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Radius of stability Region in data envelopment analysis with network structure

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Abstract

In today's world, due to the tight competition of companies, sensitivity analysis is of special importance for managers in maintaining their position. This issue can determine new strategies and distance from competitors. In this paper, a two-stage network structure was considered. The main goal is to determine a range for the inputs of each decision-making unit so that the efficiency of the entire network does not change. These changes have been made both in each entry and in all entries. With the help of data envelopment analysis models, a method has been designed to calculate a stability Region. The feasibility of this method has been checked with a set of data.

Keywords: Network Data Envelopment analysis, Stability Region, Efficiency, Sensitivity Analysis

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1. Introduction

Data envelopment analysis is a technique based on mathematical programming to calculate the relative efficiency of homogeneous decision-making units. This technique designs an efficiency boundary by comparing the inputs and outputs of the society and then measures the relative efficiency of the units by comparing each decision-making unit with the constructed efficiency boundary [1]. This topic was later extended to variable scale by Banker [2] by constructing a possibility set with efficiency. On the other hand, in order to pay more attention to the weaknesses of a decision-making unit, it is necessary to examine its network structures. In the network structure, in addition to considering the inputs and outputs of a decision-making unit, its intermediate values are also considered to check its internal structure. This causes the exact location of inefficiency to be determined and helps managers to determine the necessary strategy to improve the current situation. This issue was raised by Farr et al. (2000). Cao and Huang [3] considered a two-stage network and discussed determining the efficiency of the sub-stages based on the overall efficiency. Kao [4] extended the network structure based on relative efficiency and proposed multiple models for series and parallel network structure. Tone [5] proposed a non-radial model based on the possibility set, based on which it is possible to determine the total efficiency and the efficiency of sub-units in a two-stage network structure. Wang [6] used the approach of collective models to determine the efficiency of the whole and sub-units based on a simple two-stage structure in Chinese commercial banks. Tawana [7] presented modeling based on the determination of Malmquist's productivity index in a network structure, based on which they calculated the progress/regression of Iran's oil refineries. Ghafari developed a network structure in data envelopment analysis in imprecise environments [8]. Among other researches in this field, we can refer to Kiaei et al. (2022) who presented a new form of joint set of weights for a two-stage structure.

Each decision-making unit with a known efficiency value wants to know the range of changes in inputs or outputs. How many inputs and outputs should be so that its efficiency does not change. In other words, a confidence interval for each input or each output to achieve a suitable level of efficiency is the main goal of a decision-making unit. Due to the specific structure of data envelopment analysis models, which follows the dependence between parameters in linear programming, therefore, the sensitivity analysis method in linear programming problems cannot be applied in data envelopment analysis models. Therefore, the sensitivity analysis in data envelopment analysis models will lead to the design of an optimization problem. Among the pioneers of sensitivity and stability analysis, Charnes can be mentioned [9]. Charnes developed a sensitivity and stability analysis based on the collective model form [10]. Jahanshahlou presented a sensitivity analysis based on the conservation of efficient units in CCR and BCC models. In other words, the main problem is the maximum changes for the efficient decision-making unit that can maintain its efficiency if the parameters remain intact. This gave rise to an idea that many researchers used. That is, the maximum change to maintain membership in a particular category [11]. Soleimani proposed a method to determine the maximum changes to maintain the efficiency-to-scale classification based on this attitude and the classification of efficiency to scale for decision-making units [12]. This issue is important because a decision-making unit can know the changes needed to reach and approach the category of constant returns to scale.

Banker introduced sensitivity and stability analysis in data envelopment analysis in the presence of random data [13]. Among the recent researches in this field, we can refer to Ben Lahoul, in which they have presented a method to determine the effects of income changes on the efficiency of banks in a network structure based on auxiliary variables [14].
According to the research work done, the innovation of this research is as follows.
• Designing a model to determine the confidence interval of inputs in data coverage analysis with a network structure
• Designing a model to determine the radius of stability of inputs in data envelopment analysis with network structure
The structure of the article is as follows.

2. Background

Suppose there are n decision making units. So that the j-th decision making unit uses the input vector $X_j = (x_{1j},...,x_{mj})^t$ to produce the output vector $Y_j = (y_{1j},...,y_{sj})^t$. And also suppose $Y_j \geq 0, X_j \geq 0, Y_j \neq 0, X_j \neq 0$.
The relative efficiency of DMU_p is obtained from solving the optimization problem (1) [1].

$$\begin{aligned} Min \quad & \theta \\ s.t \quad & \sum_{j=1}^{n} \lambda_j X_j \leq \theta X_p \qquad (1) \\ & \sum_{j=1}^{n} \lambda_j X_j \geq Y_p \\ & \lambda \geq 0 \end{aligned}$$

Model (1) is known as the CCR model of the envelope form of the input nature. If you assume that θ^* is the optimal answer to the objective function of problem (1), if $\theta^* = 1$ then DMU_p is called relatively efficient and if $0 < \theta^* < 1$ then DMU_p is called inefficient.

If the goal is to determine a range for the input value of DMU_p so that the efficiency of this unit does not change, then if DMU_p is efficient, then the smaller inputs maintain the efficiency of DMU_p. But if the inputs of DMU_p increase then the performance of DMU_p may deteriorate. For this purpose, the following changes can be considered for inputs.

$$\begin{cases} X_p \to X_p + \alpha 1_m & (2) \\ X_p \to X_p + \triangle X_p & (3) \end{cases}$$

In relation (2) all inputs are increased by the same amount and in relation (3) each of the inputs can be increased independently. If equation (2) is considered as the criterion for change in inputs, then the maximum value of α is obtained from solving problem (4). (It is assumed that DMU_p were working)

$$\begin{aligned} Min \quad & \alpha \qquad (4) \\ s.t \quad & \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j X_j \leq X_p + \alpha 1_m \\ & \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j X_j \geq Y_p \\ & \lambda \geq 0 \quad j = 1,...,n,\ j \neq p \end{aligned}$$

If α^* is the optimal value of the objective function (4), then α^* is called the stability radius of DMU_p.
If equation (3) is considered as the criterion of change in inputs, then the maximum value of vector $\triangle X_p$ is obtained from the solution of model (5). (It is assumed that DMU_p were working.)

$$
\begin{aligned}
Min \quad & \triangle X_p \qquad (5)\\
s.t \quad & \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j X_j \le X_p + \triangle X_p\\
& \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j X_j \ge Y_p\\
& \lambda \ge 0 \quad j=1,\dots,n,\ j\neq p\ ,\ \triangle X_p \ge 0
\end{aligned}
$$

Note that problem (5) is a multi-objective linear programming problem that can be solved using the technique of weighted sum of objective functions. If $\triangle X_p^*$ is a Pareto optimal solution to problem (5), then suppose there is a two-stage structure in which in the first stage the j-th decision making unit consumes the input vector $X_j = \left(x_{1j},\dots,x_{mj}\right)^t$ to produce the intermediate production vector $Z_j = \left(z_{1j},\dots,z_{mj}\right)^t$ and the second stage of this unit It uses the input vector Z_j to produce the final output vector DMU_p and $X_j \ge 0, X_j \neq 0, Y_j \ge 0, Y_j \neq 0, Z_j \ge 0, Z_j \neq 0$.

The final production process is shown in Figure (1).

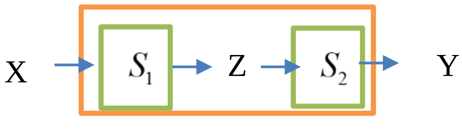

Figure 1. Two-step process

The relative efficiency of DMU_p with a two-stage structure as shown in figure (1) is obtained by solving problem (6).

$$
\begin{aligned}
Min \quad & \theta \qquad (6)\\
s.t \quad & \sum_{j=1}^{n} \lambda_j^1 X_j \le \theta X_p\\
& \sum_{j=1}^{n} \lambda_j^1 X_j \ge \sum_{j=1}^{n} \lambda_j^2 Z_j\\
& \sum_{j=1}^{n} \lambda_j^2 X_j \ge Y_p\\
& \lambda^1 \ge 0 \quad , \quad \lambda^2 \ge 0
\end{aligned}
$$

Now suppose that DMU_p is efficient with a two-stage structure, that is, the optimal solution to problem (6) is $\theta^* = 1$.

3- Proposed Method

Suppose the goal is to find the confidence interval of the t-th input from the vector X_p so that DMU_p are left. Therefore, model (7) is suggested to find this confidence interval.

$$
\begin{aligned}
Min \quad & \alpha_t\\
s.t \quad & \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^1 x_{ij} \le x_{ip}, \quad i=1,\dots,m, i\neq t\\
& \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^1 x_{ij} \le x_{tp} + \alpha_t \qquad (7)\\
& \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^1 z_{lj} \ge \sum_{j=1}^{n} \lambda_j^2 z_{lj}, l=1,\dots,k\\
& \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^2 y_{rj} \ge y_{rp}, \ r=1,\dots,s\\
& \lambda^1 \ge 0\ ,\ \ j=1,\dots,n\ ,\ \ j\neq p\\
& \lambda^2 \ge 0\ ,\ \ j=1,\dots,n\ ,\ \ j\neq p
\end{aligned}
$$

If α_t^* is the optimal value of the objective function of problem (7), then if $\alpha_t^* = 0$ then DMU_p is a non-extreme efficient (weak or strong). If $\alpha_t^* > 0$, then DMU_p was extreme efficient. The value α_t^* is called the stability radius of DMU_p at the t-th input. Obviously, the bigger the value of α_t^*, the better condition of DMU_p.

According to the model (7), the set of all the points of the set (8) will be efficient, and hence this set is also called the stability region DMU_p in the t-th input.

$$\left\{ \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \middle| \begin{array}{l} x_i = x_{ip}, i=1,\dots,m, i \neq t \,\&\, x_t = x_{tp} + \gamma \,\& \\ 0 \leq \gamma \leq \alpha_t^* \,\&\, Z = Z_p \,\&\, Y = Y_p \end{array} \right\} \quad (8)$$

Suppose the goal is to find the radius of stability of all entries DMU_p. In this case, model (9) is suggested.

$$\begin{aligned} Min \quad & \alpha & (9) \\ s.t \quad & \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j^1 x_{ij} \leq x_{ip} + \alpha, & i=1,\dots,m \\ & \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j^1 z_{lj} \geq \sum_{j=1}^{n} \lambda_j^2 z_{lj}, & l=1,\dots,k \\ & \sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j^2 y_{rj} \geq y_{rp}, & r=1,\dots,s \\ & \lambda^1 \geq 0, \ j=1,\dots,n, \ j\neq p & \\ & \lambda^2 \geq 0, \ j=1,\dots,n, \ j\neq p & \end{aligned}$$

If α^* is the optimal answer of the objective function (9), then if $\alpha^* = 0$, then DMU_p is a non-extreme efficient, and if $\alpha^* > 0$, then DMU_p is an extreme efficient. The value α^* is called the stability radius of DMU_p in the input vector X_p.

If α^* is the optimal value of the objective function (9), then the set (10) of the stability region of vector changes will be X_p out of DMU_p.

$$\left\{ \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \middle| \begin{array}{l} X = X_p + \gamma 1_m \,\&\, Z = Z_p \\ \&\, Y = Y_p \,\&\, 0 \leq \gamma \leq \alpha^* \end{array} \right\} \quad (10)$$

Obviously, the bigger α^* is, the better the condition of DMU_p will be.

4- Numerical example

In this section, based on the data of Chen [15], we examine the models proposed in this article. The data are shown in table (1). Table (2) shows the efficiency and stability radius of the decision-making units.

Table 1
Numerical example for two-stage network.

DMU	X1	X2	Z1	Z2	Y1	Y2
1	2	4	3	4	7	8
2	12	9	4	3	9	12
3	3	4	5	4	10	12
4	7	9	6	12	21	16
5	4	8	10	11	18	16

Table 2: Efficiency values and stability radius

DMU	efficiency	Efficiency status	Stability radius
1	0.727	Inefficient	-
2	0.320	Inefficient	-
3	0.900	Inefficient	-
4	0.873	Inefficient	-
5	1.000	Efficient	2.423

As you can see in Table 2, out of 5 decision-making units, only 5 DMUs are efficient and the rest of the DMUs are inefficient. Also, the stability radius for DMU5 was calculated based on the solution of model (9) and its value is equal to 2.423. This means that if each of the inputs of DMU5 is added with this number, DMU5 is still efficient. In other words, the freedom of action of DMU5 is interpreted so that even without increasing the output and adding the value of stability radius to all inputs, this decision making unit is still efficient.

5. Conclusion:

Determining the scope of changes for decision-making units is important for managers because they can have more freedom of action in determining their strategy. In other words, determining the distance with competitors by maintaining the position is effective in improving and developing the business of organizations. On the other hand, many times determining the efficiency of decision-making units without considering their internal processes and internal structure is not fruitful. Because in many cases, the inefficiency is not for the whole system, but it may be for a small part of it. Therefore, considering the internal processes and internal structure of each decision-making unit helps to determine more precisely the weaknesses and strengths of the decision-making units. In this article, two models were introduced. The first model specified the range of change of each input in maintaining efficiency, and the second model determined the radius of stability on inputs for efficient decision-making units. This issue was done on a data set and its usability was checked.

In the continuation of this research, it is possible to find the stability radius and stability area for input and output in the network structure.

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