s_r^{+} \geq 0; \quad \forall j,i,r.
\end{array} \tag{4}
$$

Which is the model additive model proposed by Charnes et al. (1985) and it can be used with nonnegative scale interval inputs and/or outputs. Model (4) with convexity constraint $\sum_{j=1}^{n}\lambda_j = 1$ is the additive model with the variable returns to scale assumption as follows:

$$\begin{aligned} \max \quad & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\ \text{s.t.} \quad & \sum_{j=1}^{n}\lambda_j x_{ij} + s_i^- = x_{io}\,; \quad i = 1,\dots,m, \\ & \sum_{j=1}^{n}\lambda_j y_{ij} - s_r^+ = y_{ro}; \quad r = 1,\dots,s, \\ & \sum_{j=1}^{n}\lambda_j = 1, \\ & \lambda_j \ge 0, s_i^- \ge 0, s_r^+ \ge 0; \quad \forall i,j,k. \end{aligned} \tag{5}$$

Model (5) which is translation invariant as demonstrated by Ali and Seiford can be used with the interval scale and/or negative interval scale input and output data. Also model (5) indicates DMU_o is BCC-efficient. In other words, the optimal solution model of (5) is zero if and only if DMU_o be BCC-efficient [2]. The model (5), however, does not provide an efficiency measurement for DMUs which are not BCC-efficient.

3. A modified slacks-based measure model to deal negative interval outputs and inputs

The following modified slacks-based measure (MSBM), proposed by Sharp et al (2007), also provides efficiency measures and obviates the problems associated with zero weights being assigned to inputs or outputs.

$$\begin{aligned} min \quad & \frac{1-\frac{1}{m}\sum_{i=1}^{m}\frac{w_i s_i^-}{R_{i0}^-}}{1+\frac{1}{s}\sum_{r=1}^{s}\frac{v_r s_r^+}{R_{r0}^+}} \\ s.t. \quad & \sum_{j=1}^{n}\lambda_j x_{ij} + s_i^- = x_{io} \quad i = 1,\dots,m \\ & \sum_{j=1}^{n}\lambda_j y_{ij} - s_r^+ = y_{ro} \quad r = 1,\dots,s \\ & \sum_{j=1}^{n}\lambda_j = 1 \\ & \sum_{i=1}^{m} w_i = 1, \quad \sum_{r=1}^{s} v_r = 1 \\ & \lambda_j \ge 0\,, s_i^- \ge 0\,, s_r^+ \ge 0 \quad \forall j,i,r. \end{aligned} \tag{5}$$

Where $R_{io}^- = x_{io} - \min_j\{x_{ij}\},\ i = 1,\dots,m;\ R_{ro}^+ = \max_j\{y_{rj}\} - y_{ro},\ r = 1,\dots,s.$

To avoid problems with zeros, the slacks corresponding to variables with $R_{io}^- = 0$, for some $i = 1,\dots,m$, and/or $R_{ro}^+ = 0$, for some $r = 1,\dots,s$, are ignored. The ratio of virtual effect interval scale data and observed interval scale data plays a central role in the calculations whose provide efficiency measure in the objective function of model (5), thus we cannot use the model with interval scale data.

Following Tone (2001), the fractional programming problem represented by model (5) can be transformed to the following problem

$$
\begin{aligned}
\max \quad & \lambda - \sum_{i=1}^{m} \frac{w_i \alpha_i^-}{R_{io}} \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} - \alpha_i^- = x_{io} \quad i = 1, \dots, m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} + \beta_r^+ = y_{ro} \quad r = 1, \dots, s \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda + \frac{1}{|R_{out}|} \sum_{r=1}^{s} \frac{v_r \beta_r^+}{y_{io}} \\
& \sum_{i=1}^{m} w_i = 1, \sum_{r=1}^{s} v_r = 1 \\
& \lambda > 0; \mu_j \geq 0, \alpha_i^- \geq 0, \beta_r^+ \geq 0, \forall j, i, r.
\end{aligned} \tag{6}
$$

Also, a modified version of model (6) is proposed as follows:

$$
\begin{aligned}
\min \quad & \lambda - \frac{1}{|R_{inp}|} \sum_{i \epsilon R_{inp}} \frac{w_i \alpha_i^-}{R_{io}} - \varepsilon \left(\sum_{i \epsilon I_{inp}} \alpha_i^- + \sum_{r \epsilon I_{out}} \beta_r^+ \right) \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} - \alpha_i^- = x_{io} \quad i = 1, \dots, m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} + \beta_r^+ = y_{ro} \quad r = 1, \dots, s \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda + \frac{1}{|R_{out}|} \sum_{r \epsilon R_{out}} \frac{v_r \beta_r^+}{y_{io}} = 1 \\
& \sum_{i=1}^{m} w_i = 1, \sum_{r=1}^{s} v_r = 1 \\
& \lambda > 0; \mu_j \geq 0, \alpha_i^- \geq 0, \beta_r^+ \geq 0, \forall j, i, r,
\end{aligned} \tag{7}
$$

to deal with unrestricted-in-sign interval scale input and/or output data as well as to take account of all inefficiencies of input and output data, if any.

4. DEA weighed-additive models

The following weighted-additive model, proposed by Pastor and Ruiz (2005), allow us also indicates DMU_o is CCR-efficient and provides efficiency measures

$$
\begin{aligned}
\max \quad & \frac{1}{m+s} \left(\sum_{i=1}^{m} \frac{s_i^-}{R_{i0}^-} + \sum_{r=1}^{s} \frac{s_r^+}{R_{r0}^+} \right) \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io}; \quad i = 1, \dots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{ij} - s_r^+ = y_{ro}; \quad r = 1, \dots, s, \\
& \lambda_j \geq 0 , s_i^- \geq 0 , s_r^+ \geq 0; \quad \forall j, i, r,
\end{aligned} \tag{12}
$$

where

$$\mathrm{R}_{\mathrm{io}}^{-} = x_{io} - \min_{j}\{x_{ij}\} \quad i = 1, \dots, m,$$
$$\mathrm{R}_{\mathrm{ro}}^{+} = \max_{j}\{y_{rj}\} - y_{ro} \quad r = 1, \dots, s.$$

To avoid problems with zeros, the slacks corresponding to variables with $\mathrm{R}_{\mathrm{io}}^{-} = 0$, for some $i = 1, \dots, m$, and/or $\mathrm{R}_{\mathrm{ro}}^{+} = 0$, for some$r = 1, \dots, s$, are ignored.

The ratio of virtual effect interval scale data and observed interval scale data plays a central role in the calculations whose provide efficiency measure in the objective function of model (12), thus we cannot use the model with interval scale data. A modified version of weighted-additive model (12) proposed as follows, allows us also deal with nonnegative interval scale data and provides efficiency measure.

$$\begin{array}{ll} \max & \frac{1}{|R_{inp}|+|R_{out}|}\left(\sum_{i\epsilon R_{inp}} \frac{s_i^-}{R_{i0}^-} + \sum_{r\epsilon R_{out}} \frac{s_r^+}{R_{r0}^+}\right) - \varepsilon\left(\sum_{i\epsilon I_{inp}} s_i^- + \sum_{r\epsilon I_{out}} s_r^+\right) \\ & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io}; \quad i = 1, \dots, m, \\ \mathrm{s.t.} & \sum_{j=1}^{n} \lambda_j y_{ij} - s_r^+ = y_{ro}; \quad r = 1, \dots, s, \\ & \lambda_j \geq 0\,, s_i^- \geq 0\,, s_r^+ \geq 0; \quad \forall j, i, r, \end{array} \tag{13}$$

Model (13) with convexity constraint $\sum_{j=1}^{n} \lambda_j = 1$ can be used with unrestricted-in-sign interval scale input and/or output data. Also, model (13) takes account of all inefficiencies of input and output data (if any) both to indicate the efficiency measure and to provide the efficient projection of $\mathrm{DMU}_{\mathrm{o}}$. In addition, assuming that $(\lambda_1^*, \dots, \lambda_n^*), (s_1^{-*}, \dots, s_m^{-*}), (s_1^{+*}, \dots, s_s^{+*})$ be an optimal solution of model (13) production possible $(\sum_{j=1}^{n} \lambda_j^* \boldsymbol{x}_j\,, \sum_{\mathrm{j}=1}^{\mathrm{n}} \lambda_j^* \boldsymbol{y}_j)$ is BCC-efficient.

5. Conclusion

In this paper it is presented a systematic investigation of the problem of interval scale data in DEA. The problem with interval scale variables arises from the fact that rations of measurement on such a scale are meaningless. Consequently, a model of DEA can be used to deal with interval scale outputs (inputs) in which the ratio of virtual effect outputs (inputs) and observed outputs (inputs) does not have any role in the calculations. Also, it is pointed out that the interval scale variables are invariant under any translation transformation; therefore, a translation invariant DEA model can be used to handle negative interval scale variables. It is also discussed a general approach of deriving DEA models to deal with unrestricted-in-sign interval scale input and/or output data.

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