Available online at http://ijdea.srbiau.ac.ir

Int. J. Data Envelopment Analysis (ISSN 2345-458X)

Vol. 9, No. 3, Year 2021 Article ID IJDEA-00422, Pages 31-46
Research Article

International Journal of Data Envelopment Analysis

Science and Research Branch (IAU)

Gradual Improvement of Benchmarking in Data Envelopment Analysis Using Gradient Line Method

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Received 18 January 2021, Accepted 19 April 2021

Abstract
Data envelopment analysis (DEA), firstly, checks whether decision making units (DMUs) are efficient or inefficient and then it introduces a benchmark for inefficient DMUs. This benchmark is of significant importance for managers and decision-makers. There are different methods for benchmarking one of which is the gradient line method. This method has a major problem which is that the benchmark introduced by this method is not always Pareto efficient. Having given an example, this problem is commented on in this article. On the other hand, the application of gradient line is effective on gradual improvement of efficiency because the introduced equation is in such a way that for reducing a certain amount of inputs, the largest expansion is given to outputs. Finally, we demonstrated that by using gradient line in gradual improvement method, there is no need any more to ask the managers for improvement bounds of inputs and outputs in any level and it is enough for the manager to state the highest efficiency improvement amount he expects in each step.

Keywords: Data envelopment analysis (DEA), gradual improvement, gradient line method, benchmark

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1. Introduction

Data envelopment analysis (DEA) is a non-parametric mathematical programming approach which is applied for evaluating the relative efficiency of the homogeneous DMUs which consist of multiple inputs for producing multiple outputs. In 1957, Farrell suggested a method for measuring the efficiency of the DMUs including one input and one output. Charnes, Cooper and Rhodes (1978), Banker et al. (1984) developed the Farrell's view and presented a model which was able to measure the efficiency of the DMUs including multiple inputs and outputs. This technique is called Data Envelopment Analysis (DEA). Charnes, Cooper and Rhodes proposed model known as the CCR which has led to many articles with different models including non-radial ones such as additive and SBM. Benchmark is achieved by projecting the evaluated DMU on the efficient frontier. In the radial models which are input-oriented, we can achieve the benchmark by reducing just the inputs. This is functional when the policy of decision makers is to reduce the inputs such as resources, work force etc. When the focus is on expanding the outputs, for example, increasing the production, the model is called output-oriented. If you reduce the inputs and expand the outputs simultaneously, non-radial models such as the additive and SBM models will be used. The benchmark introduced by the radial model may be on the strong or weak frontier, but the benchmark introduced by the non-radial models will be on the frontier of the strong efficient. Different studies for finding suitable benchmarks have been studied. Such as: Navabakhsh et al. (2007), Hosseinzadeh Lotfi et al. (2007), Mohammadi et al. (2009), Jahanshahloo et al.(2012a), Jahanshahloo et al.(2012b), Payan et al. (2014) [1-6].

Sometimes, there is a big difference between the evaluated DMU and the introduced benchmark; furthermore, occasionally, achieving the target in one step is out of reach for managers. To fix the problem, it was proposed to project the evaluated DMU as close as possible to the frontier Frei and Harker (1999), González and Álvarez (2001), Silva Portela et al. (2003), Aparicio et al. (2007), Jahanshahloo et al. (2012), Jahanshahloo et al. (2013). Lozano et al. (2005) suggested that the inefficient DMUs achieve the efficiency frontier in a sequence of steps. This method is known as the gradual improvement [7-12].

Maital et al. (1999) introduced a path equation for improving the efficiency of the evaluated DMUs. They defined the path equation of the objective function so that the objective function maximizes locally; therefore, they increased the efficiency locally by moving the DMUs along the gradient line (without having to introduce any benchmark in advance). They claimed that the suggested path for the objective function which is an equation of ellipse is optimal for improving the efficiency. The gradient method is also able to introduce a DMU whose efficiency is better than before in a certain percentage [13].

The benchmark produced by Maital et al. proposed model will be on the efficient frontier of CCR. One of the issues addressed in this paper is that this benchmark is not necessarily a strong efficient DMU; since all the inputs are reduced in a same parameter and all the outputs are expanded in a same proportion. Therefore, it is possible to have an input (output) which is able to reduce (expand) more than the mentioned proportion. So, the suggested benchmark by gradient will not be necessarily a strong efficient DMU. Then, applying the gradual improvement method provided by Lozano and Villa, weak efficient DMUs are gradually projected on the strong frontier [13].

In the next section, we try to use the equation of ellipse suggested in the gradient line approach in the gradual improvement method. This means that, instead of asking all inputs bounds and the output bounds from the managers like what happens in Lozano and Villa method, just input bounds will be determined by managers and then the output bounds will achieve by applying the ellipse equation presented by Maital et al. As the gradient line method gives the highest possible expansion to outputs for reducing a certain amount of inputs, it is expected that the gradual improvement method using ellipse equation would suggest a better benchmark than before, in each step. Also, this trend has been done for the combined oriented and the benchmark produced by the gradual improvement method the combined oriented and the equation of ellipse has been compared in the first step. Other advantage of using gradient line in gradual improvement is that instead of asking inputs and outputs bounds from the manager in each step, all you have to ask from the manager is the amount of efficiency improvement expected in each step [13].

This paper is divided as follows: in the second section a brief overview of data envelopment analysis, gradient line method for finding a benchmark, as well as the gradual improvement method are expressed. Gradient line weakness accompanied by an example and also using gradient line method in the gradual improvement are discussed in the third. Finally, conclusions are mentioned in the fourth section.

2. The Gradient line and gradual improvement methods in DEA

2.1. Data Envelopment Analysis

If there are n DMUs, m inputs and s outputs; their possible production set (PPS) with Constant Returns to Scale is as below:

$$T_{CCR} = \left\{ \begin{array}{c} (X,Y): X \geq \sum_{j=1}^{n} \lambda_j X_j, Y \leq \sum_{j=1}^{n} \lambda_j Y_j, \\ \lambda_j \geq 0, j = 1,...,n \end{array} \right\} \quad (1)$$

Input-oriented tries to reduce the value of inputs as much as possible so as to keep DMUs in possible production set. So, regarding PPS, the CCR model of the input oriented would be:

$$\begin{aligned} &Min \quad \theta \\ &s.t. \\ &\sum_{j=1}^{n} \lambda_j X_j \leq \theta X_p \qquad (2) \\ &\sum_{j=1}^{n} \lambda_j Y_j \geq Y_p \\ &\lambda_j \geq 0 \qquad j = 1,...,n \end{aligned}$$

Having the optimal value of the objective function equal to one is a necessary condition for Pareto efficiency of evaluated DMU, but it is not a sufficient condition because it is possible to have the evaluated DMU on the weak frontier, which means that some inputs (outputs) may still reduce (expand).

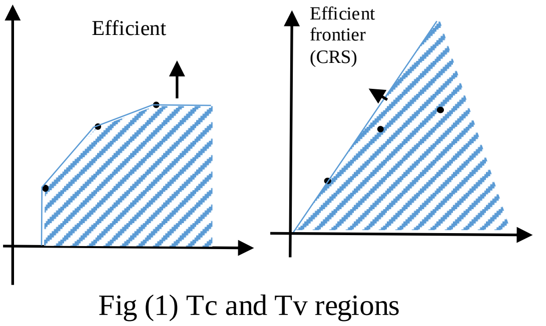

Fig (1) Tc and Tv regions

Fig (1) shows efficient frontier of production possibility set with Variable and also Constant Returns to Scale (VRS and CRS) for DMUs with one input and one output.

In the combined-oriented, the inputs and the outputs are improved in the same ratio in order to achieve the frontier.

$$\begin{aligned} &\max \theta \\ &s.t. \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{ip}(1-\theta), \ \mathrm{i}=1,...,\mathrm{m} \\ &\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rp}(1+\theta), r=1,...,s \\ &\lambda_j \ge 0. \end{aligned} \quad (3)$$

The non-radial models both reduce the inputs and expand the outputs to achieve the frontier. For example, the weighted additive model is as below:

$$\begin{aligned} &Max \quad \delta_o = \sum_i \frac{s_i^-}{x_{io}} + \sum_r \frac{s_r^+}{y_{ro}} \\ &s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} - s_i^- \ \forall i \\ &\sum_{j=1}^{n} \lambda_j y_{rj} = x_{ro} - s_i^+ \ \forall i \\ &\lambda_j, s_i^-, s_r^+ \ge 0 \quad \forall j, i, r \end{aligned} \quad (4)$$

2.2. The gradient line in DEA

Maital et al. (1999), to improve inefficiency, presented a method which does not need to specify other DMUs. Calculating the partial derivation of the efficiency function, they tried to put the evaluated DMU along an ellipse path. Moving in this direction, the efficiency is improved locally. The presented path equation of ellipse is independent of the orientation of the model. Also each DMU has its own equation curve path. In a practical sense, slope approximation provides the chance to decide how much change may be made in the inputs or outputs so that we can improve on efficiency for a certain percentage [13].

First, P_o of the two-dimensional space is considered as $P_o = \{(\tilde{X}, \tilde{Y}) : \tilde{X} = \alpha X_o, \tilde{Y} = \beta Y_o\}$ Then the gradient of the objective function is calculated:

$$grad \quad E = (E_{X_i}, E_{Y_r}, \forall i, \forall r)$$

Then, the calculated partial derivation is projected onto the P_o plane. The equation which moving can improve the efficiency of the evaluated DMU is presented as below. This equation represents an ellipse directed along size coordinate axes with half-axes equal to k_α and k_β correspondingly.

$$\frac{\alpha^2}{k_\alpha^2} + \frac{\beta^2}{k_\beta^2} = 1 \quad (5)$$

$$k_\alpha = \left[\frac{\sum_i x_{io}^2 + \sum_i y_{ro}^2}{\sum_i x_{io}^2} \right]^{1/2} \quad (6)$$

$$k_\beta = \left[\frac{\sum_i x_{io}^2 + \sum_i y_{ro}^2}{\sum_i y_{ro}^2} \right]^{1/2} \quad (7)$$

The α and β values which help achieve the benchmark on the frontier are produced by the following equations:

$$\alpha_{\substack{corresponding \\ to \quad frontier}} = \frac{k_\alpha k_\beta E_o}{\sqrt{k_\alpha^2 + (k_\beta E_o)^2}} \quad (8)$$

$$\beta_{\substack{corresponding \\ to \quad frontier}} = \frac{\alpha_{\substack{corresponding \\ to \quad frontier}}}{E_o} = \frac{k_\alpha k_\beta}{\sqrt{k_\alpha^2 + (k_\beta E_o)^2}} \quad (9)$$

Gradient line method has the potentiality to introduce the point whose efficiency value can be increased to a certain percentage comparing to the evaluated DMU's efficiency. For example, if we would like to increase the evaluated DMU's efficiency to 5%, we can put K= 0.05 in the formula $E_o = \frac{1}{1+k}$ and gain the values of corresponding α and β and introduce a benchmark which has improved 5% in efficiency compared with the previous status. This has conducted for the data in the table (1) that shows the coordinates of eleven DMUs with two inputs and two outputs. To achieve the 5% improvement, we put $E_o = \frac{1}{1+0.05} = 0.95$ and found the α and β values using (8) and (9). See table (2).

Table (1) Data of observed DMUs

DMU.No	1	2	3	4	5	6	7	8	9	10	11
I1	2	2	5	10	10	3.5	5	9	12	8	3
I2	12	8	9	4	6	6.5	3	5	9	3	4
O1	4	3	2	2	2	1	2	8	5	3	8
O2	6	5	2	3	8	12	8	7	3	8	9

Table (2) α and β values for 5% improvement in efficiency for the monitored data

	DMU1	DMU2	DMU3	DMU4	DMU5	DMU6	DMU7	DMU8	DMU9	DMU10	DMU11
α	0.99	1	1	0.99	0.98	0.96	0.97	0.97	0.99	0.98	0.96
β	1.04	1	1	1.04	1.03	1.01	1.02	1.02	1.04	1.02	1.01

2.3. Gradual improvement in efficiency

Sometimes, the target points or benchmarks which are determined by the DEA models are significantly different from the evaluated DMU. Therefore, inputs (outputs) generally have to reduce (expand) to a large degree in order to reach the efficient frontier. In such a case, as the DMUs need significant changes to reach the introduced benchmark, they are faced with many problems.

Regarding this, Lozano and Villa (2005) presented TEIP model that is used for inefficient DMUs. This model gradually takes the evaluated DMU to the frontier and it reduces (expands) input (output) a little at every step. To do this, the additive model Tv has been applied that the achieved benchmark by this method is strongly efficient [11].

α_{io}^{-} **:** The determined bound for the value reduction of the i[th] input in the DMU_O

β_{ro}^{+} **:** The determined bound for the value expansion of the r[th] output in the DMU_O

Whenever t reflects the intermediate target points and s_i^- denotes the amount of reduction of inputs i and s_r^+ shows increase amount of output r, the model TEIP is expressed as below:

$$
\begin{aligned}
Max \quad & \delta_o = \sum_i \frac{s_i^-}{x_{io}} + \sum_r \frac{s_r^+}{y_{ro}} \\
s.t. \quad & \\
& \sum_{j=1}^{n} \lambda_{jt} x_{ij} = x_{io}^{t-1} - s_{it}^- \qquad \forall i \\
& \sum_{j=1}^{n} \lambda_{jt} y_{rj} = y_{ro}^{t-1} + s_{rt}^+ \qquad \forall r \qquad (10) \\
& s_{it}^- \le \alpha_{io}^- \, x_{io}^{t-1} \qquad \forall i \\
& s_{rt}^+ \le \beta_{ro}^+ \, y_{ro}^{t-1} \qquad \forall r \\
& \lambda_j, s_i^-, s_r^+ \ge 0 \qquad \forall j,i,r
\end{aligned}
$$

The optimal solution for the above model, calculates the intermediate target point of the next level as below:

$$
\begin{aligned}
x_{io}^{t} &= \sum_{j=1}^{n} \lambda_{jt}^{*} x_{ij} = x_{io}^{t-1} - (s_{it}^{-})^{*} \qquad \forall i \\
y_{ro}^{t} &= \sum_{j=1}^{n} \lambda_{jt}^{*} y_{rj} = y_{ro}^{t-1} - (s_{rt}^{+})^{*} \qquad \forall r
\end{aligned} \qquad (11)
$$

The optimal value of the objective function mentioned above is equal to zero at the final step. The model TEIP produces a limited sequence of levels in a way that the introduced DMU at each level overcomes the preceding one. Therefore, the efficiency necessarily improves at each level and gradually approaches the frontier until the final solution reaches the efficiency frontier.

Lozano and Villa (2005) presented a method to put the DMUs, which are efficient but are not in most productive scale size (MPSS), in MPSS frontier. The concept of MPSS was first discussed by banker (1984). The strategy of model is also based on the gradual changes and it is known as SEIP. The algorithm of this method is as follows. First, an MPSS DMU is considered as the final target. To do this, the additive models for both T_v and Tc technologies are solved. If the optimal objective function of problems equals to zero, the evaluated DMU lies on the MPSS frontier. There are different criteria for choosing an MPSS out of the achieved MPSS DMUs. Lozano and Villa applied the criteria below to choose the final target point [10,15]:

$$o^{'} = \arg \min_{\{j|j\in MPSS\}} \left\{ \sum_{i} \frac{|x_{io}-x_{ij}|}{x_{io}} + \sum_{r} \frac{|y_{ro}-y_{rj}|}{y_{ro}} \right\} \quad (12)$$

After selecting the DMU $O^{'}$ as the final target point, a sequence of points on a line that connects the point *O* to the point $O^{'}$ is determined by the SEIP. To this end, the following bounds are required.

α^{+}_{io} **:** The determined bound for the value expansion of the i^{th} input in the DMU_O

β^{-}_{ro} **:** The determined bound for the value reduction of the r^{th} output in the DMU_O

Whenever t represents the level of running the model, SEIP model is presented as follows:

$$\begin{aligned}
& Min \quad \lambda_{ot} \\
& s.t. \\
& \quad \lambda_{ot} x_{io} + \lambda_{o't} x_{io'} = x^{t-1}_{io} + s^{+}_{it} - s^{-}_{it} && \forall i \\
& \quad \lambda_{ot} y_{ro} + \lambda_{o't} y_{ro'} = y^{t-1}_{ro} + s^{+}_{rt} - s^{-}_{rt} && \forall r \\
& \quad \lambda_{ot} + \lambda_{o't} = 1 \\
& \quad s^{+}_{it} \le \alpha^{+}_{io}\, x^{t-1}_{io} && \forall i \qquad (13) \\
& \quad s^{-}_{it} \le \alpha^{-}_{io}\, x^{t-1}_{io} && \forall i \\
& \quad s^{+}_{rt} \le \beta^{+}_{ro}\, y^{t-1}_{ro} && \forall r \\
& \quad s^{-}_{rt} \le \beta^{-}_{ro}\, y^{t-1}_{ro} && \forall r \\
& \quad \lambda_{ot}, \lambda_{o't}, s^{-}_{it}, s^{+}_{it}, s^{+}_{rt}, s^{-}_{rt} \ge 0 && \forall j,i,r
\end{aligned}$$

Changes in inputs and outputs will not exceed the determined bound. As the value of λ_{ot} is reduced in each level, intermediate points gradually grow distant from the DMU_O and approach the $DMU_{O^{'}}$ .Finally, they are projected onto the $DMU_{O^{'}}$ in an MPSS frontier.

3. Applying the gradient line method in the process of gradual improvement.

In this section first the weakness of gradient line method is discussed next applying it to gradual improvement method is proposed.

3.1. Gradient line method weakness in finding the benchmark

As noted earlier, gradient line method uses an equation of ellipse to introduce a benchmark. The target point is presented as $(\alpha X_p, \beta Y_p)$ in which all the inputs have changed by the parameter α and all the outputs by the parameter β [14].

The CCR efficiency and Slack Based Measure (SBM) (Tone 2001) efficiency values of data of table (1) are displayed in the table (3). The equation of ellipse and also α and β value for each DMU were calculated separately in accordance with the method of gradient line. The α and β values are displayed in the second and third rows of the table (4) and the resulting projection point in the fourth to seventh rows. The CCR efficiency value of these projection points which are displayed in the eighth row of the table (4) are all equal to one but the SBM efficiency value of projection points for the DMUs 1, 2, 3, 4, 8 and 9 is less than one and that means that these benchmarks are not strongly efficient. Thus, the presented projection points by gradient line method are on the CCR efficient frontier but they are not necessarily strongly efficient.

Table (3) the CCR and SBM efficiency values of the observed data

DMU.No	1	2	3	4	5	6	7	8	9	10	11
CCR	0.966	0.796	0.15	0.311	0.5	1	1	0.8	0.278	1	1
SBM	0.52	0.45	0.12	0.19	0.19	1	1	0.5	0.15	1	1

Table (4) the efficiency value of the projection points produced by gradient line model

DMU.No		1	2	3	4	5	6	7	8	9	10	11
α*		0.99	0.92	0.5	0.72	0.71	1	1	0.88	0.62	1	1
β*		1.03	1.15	3.31	2.31	1.41	1	1	1.1	2.25	1	1
I1		1.98	1.83	2.48	7.18	7.07	3.5	5	7.92	7.49	8	3
I2		11.89	7.33	4.47	2.87	4.24	6.5	3	4.4	5.61	3	4
O1		4.1	3.45	6.63	4.62	2.83	1	2	8.8	11.23	3	8
O2		6.16	5.75	6.63	6.93	11.3	12	8	7.7	6.74	8	9
CCR	efficiency value of benchmark presented by gradient line	1	1	1	1	1	1	1	1	1	1	1
SBM		0.622	0.688	0.819	0.8	1	1	1	0.62	0.544	1	1

Table (5) The benchmark obtained by the CCR method and it's SBM efficiency values

DMU.No	1	2	3	4	5	6	7	8	9	10	11
I1	1.93	1.59	1	3	5	3.5	5	7.2	3.33	8	3
I2	11.59	6.37	1	1	3	6.5	3	4	2.5	3	4
O1	4	3	2	2	2	1	2	8	5	3	8
O2	6	5	2	3	8	12	8	7	3	8	9
SBM	0.622	0.69	0.8192	0.8	1	1	1	0.6198	0.5435	1	1

We see that the projection points produced for the DMUs 1, 2, 3, 4, 8 and 9 using gradient line method are not on the strong efficient frontier.

The reason for this is that all inputs are reduced by the parameter α and all outputs are expanded by the parameter β in the gradient line method, so there may be an input (output) which can be reduced (expanded) more than the mentioned ratio, in other words, there may be positive slack variables. In other words, the ellipse path equation is not flexible. Therefore, the presented benchmark by gradient line model is not necessarily a strongly efficient DMU.

The first four rows in the table (5) shows the benchmark produced by the CCR method and their corresponding SBM efficiency value is in the last row. These benchmarks can be compared with the benchmarks produced by the gradient line method which are presented in the fourth to seventh rows in the table (4) and their SBM efficiency score is in the last row.

Regarding the tables (4) and (5), it seems that benchmarks produced by both the CCR model and gradient line method have equal SBM efficiency values and that is because of the positive slack variables which cannot be removed by any of these two models.

3.2. Gradual improvement for inefficient DMUs by using the gradient line.

In Lozano and Villa (2009) gradual improvement method which was mentioned in section (2.3), the parameters α and β are stated by the manager in order to find maximum input reduction and maximum output expansion. Regarding the two advantages of gradient lines method which are [11]:

1- Moving on the ellipse path, for reducing a certain amount of inputs, the highest amount of output expansion is obtained.

2- By just being aware of the amount of the efficiency increase, considered by the manager, the amounts of the parameters α and β can be calculated using the equations (8) and (9).

It is suggested that, in each step of gradual improvement, input reduction and output expansion bounds are determined by gradient lines. Applying ellipse path in gradual improvement has two advantages:

1- For reducing a certain amount of inputs, a bound with maximum possible expansion is obtained for the output.

2- In the cases where stating α and β parameters, that is input and output reduction and expansion bound amount, is not possible for the manager; it is enough for the decision maker to just state efficiency improvement amount considered for each step in gradual improvement method.

The second advantage is important in terms of management and application because if the Decision Maker wants the evaluated-DMU efficiency to be increased in a certain amount, the proposed algorithm helps them to consider to what extent the inputs and outputs should be changed at most. Consequently, the Decision Maker does not need to introduce α and β parameters. Therefore, the gradual improvement, and as a result the benchmark, is introduced regarding efficiency increase percentage considered by the Decision Maker.

The disadvantage, however, is that the introduced benchmark in gradual improvement method with gradient lines is not necessarily Pareto efficient and this drawback is due to the efficiency score stated in gradient lines method which is not necessarily Pareto efficient.

In this section, first we are going to once calculate the values of α and β in the gradual improvement method by using the gradient line method; then, compare the produced results with the α and β produced by using combined oriented in which α = β. As gradient line method proposes the highest output expansion for reduction of a certain values of inputs, it is predicted that the gradual improvement method using the α and β produced by gradient line method presents a better benchmark in each step comparing to the other methods such as combined oriented. Thus, this is discussed by providing an example.

Table (6) the results produced by the gradual improvement model, calculating β (for α = 0.2) using the ellipse path equation of gradient line method

DMU	SBM efficiency score of evaluated DMU	The amount of β in first step	SBM efficiency score of projected point in first step	The amount of β in second step	SBM efficiency score of projected point in second step	The amount of β in third step	SBM efficiency score of projected point in third step	The amount of β in fourth step	SBM efficiency score of projected point in fourth step	The amount of β in fifth step	SBM efficiency score of projected point in fifth step	The amount of β in sixth step	SBM efficiency score of projected point in sixth step	The amount of β in seventh step	SBM efficiency score of projected point in seventh step
1	0.52	1.93	0.57	1.64	0.63	1.43	0.68	1.29	0.75	1.20	0.82	1.14	0.92	1.10	1
2	0.45	1.71	0.67	1.32	0.73	1.22	0.81	1.15	0.90	1.11	1.00	-	1.00	-	1
3	0.12	3.70	0.53	1.29	0.86	1.12	0.96	1.09	1.00	-	1.00	-	1.00	-	1
4	0.19	3.09	0.63	1.29	1.00	-	1.00	-	1.00	-	1.00	-	1.00	-	1
5	0.19	1.71	0.73	1.20	0.80	1.19	0.92	1.18	1.00	-	1.00	-	1.00	-	1
6	1.00	-	1.00	-	1.00	-	1.00	-	1.00	-	1.00	-	1.00	-	1
7	1.00	-	1.00	-	1.00	-	1.00	-	1.00		1.00	-	1.00	-	1
8	0.5	1.38	0.65	1.20	0.90	1.12	1.00	-	1.00	-	1.00	-	1.00	-	1
9	0.15	2.71	0.43	1.40	0.60	1.17	0.83	1.11	1.00	-	1.00	-	1.00	-	1
10	1.00	-	1.00	-	1.00	-	1.00	-	1.00	-	1.00	-	1.00	-	1
11	1.00	-	1.00	-	1.00	-	1.00	-	1.00	-	1.00	-	1.00	-	1

Regarding the manager's idea, considered α = 0.2 and β is calculated using the equation of ellipse in the gradient line method. (k_α and k_β are parameters, α is a certain value, so β is achieved by the formula (5)). Then, we put β-1 as the β related to the gradual model. It means that we calculated the expansion range of Y in the gradual model regarding the gradient model. In other words, the output slack constraint in each level is as follows: $S^+ \leq (\beta - 1)Y_p$.

We ran the model for eleven DMUs with two inputs and two outputs. Putting $\alpha = 0.2$, the reduction range of inputs would be 0.2 X_p at most and the expansion range in each step is achieved by the path equation of ellipse:

$$\beta = k_\beta \sqrt{1 - \frac{\alpha^2}{k_\alpha^2}} \qquad (14)$$

All the DMUs were projected onto the strong efficient frontier in at most seven steps. As it is seen in the table (6), the SBM efficiency value is expanding on each iteration until a strong efficient target is proposed for each DMU.

Table (7) the results produced by the gradual improvement model (10), calculating $\alpha = \beta = 0.2$ using the path equation of combined oriented.

In gradual improvement method, the bounds related to inputs and outputs are determined in each step based on the manager preference. In this section, the manager has been asked just for the inputs bounds and the maximum output bound amount has been obtained according to the gradient lines method.

Table (7) the results produced by (10) the gradual improvement model, calculating using the path equation of combined oriented α =β = 0.2

DMU.No.	SBM efficiency score of evaluated DMU	Efficiency score of projected point of 1th step	Efficiency score of projected point of 2th step	Efficiency score of projected point of 3th step	Efficiency score of projected point of 4h step	Efficiency score of projected point of 5th step	Efficiency score of projected point of 6th step	Efficiency score of projected point of 7th step
1	0.52	0.57	0.63	0.68	0.75	0.82	0.92	1.00
2	0.45	0.67	0.74	0.82	0.92	1.00	1.00	1.00
3	0.12	0.18	0.26	0.39	0.58	0.86	0.97	1.00
4	0.19	0.27	0.39	0.54	1.00	1.00	1.00	1.00
5	0.19	0.28	1.00	1.00	1.00	1.00	1.00	1.00
6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
7	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
8	0.50	0.61	0.75	1.00	1.00	1.00	1.00	1.00
9	0.15	0.21	0.30	0.43	0.62	0.78	1.00	1.00
10	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
11	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Table (8) the number of steps for evaluated DMUs in gradual improvement method to reach strong efficient frontier.

DMU.No	Using β obtained from ellipse equation of gradient line	Using path equation of combined oriented α, β
DMU1	7	7
DMU2	5	5
DMU3	4	7
DMU4	2	4
DMU5	4	2
DMU6	1	1
DMU7	1	1
DMU8	3	3
DMU9	4	6
DMU10	1	1
DMU11	1	1

In the gradual improvement method, α = 0.2 (regarding manager's idea) was put in the constraints $S^- \le \alpha X_p, S^+ \le \beta Y_p$ and β was calculated by using the equation of ellipse in the gradient line method. The results for the eleven observed DMUs showed that they reached the efficiency frontier in at most seven steps. Then, we calculated α and β of the gradual improvement method by using the path equation of combined oriented. As the input reduction and output expansion are in the same ratio in the combined oriented, which means $\alpha = \beta$, we put $\alpha = \beta = 0.2$ to compare it with the previous method. Then, the gradual improvement method was run for this α and β and as it is seen in the table (7) the DMUs finally reached the efficiency frontier in at most seven steps.
In the table (8), the number of steps each DMU has taken in each strategy to reach the frontier has been displayed. When we used gradient line method, the DMUs 3, 4 and 9 were projected onto the strongly efficient frontier in less step numbers; however, the DMU 5 was not like this.
The fourth column in the table (6) and the third column in the table (7), respectively, show the SBM efficiency value of the benchmarks in the first step of the gradual improvement method using gradient line and combined oriented. As you see, the efficiency values in the fourth column of the table (6) are greater than or equal to the values in the third column of table (7). It means in the first step of the gradient method, for a certain amount of input reduction, the outputs have expanded more in comparison to the combined oriented method. Not that the next steps, as the presented projection points of the two methods are different, they cannot be compared.
Therefore, in the first step of the gradual improvement, the gradient line method is preferred since it introduces the highest possible amount for outputs.

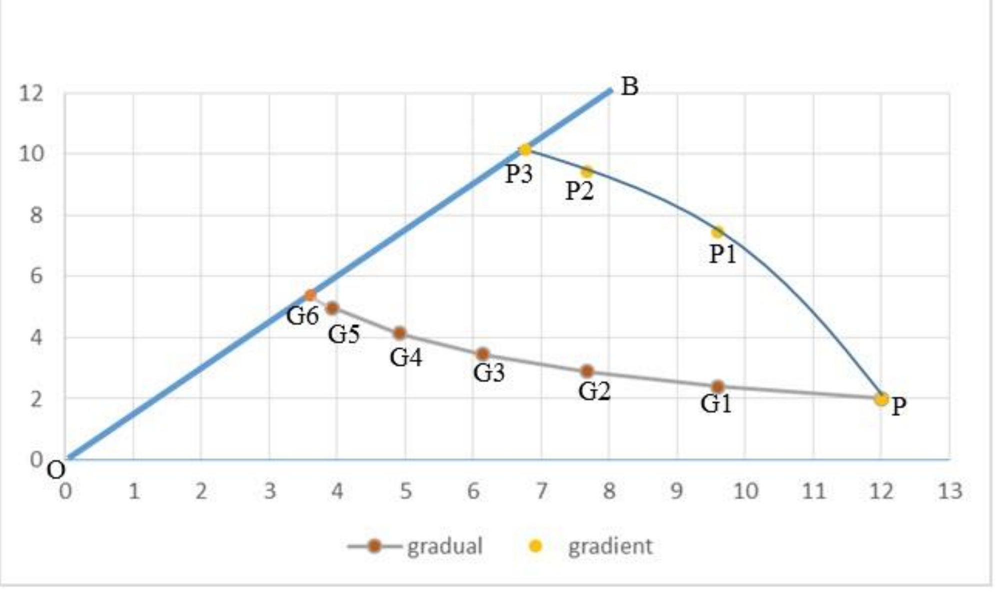

Fig (2) improving the efficiency of evaluated DMUs by two different methods

In Fig (2) the P-G6 path advances the gradual improvement by α and β produced by the combined oriented model. In other words, inputs and outputs improve to the extent not to exceed the bounds βY_p *and* αX_p in each step which α = β. In the P-P3 curve path, to improve the evaluated DMU gradually, we used gradient line method that the input bounds were considered α = 0.8 (regarding the manager's decision) and the maximum output value βY_p was achieved by equation (14). This process was repeated in each level.

As mentioned above in Fig (2) G1 and P1 are proposed benchmark in the first step of gradual improvement that bounds of improvement of inputs and outputs are achieved by combined oriented method and gradient line method respectively. Which by reducing inputs at most αX_p, the outputs of gradient line method (point P1) have expanded more.

As mentioned before, one of the advantages of applying elliptic path in gradual improvement method is that instead of asking Decision Maker for all the input and output bounds in each step of gradual improvement method, it is enough to ask for the percentage of efficiency increase considered by each DMU in each step; then, input and output reduction bounds can be determined by applying gradient lines method, as well as, formulas (8) and (9). Therefor in gradual improvement method if the Decision Makers' point of view is based on efficiency changes or if they don't have enough information about the change bounds of inputs and outputs in each step, using this technique seems to be more logical.

For example, for the data in table (1), if the Decision Maker wants %5 increase in efficiency score in the first step, inputs and outputs bound amount in gradual improvement method can be calculated based on what was mentioned in section (2.2). So improving efficiency score to 5%, based on table (2) the input and output bounds equals to $S^- \leq (\alpha)X_p$ and $S^+ \leq (\beta - 1)Y_p$.

To sum up, since Introducing the bounds in each step may be difficult or impossible for the manager, so the bounds in gradual improvement method can be obtained in each step by just being informed of maximum efficiency amount improvement considered by the manager. Therefore, the gradual improvement is totally followed based on the manager's will.

As is clear, in Gradual improvement method the aim is to reach the evaluated DMU to the efficient frontier in a few steps and the manager determines the bounds of input reduction and output expansion in each step (call it A procedure). On the other hand, in this paper we discussed about finding these bounds by using gradient line method and considering managers' (call it B procedure). We suggest using both point of view. Which means to find the minimum of the bounds obtained by procure A and B and consider them as the bounds of input reduction and output expansion in each step of gradual line method. So the managers' will about the efficiency advancement in each step for each DMU is achieved.

According to the Lozano and Villa's method (2010), we do the same to help the efficient DMUs to reach the MPSS. First, the evaluated DMU is projected onto the frontier using the gradual improvement model which bounds of inputs and outputs are obtained by gradient line method. Then, the MPSS DMUs are identified. As you see, the DMUs 6, 7, 10 and 11 which are displayed in the last column in the table (9) are MPSS. Then, using gradual improvement and the SEIP model and putting optionally $\beta^+_{ro} = \beta^-_{ro} = 0.2$ $\alpha^+_{io} = \alpha^-_{io} = 0.2$ we solve the model (13). As it is seen in the table (9), the DMUs 3, 4 and 5 has reached MPSS in two steps. However, the DMUs 1, 2 and 9 has reached MPSS in six steps. The second column in the table (10) displays the proposed benchmark by formula (12) for inefficient DMUs.

Table (9) Most Productive Scale Size (MPSS) DMUs

	SBM efficiency value in Tc	SBM efficiency value in Tv	
DMU1	10.67	0	
DMU2	867	0	
DMU3	26.67	20	
DMU4	19	19	
DMU5	21	16	
DMU6	0	0	MPSS
DMU7	0	0	MPSS
DMU8	11.5	9	
DMU9	35.5	23	
DMU10	0	0	MPSS
DMU11	0	0	MPSS

Table (10) Projecting inefficient DMUs to their benchmarks.

Inefficient DMUs	The argument for the corresponding inefficient DMU	Optimal value of the objective function of model (13)					
		1^{th} step	2^{th} step	3^{th} step	4^{th} step	5^{th} step	6^{th} Step
DMU1	DMU11	0.79	.054	.033	.016	0.3	0
DMU2	DMU6	0.82	0.60	0.39	0.23	0.11	0
DMU3	DMU11	0.44	0	-	-	-	-
DMU4	DMU10	0.43	0	-	-	-	-
DMU5	DMU10	0.32	0	-	-	-	-
DMU8	DMU11	0.68	0.42	0.21	0.05	0	-
DMU9	DMU10	0.73	0.51	0.33	0.19	0.08	0

4. Conclusion

Maital et al. (1999) proposed a method for finding the projection point by using gradient line. The evaluated DMU projects on the CCR frontier by introducing a path which uses equation of ellipse. Using the provided example, we discussed that this projection is not necessarily strongly efficient.

In Gradual improvement method which is proposed by Lozano and Villa (2005), the manager determines the bounds of input reduction and output expansion in each step. In this paper we suggested using the gradient line method provided by Maital et al. (1999) as tool to introduce the bounds of gradual improvement method. As the method of gradient line proposes the highest output expansion per certain amount of input reduction, therefore, we expected that this method proposes the best path for gradual movement toward the frontier. What we expected was achieved in the first step and the efficiency value of newly proposed benchmark compared to the prior gradual improvement method was either better or the same as the previous value, in the first step. However, as the starting point in the next steps of the models are not the same (each method proposes a distinct benchmark) the results are not comparable in the next steps. In the same way, we used the radial orientation in the gradual improvement once again to move toward the efficient frontier; comparing its results with moving in the elliptic path indicates, again, that the gradual improvement in the path equation of ellipse in the first step introduces a better benchmark.

Using gradient line method in gradual improvement method has two main advantages. First of all, in gradual improvement method, determining the bounds in each step may be difficult or even impossible. Moreover, when the manager is looking for a special percentage of efficiency improvement and the amount of he/she cares about the maximum improvement in each step the suggested method is useful.

Then the DMU which had reached the frontier gradually by gradient line direction was taken to the MPSS by the proposed method by Lozano and villas.

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