



## Basic ratio-based DEA models

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### Abstract

In this paper, different input-oriented ratio-based DEA (DEA-R-I) models are proposed. By presenting the envelopment-additive-enhanced Russell model based on the DEA-R-I form, each decision making unit is evaluated and its efficiency score is calculated. Also, by presenting the central resource allocation model based on the DEA-R-I form, a suitable benchmark for all DMUs is proposed by solving a linear programming problem. Finally, the efficiency scores obtained for the data of 20 commercial bank branches by each of the proposed models are compared.

*Keywords* : DEA, DEA-R, Efficiency.

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### 1 Introduction

Data Envelopment Analysis (DEA), by Charnes et al. [2], is a method for evaluating the relative efficiency of comparable entities referred to as Decision Making Units (DMUs). DEA forms a production possibility frontier, or an efficient surface. If a DMU lies on the surface, i.e., there is no other DMU that can either produce the same outputs by consuming less inputs (input-oriented DEA) or produce more outputs by consuming the same amount of inputs (output-oriented DEA), it is referred to as an efficient unit, otherwise inefficient. DEA also provides efficiency scores and reference units for inefficient DMUs. The efficiency score tells the percentage by which a DMU should decrease its inputs (input-oriented DEA) or increase its outputs (output oriented DEA) in order to become efficient. Tracey and Chen [11] first proposed that weight hypothesis may lead to underestimation from additional weight restrictions. However, CCR, based on, implies inherent weight restrictions, and has never been extensively discussed. Such restrictions may lead to a failure to represent the relations of certain weights, thus, output-oriented DEA-R was developed to address such problems ([7]). Since unnecessary and unreasonable weight hypothesis would cause CCR to underestimate the efficiency of a DMU, an input-oriented DEA-R model was developed.

The DEA-R-I model was presented by Despic et al. [7] with weights restrictions. Later, Wei et al. worked on the subject in [3, 4]. In the present paper, basic DEA models such as the envelopment, additive, and enhanced Russell models will be proposed based on the DEA-R-I form. In section 2, a review of DEA is provided. In section 3, basic DEA-R-I models are proposed. Section 4 contains an application of the proposed model by a numerical example for 20 bank branches. Final Section 5 contains conclusions and starting points for future research.

### 2 Data Envelopment Analysis

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Consider  $n$  DMUs with  $m$  inputs and  $s$  outputs. The input and output vectors of DMU <sub>$j$</sub>  ( $j = 1, \dots, n$ ) are  $X_j = (x_{1j}, \dots, x_{mj})^t, Y_j = (y_{1j}, \dots, y_{sj})^t$  where  $X_j \geq 0, X_j \neq 0, Y_j \geq 0, Y_j \neq 0$ .

By using the variable returns to scale, convexity, and possibility postulates, the non-empty production possibility set (PPS) is defined as follows:

$$T_c = \left\{ (X, Y) : X \geq \sum_{j=1}^n \lambda_j X_j, Y \leq \sum_{j=1}^n \lambda_j Y_j, \lambda_j \geq 0, j = 1, \dots, n \right\}$$

By the above definition, the CCR model proposed by Charnes et al. [2] is as follows:

$$\begin{aligned} \text{Min} \quad & \theta - \varepsilon \left[ \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \right] \\ \text{S.t} \quad & \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = \theta x_{ip}, \quad i = 1, \dots, m \\ & \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{rp}, \quad r = 1, \dots, s \\ & \lambda_j \geq 0, \quad j = 1, \dots, n \\ & s_i^- \geq 0, \quad i = 1, \dots, m \\ & s_r^+ \geq 0, \quad r = 1, \dots, s. \end{aligned} \tag{1}$$

Clearly, the evaluated DMU <sub>$p$</sub>  is efficient if and only if  $\theta^* = 1$  and all slack variables in the optimal solution are zero in problem (1).

### 2.1 The DEA additive model

The additive model (ADD) maximizes the  $L_1$  distance of the DMU under analysis to the projected point on the efficient frontier.

$$\begin{aligned}
& \text{Max} \quad \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+ \\
& \text{s.t.} \quad \sum_{j=1}^n \lambda_j x_{ij} + s_i^- = x_{ip}, \quad i = 1, \dots, m \\
& \quad \quad \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ = y_{rp}, \quad r = 1, \dots, s \\
& \quad \quad \lambda_j \geq 0, \quad j = 1, \dots, n \\
& \quad \quad s_i^- \geq 0, s_r^+ \geq 0, \quad i = 1, \dots, m, \quad r = 1, \dots, s.
\end{aligned} \tag{2}$$

Clearly, the evaluated  $DMU_p$  is efficient if and only if  $\theta^* = 1$  and all slack variables in the optimal solution are zero in problem (2).

## 2.2 Enhanced Russell model

Koopmans [8] defined an input-output vector as technically efficient if and only if increasing any output or decreasing any input is possible only by decreasing some other output or increasing some other input, respectively. A nonradial Pareto-Koopmans measure of technical efficiency for  $DMU_p$  can be computed as follows:

$$\begin{aligned}
& \text{Min} \quad \frac{\frac{1}{m} \sum_{i=1}^m \theta_i}{\frac{1}{s} \sum_{r=1}^s \varphi_r} \\
& \text{s.t.} \quad \sum_{j=1}^n \lambda_j x_{ij} \leq \theta_i x_{ip}, \quad i = 1, \dots, m \\
& \quad \quad \sum_{j=1}^n \lambda_j y_{rj} \geq \varphi_r y_{rp}, \quad r = 1, \dots, s \\
& \quad \quad \lambda_j \geq 0, \quad j = 1, \dots, n \\
& \quad \quad \varphi_r \geq 1 \quad r = 1, \dots, s \\
& \quad \quad \theta_i \leq 1 \quad i = 1, \dots, m
\end{aligned} \tag{3}$$

In model (3) we suppose that all inputs and outputs are of the same importance in determining the efficiency. That is to say, the priority in increasing or decreasing the outputs or the inputs, respectively, is the same for all inputs and outputs.

### 3 Input-oriented DEA-R model

Despic et al. [7] proposed a new mathematical model for efficiency score, which combines the DEA methodology with the idea of ratio analysis.

$$\begin{aligned}
 & \text{Max} \quad \Delta \\
 & \text{S.t.} \quad \sum_{r=1}^s \sum_{i=1}^m w_{ir} \left( \frac{x_{ij}/y_{rj}}{\frac{x_{ip}}{y_{rp}}} \right) \geq \Delta, \quad j = 1, \dots, n \quad (4) \\
 & \quad \quad \sum_{r=1}^s \sum_{i=1}^m w_{ir} = 1 \\
 & \quad \quad w_{ir} \geq 0, \quad \Delta \geq 0 \quad i = 1, \dots, m \quad r = 1, \dots, s
 \end{aligned}$$

Model (4) is a DEA-R-I model that calculates the DEA-R-I efficiency score of  $DMU_p$ .

$$\begin{aligned}
 & \text{Min} \quad \theta_R \\
 & \text{s.t.} \quad \sum_{j=1}^n \lambda_j \left( \frac{x_{ij}/y_{rj}}{\frac{x_{ip}}{y_{rp}}} \right) \leq \theta_R, \quad i = 1, \dots, m \quad r = 1, \dots, s \quad (5) \\
 & \quad \quad \sum_{j=1}^n \lambda_j \geq 1, \\
 & \quad \quad \lambda_j \geq 0, \quad j = 1, \dots, n.
 \end{aligned}$$

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**Definition 1.**  $DMU_p$  is R-CCR-I-efficient (input-oriented CCR –R-efficient) if and only if  $\theta_R^* = 1$ .

Model (5) has the following properties.

- 1) The efficiency and super-efficiency scores obtained by this model are greater than or equal to those of the CCR model.
- 2) The efficiency scores of the model in the input and output orientations are not necessarily equal.
- 3) In a situation involving no weight restrictions, the input-target improvement strategy given by DEA-R-I is always better than the CCR-I model.
- 4) When DEA-R-I weights are concentrated on one output, the CCR-I efficiency and DEA-R-I efficiency are the same.
- 5) When DEA-R-I weights are not concentrated on multiple outputs, the CCR-I efficiency and DEA-R-I efficiency are the same, with the exception of, every times of corresponding output of targeted DMU. (See [3, 4, 5]).

The two-phase model based on the DEA-R-I model, considering slack variables, is proposed as follows.

$$\begin{aligned}
 & \text{Min} \quad \theta - \varepsilon \left( \sum_{r=1}^s \sum_{i=1}^m S_{ir} \right) \\
 & \text{s.t.} \quad \sum_{j=1}^n \lambda_j \left( \frac{x_{ij}/y_{rj}}{\frac{x_{ip}}{y_{rp}}} \right) + S_{ir} = \theta, \quad i = 1, \dots, m \quad r = 1, \dots, s \\
 & \quad \sum_{j=1}^n \lambda_j \geq 1, \\
 & \quad S_{ir} \geq 0 \quad i = 1, \dots, m \quad r = 1, \dots, s \\
 & \quad \lambda_j \geq 0, \quad j = 1, \dots, n.
 \end{aligned} \tag{6}$$

The additive model based on the DEA-R-I model for distinguishing between efficient and inefficient

$$\begin{aligned}
 & \text{Max} \quad \sum_{r=1}^s \sum_{i=1}^m S_{ir} \\
 & \text{s.t.} \quad \sum_{j=1}^n \lambda_j \left( \frac{x_{ij}/y_{rj}}{\frac{x_{ip}}{y_{rp}}} \right) + S_{ir} = 1, \quad i = 1, \dots, m \quad r = 1, \dots, s \tag{7}
 \end{aligned}$$

units is suggested as follows.

$$\begin{aligned}
 & \sum_{j=1}^n \lambda_j \geq 1, \\
 & \quad S_{ir} \geq 0 \quad i = 1, \dots, m \quad r = 1, \dots, s \\
 & \quad \lambda_j \geq 0, \quad j = 1, \dots, n.
 \end{aligned}$$

The enhanced Russell model based on the DEA-R-I model is suggested as follows.

$$\begin{aligned}
 & \text{Min} \quad \frac{1}{ms} \sum_{r=1}^s \sum_{i=1}^m \theta_{ir} \\
 & \text{s.t.} \quad \sum_{j=1}^n \lambda_j \left( \frac{x_{ij}/y_{rj}}{\frac{x_{ip}}{y_{rp}}} \right) \leq \theta_{ir}, \quad i = 1, \dots, m \quad r = 1, \dots, s \tag{8} \\
 & \quad \sum_{j=1}^n \lambda_j \geq 1, \\
 & \quad \theta_{ir} \leq 1 \quad i = 1, \dots, m \quad r = 1, \dots, s \\
 & \quad \lambda_j \geq 0, \quad j = 1, \dots, n.
 \end{aligned}$$

The central resource allocation model based on the DEA-R-I model is suggested as follows.

$$\begin{aligned}
 & \text{Min } \theta \\
 & s.t. \sum_{k=1}^n \sum_{j=1}^n \lambda_{kj} \left( \frac{\frac{x_{ij}}{y_{rj}}}{\frac{\sum_{j=1}^n x_{ij}}{\sum_{j=1}^n y_{rj}}} \right) \leq \theta, \quad i = 1, \dots, m \quad r = 1, \dots, s \quad (9) \\
 & \sum_{j=1}^n \lambda_{kj} \geq 1, \quad k = 1, \dots, n \\
 & \lambda_{kj} \geq 0, \quad k = 1, \dots, n, j = 1, \dots, n.
 \end{aligned}$$

#### 4 An application

We consider the data of 20 bank branches with three inputs and eight outputs, in the following table:

Table 1.

Outputs and Inputs of bank branches.

Inputs:	Outputs:
1. Interest paid	1. Interest-free saving account
2. Level of education of the staff	2. Current account
3. Demand	3. Short-term
	4. long-term
	5. Other
	6. Loans
	7. Interest received
	8. Banking fees

Table 2.

Outputs of bank branches.

DMUs	$O_1$	$O_2$	$O_3$	$O_4$	$O_5$	$O_6$	$O_7$	$O_8$
1	709874	2610439	172126	342833	228993	2761590	59161.51	9589.48
2	73560	317617	2326	26234	85324	252120	3504.68	180.38
3	435248	489666	6506	79194	824241	2043899	5817.55	760.9
4	29165	971640	1559	46160	354722	452931	622.48	244.75
5	216148	158130	845	90278	12256	552673	49570.82	20.7
6	514881	508735	103257	69604	396716	1751591	34115.58	1234.52
7	15676	172947	822	7373	287183	537567	1119.32	143.02
8	155799	269617	5451	83400	71509	1534505	10163.75	1868.82
9	13707	163397	5759	86775	135464	4829312	13520.68	1024.48
10	666679	242971	2319	5148	16992	2949072	21773.84	22.1
11	106682	408506	2439	173299	20643	1808353	2434.86	1193.46
12	190455	174657	9402	110027	69802	510656	2594.17	531.59
13	337298	293580	12340	148121	24997	768622	1946.46	94.83
14	133062	131262	15723	68920	150625	422274	422.05	461.59
15	152552	189936	11778	41074	420668	729915	2269.94	376.75
16	245412	106089	3651	12176	118734	448984	1200.53	102.69
17	392104	646367	6241	30690	375109	2144785	10428.37	295.05
18	27880	196292	2034	6021	6122	28007	365.28	144.75
19	204028	502730	3511	38927	1100502	1977217	8529.12	731.69
20	52156	71458	12850	14869	15344	277254	2761.24	162.163

Table 3

Inputs of bank branches.

DMUs	$I_1$	$I_2$	$I_3$
1	12963.87	37.75	134529
2	1483.69	22.74	10244
3	11756.02	25.5	42668
4	866.86	20.94	44128
5	4545.92	14.43	13043
6	9139.03	18.86	87981
7	308.67	25.66	97763
8	3185.7	26.27	7629
9	832.02	21.91	430513
10	11589.09	16.75	7859
11	2886.74	21.2	1192
12	3880.46	23.67	53209
13	6269.71	21.85	27506
14	2616.9	21.56	17988
15	2600.23	37.5	50229
16	4257	24.5	32618
17	6179.39	25.72	41817
18	528.92	16.28	22262
19	3679.91	28.76	16795
20	916.42	30.4	30326

Table 4

Efficiency score for commercial bank branches

DMUs	Model (1)	Model (2)	Model (5)	Model (7)	Model (8)
1	1.0000	0.0000	1.0000	0.0000	1.0000
2	1.0000	0.0000	1.0000	0.0000	1.0000
3	1.0000	0.0000	1.0000	0.0000	1.0000
4	1.0000	0.0000	1.0000	0.0000	1.0000
5	1.0000	0.0000	1.0000	0.0000	1.0000
6	1.0000	0.0000	1.0000	0.0000	1.0000
7	1.0000	0.0000	1.0000	0.0000	1.0000
8	1.0000	0.0000	1.0000	0.0000	1.0000
9	1.0000	0.0000	1.0000	0.0000	1.0000
10	1.0000	0.0000	1.0000	0.0000	1.0000
11	1.0000	0.0000	1.0000	0.0000	1.0000
12	0.931985	2.31811E+6	0.9886	10.7747	0.5511
13	1.0000	0.0000	1.0000	0.0000	1.0000
14	0.963303	0.0000	1.0000	0.0000	1.0000
15	1.0000	0.0000	1.0000	0.0000	1.0000
16	0.908524	2.27890E+6	0.9085	14.8423	0.3816
17	1.0000	0.0000	0.0000	0.0000	1.0000
18	0.973695	0.0000	1.0000	0.0000	1.0000
19	1.0000	0.0000	1.0000	0.0000	1.0000
20	1.0000	0.0000	1.0000	0.0000	1.0000

## 5 Conclusion

Regarding the fact that basic DEA models, such as the input-oriented envelopment form, the additive model, etc., play a fundamental role in many DEA contexts, using these models in the DEA-R-I form can be very important for comparing the efficiency –projection of DMUs and for their ranking. Further concentration on DEA-R-I models maintains the value of basic DEA models. It can also help to develop and extend different contexts in performance evaluation of DMUs. Ranking of DMUs and finding a suitable benchmark by using the proposed models can be considered for further studies.

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