Available online at http://ijdea.srbiau.ac.ir

Int. J. Data Envelopment Analysis (ISSN 2345-458X)

Vol. 8, No. 4, Year 2020 Article ID IJDEA-00422, 22 pages
Research Article

International Journal of Data Envelopment Analysis Science and Research Branch (IAU)

Using Multi-Objective Linear Programming (MOLP) and Data Envelopment Analysis (DEA) models in Non-Discretionary Performance Measurement

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Received 1 July 2020, Accepted 20 October 2020.

Abstract

Inverse DEA (InvDEA) models put one step forward, in contrast with the DEA models by estimating required input level for producing a perturbed output level with the current efficiency status. In many real world applications, decision-makers face non-discretionary factors which can be hardly controlled by the Decision Making Units (DMUs). However, these types of factor are not dealt in the inverse DEA problems. Thus, the current covers the research methodological gap of the literature by developing mathematical foundation of the InDEA models capable of dealing with non-discretionary factors. To do this end, an MOLP model along with its required constraints is developed to be linked with the developed models. A numerical example and a real-world case study are provided to illustrate the proposed models and demonstrate their applicability and validity for the real world problems.

Keywords: Data Envelopment Analysis (DEA); Decision Making Unit (DMU); Inverse DEA (InvDEA); multi-objective linear programming (MOLP); Non-discretionary

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1. Introduction

Performance measurement is an important topic which has been vastly investigated in previous research but less defined [1]. According to previous studies on this topic, performance measurement is the process of quantifying an activity or an action [2]. In other words, performance measurement process aims to quantity activities first and then develop an assessment tool to measure them. Today's competitive markets make it necessary for any business to assess its performance. In addition, managers, researchers and practitioners have acknowledged the necessity of monitoring their performance in different periods. In this regard, there are different approaches to translate different metrics of performance measurement systems to an understandable score. These approaches vary from simple monitoring of assigned costs to complex process of sustainable performance assessment. However, some managers might be interested in reverse process. In other words, they prefer to set their desired performance score and try to vary the inputs or outputs to achieve this objective. Fortunately, the performance measurement topic can address both types of problem as they have been addressed in previous literature.

With the advent of data science, different applications of numerous data sets can be considered as a proper approach to investigate different scenarios in the process of performance measurement. Among different types of models used to assess the performance, DEA has been vastly investigated in previous literature [3]. This model is firstly developed by Charnes et. al [4] as a non-parametric approach to assess the relative performance of different DMUs with common input and outputs. Although the DEA has been vastly investigated in previous literature, a few studies have been conducted on InvDEA and its integration with MOLP models. In this regard, developing appropriate model, its integration with MOLP and finally solving by appropriate approaches can contribute to enrich the performance measurement problem. While there are many business managers who are interested in fixing their performance to a predetermined score and set the related inputs/outputs, previous studies have not adequately investigated this concern. This ignorance is mainly due to the inherent complexities which are linked with InvDEA models. In addition, solving MOLP models is a complex process which is less studied in conjunction with DEA models. However, determining the quantities of inputs and outputs to have a constant score of performance is beneficial with regard to following justifications:

1. Many real world problems aim to set an ultimate goal for the performance
2. Setting a high score of companies' performance need a precise understanding on the required inputs and outputs

Therefore, developing an integrated MOLP-DEA to investigate the non-discretionary factors in performance measurement problem is the main aim of this research. Following research questions are addressed in this study:

1. How to develop an InvDEA model in presence of both discretionary and non-discretionary inputs?
2. How to change discretionary inputs to preserve the efficiency score in case of perturbing the output value of a DMU?
3. How to link InvDEA and MOLP to enrich the performance measurement problem?
4. How to solve the integrated model using WSA?

The scope of this research is limited to application of DEA, InvDEA, MOLP and their integration in performance measurement problem. However, the developed model, research methodology and solution approach can be applied in any business which is interested in performance measurement topics. This research contributes in developing an integrated MOLP-DEA model to investigate non-discretionary factors of performance measurement problem. It provides novel idea to this area which can be easily extended to other similar problems. As a matter of fact, the current article develops mathematical foundation of InDEA capable of dealing with non-discretionary factors in the process of estimating required inputs for producing perturbed output with an unchanged efficiency level. Therefore, from both research and practice viewpoints, this study contributes to the problem of performance measurement as, to the best of authors' knowledge, no comparable study has been done on this topic.

The remainder of this paper is organized as follows. Section 2 develops the related literature. The research methodology and its different phases are discussed in Section 3. This is followed by model development in Section 4. Following, Section 5 discusses the numerical example followed by the obtained results of a real case study in Section 6. Finally, the concluding remarks and future research directions are provided in Section 7.

2. Literature Review

2.1 DEA

DEA is a non-parametric technique to measure and assess the relative performance of different entities called DMUs with common inputs and outputs [4]. This model is mainly based on common efficiency approach of single-input, single-output ratio which is converted to a multiple-input, multiple-output by the aid of a weighted sum ratio of outputs to inputs. A DMU is considered efficient when the output of this ratio is equal to one (1). In other words, the score of one is assigned to efficient units on the frontier. DEA assesses the relative performance of peer DMUs with multiple inputs and outputs. The methodology has been vastly investigated in different area of science including decision making, service industries, maintenance planning, manufacturing and business [5]. With the advent of its application in these areas, DEA has become a famous approach to deal with real world problems. In addition, different types of data with inaccurate, qualitative, outliers and undesirable factors have been used in DEA models.

2.2 Non-discretionary inputs

The classic form of DEA assumes all the inputs and outputs data to be discretionary. In other words, the standard DEA assumes that the considered inputs and outputs data can be controlled by the managers of DMU to be varied to its discretion. Though, there are many circumstances when the data are non-discretionary or exogenously fixed. In this regard, the inputs and outputs cannot be controlled by the managers of DMUs. These problems have been considered in previous literature [6, 7, and 8]. These type of data are very important and applicable in business, public and manufacturing sectors. There is a direct relation between the non-discretionary inputs, efficiency scores and the quantity of DMUs which are considered as efficient. In other words, as the quantity of non-discretionary inputs increases, the efficiency scores are biased upward and the number of efficient DMUs is also increased. To handle this issue, previous literature has suggested to apply multi-stage models. As an example, Ray [7] developed a methodology to apply the standard DEA models for the first stage using discretionary inputs and outputs only. In another example, Ruggiero [9 and 10] developed a DEA-based study to address the same concerns. In another study, Ruggiero [11] developed the available literature on the problem by investigating the correlation possibility among efficiency and non-discretionary factors. According to this research, the current estimates of DEA efficiency is biased upward when the true technical efficiency is negatively correlated with nondiscretionary inputs. To address this issue, this research developed a corrected model to handle the problem.

2.3 InvDEA

The InvDEA has been vastly investigated in previous literature. In this regard, different studies have considered it in different types of problems. In an example, Wei et al. [12] applied the inverse optimization problem concept to formulate the InvDEA. This research is mainly influenced by a study conducted by Zhang and Chi [13] which developed a project assessment system based on DEA methodologies. As mentioned, the InvDEA aims to keep a constant score for efficiency in the existence of data perturbation. In other words, it determines the extra inputs which are required to increase the outputs while keeping the efficiency of DMU unchanged. In another study on this topic, Yan et al. [14] developed the conducted research of Wei et al. [12]. In addition to discussed studies, there are some studies focusing on constant inputs, varying outputs and unchanged efficiency score [15, 16, 17, 18 and 19].

The InvDEA models have been modified to be applied in different case studies. For example, Jahanshahloo et al. [15] modified InvDEA models to assess the sensitivity of efficient and inefficient DMUs. To do so, the non-dominated solution of the modified version of InvDEA are applied to attain the upper and lower bounds of variation range for both inputs and outputs. In another research on this topic, Hadi Vencheh et al. [17] predicted the input and output levels of a specific DMU allowing for the alteration of its input/output levels under preserving the efficiency index. Different presentations of DEA are available in the literature. As a famous one, Banker et. al. [20] (BCC) has been investigated in previous literature. As an example, the Inverse BCC has been studied in some studies [18]. According to the developed model of this research, it is possible for the developed Inverse BCC to preserve the relative efficiency values of all DMUs in a new production possibility set including all current DMUs and a perturbed DMU with new values of inputs and outputs. Similar to what has been focused in this research, Lertworasirikul [18] developed a MOLP model to be applied in the process of finding the inputs. Following, the model of [18] is developed by Fathi and Izadikhah [21]. This research did not consider the discretionary and non-discretionary data. According to [19], there are many issues linked with the developed proof of Lertworasirikul's theorem [18]. In addition, according to this study, there are some ambiguities in application of MOLP. Different assumptions have been linked with the considered problem. For example, Jahanshahloo et al. [15] applied the inter temporal dependence assumption to be linked with InvDEA model. There are some recent studies focusing on application of InvDEA model in mergers [22, 23, 24 and 25]. Ghiyasi and Khoshfetrat [26] developed an inverse DEA model that considers the uncertain environment. Therefore, two multi-objective linear programming (MOLP) models are proposed to estimate the required upper/lower inputs, producing requested outputs and preserving the efficiency scores. Zhang and Cui [27] investigate the non-radial DEA models in the inverse DEA framework. They proposed some non-radial inverse DEA model for input-output estimation while keeping the efficiency scores. In another recent research, Kalantary and Saen [28] developed a network dynamic DEA model to investigate the supply chain's sustainability. In addition, this study has developed an Inverse network DEA model in dynamic environment. Lertworasirikul et. al. [18] dealt with variable returns to scale for the inverse DEA models which was modified by Ghiyasi [19] later on. Hu [29] dealt with the role of slacks when estimating input and output in the inverse DEA models. Kalantary & Saen [28] proposed an inverse dynamic network DEA model for assessing the sustainability of supply chains. In another research [30], authors proposed an inverse version of network dynamic range adjusted measure for sustainability assessment of supply chains. Hassanzadeh et. al. [31] considered both positive and negative data and proposed an inverse DEA model for assessing suitability of OECD countries. They dealt with negative data by using SORM model [32]. Chen et. al. [33] investigated the investment problem of sustainable development in China utilizing an inverse DEA model that is capable of dealing with undesirable outputs. Emrouznejad et. al. [34] proposed an inverse DEA model for allocation of CO_2 emission for different chines regions. Lin et. al. [35] utilized inverse DEA models for assessing container ports' efficiency and investigating their resource consumption in the presence of undesirable outputs.

2.4 Summary and Identification of Research Gaps

In summary, an efficient approach to integrate MOLP and DEA models to investigate and apply non-discretionary factors in performance measurement problem is not developed yet. In this regard, to fill the gap of previous literature, this study is an attempt to provide a framework for InvDEA with presence of non-discretionary data to evaluate the efficiency scores of DMUs. Providing an effective framework is important as the conditions of business environments might change during the time. By the best knowledge of the authors, there is no works in the existing literature to deals with the non-discretionary factors in the InDEA frameworks so far.

In fact, proposed models in the current paper contributes the literature by developing the mathematical basis of the InDEA model capable of dealing with the non-discretionary data. This yields an MOLP that estimate the required input level of perturbed output that may be non-discretionary.

The developed methodology guarantees unchanged efficiency score not only for the evaluated DMU but also for other DMUs. This study provides a simple and practical approach to assist managers to deal with non-discretionary data. The developed approach is applicable in efficiency analysis, resource estimation and long term performance assessment in the presence of both discretionary and non-discretionary data. To construct the framework, a MOLP model is developed to distinguish how the inputs should be changed when the outputs are changed to maintain the previous efficiencies. Several models are considered to attain the purpose of this study. In addition, a numerical example and a case study have been applied to show the practicality and validity of the developed mathematical model.

3. Research Methodology

As discussed before, the main aim of this research is to show the application of an integrated MOLP-DEA approach to investigate the non-discretionary factors of performance measurement. Figure 1 depicts different process of achieving the objectives of this research. As shown in this Figure, this research has been completed in three linked phases as follows:

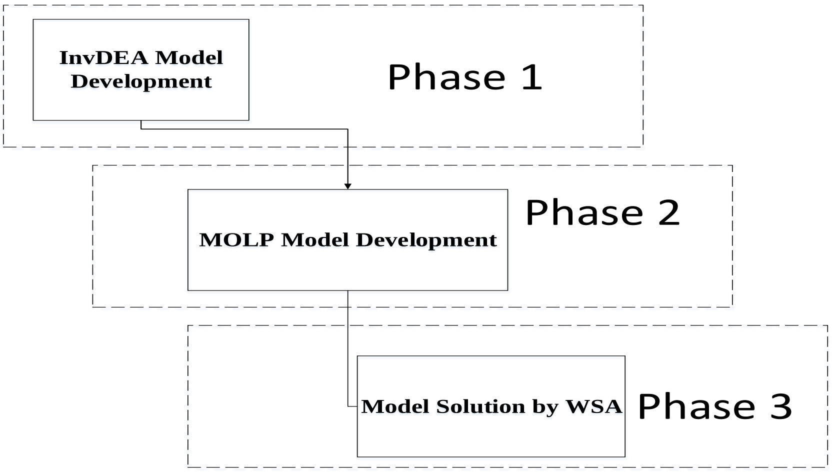

Figure 1: Research procedure

Phase 1: InvDEA Model Development

Firstly, an InvDEA model is developed to address the non-discretionary data of different business. As mentioned, against the classic DEA models, the InvDEA can determine the required quantities of inputs and outputs to have a fix score of performance. Therefore, as this is common for a manager to be interested in fixing the performance of company to a predetermined score, this phase aims to develop an InvDEA model to address this concern.

Phase2: MOLP Model Development

Following the development process of InvDEA model, this phase develops a MOLP model to be linked with the obtained output of the previous phase. The developed integrated InvDEA-MOLP model can be applied to assess the performance of businesses or any company with non-discretionary factors.

Phase 3: Model Solution by WSA

Finally, a numerical example and a real case study are applied to check the applicability and validity of the integrated model. There are numerous approaches to solve these types of models. However, there are many circumstances when the decision makers prefer to assign different weights to different objectives. In this regard, the developed integrated model is solved using a WSA to obtain the decision variables from the input parameters. A numerical example and a real-world case study are applied to demonstrate the application of the model.

4.Model Development

This sections discusses the process of model development. The section has been divided based on each specific model. In addition, different steps of the model development are comprehensively discussed in each sub-section.

4.1. Classical DEA model and non-discretionary inputs

Assume n DMUs, $DMU_j, \quad j \in J = \{1,2,...,n\}$, that consume p-dimensional discretionary inputs of $x_j \in R_+^p$ and q-dimensional non-discretionary inputs of $z_j \in R_+^q$ for producing s-dimensional outputs $y_j \in R_+^s$. Let $X \in R_+^{p\times n}$ and $Z \in R_+^{q\times n}$ be the discretionary and non-discretionary input matrix of all DMUs, respectively and $Y \in R_+^{s\times n}$ be the output matrix of all DMUs. A general production technology in presence of just discretionary input, and in presence of both discretionary and non-discretionary inputs may be considered as follows:

$T = \{(x, y) \in T^{p+s} \mid x \text{ can produce } y\}$ and

$T = \{(x, z, y) \in T^{p+q+s} \mid (x, z) \text{ can produce } y\}$

4.2. DEA model with regular inputs

Different returns to scale properties may be considered for the production technology. The variable returns to scale properties is assumed to have no loss of generality for the production technology. Therefore, for the case of regular inputs, the Production Possibility Set (PPS) can be found as follows:

$$T_{VRS} = \left\{ \begin{array}{c} (x,y) \Big| x \geq \sum_{j\in J} \lambda_j x_j, y \leq \sum_{j\in J} \lambda_j y_j, \sum_{j\in J} \lambda_j = 1, \\ \lambda_j \geq 0, \forall j \in J \end{array} \right\}$$

The following basic Farrell input measure can be used to measure the efficiency score of DMUo, where $o \in \{1,2,...,n\}$

$\min\{\theta \mid (\theta x_o, y_o) \in T_{VRS}\}$

This yields the following linear programming model, which is known as BCC model in DEA literature:

$$\begin{array}{ll} Min & \theta \\ s.t. & X\lambda \leq \theta x_o \qquad (1) \\ & Y\lambda \geq y_o \\ & 1^T \lambda = 1 \\ & \lambda \geq 0. \end{array}$$

The above model is contracting inputs to remove their waste which is caused by the investigated DMU. This is equal to no waste of input when the contraction process is not possible. In addition, in these cases, the investigated DMU uses its inputs in an (weak) efficient way. Or else, it is possible to have some waste of inputs which causes the investigated DMU to be inefficient. This description is summarized by the following definition adopted from Technical Efficiency [36].

Definition. Assume $(\lambda^*, \theta_o^{D*})$ is the optimal solution of the above model, where superscript of D is referred to a case with just discretionary inputs. If, then DMUo is efficient, otherwise DMUo is called to be inefficient.

4.3. DEA model with non-discretionary inputs

In presence of both discretionary and non-discretionary inputs, the PPS of T_{VRS} needs to be modified as follows:

$$T_{VRS}^{ND} = \left\{ \begin{array}{c} (x,z,y) \Big| (x,z) \geq (\sum_{j\in J} \lambda_j x_j, \sum_{j\in J} \lambda_j z_j)\ ,\ y \leq \sum_{j\in J} \lambda_j y_j, \\ \sum_{j\in J} \lambda_j = 1,\ \lambda_j \geq 0\ \ \forall j \in J \end{array} \right\}$$

A modified Farrell efficiency measure dealing with both discretionary and non-discretionary inputs is proposed as follows:

$\min\{\theta \mid (\theta x_o, z, y_o) \in T_{VRS}^{ND}\}$

Considering the modified Farrell input efficiency measure to deal with both discretionary and non-discretionary inputs, the following linear programming model is developed:

$$\begin{array}{ll} Min & \theta \\ s.t. & X\lambda \leq \theta x_o \qquad (2) \\ & Z\lambda \leq z_o \\ & Y\lambda \geq y_o \\ & 1^T \lambda = 1 \\ & \lambda \geq 0 \end{array}$$

In contrast with model (1), the non-discretionary inputs are not scaled down in the model (2) since there is no control on these types of inputs.

4.4 Inverse DEA model and non-discretionary inputs

This section deals with the InvDEA models. The next subsection presents the InvDEA model which is capable to deal with just discretionary inputs. The second subsection proposes an InvDEA model to consider both discretionary and non-discretionary inputs in the process of input estimation.

4.5. Inverse DEA model with regular inputs

Assume that DMUo changes its output from y_o to $y_o + \Delta y_o$, where $\Delta y_o \in R^s$. In order to estimate the required inputs to guarantee the unchanged efficiency score, following InvDEA model which only considers discretionary inputs can be developed:

$$\begin{aligned} Min \quad & \Delta x \\ s.t. \quad & X\lambda \le \theta_o^{D*}(x_o + \Delta x) \qquad (3) \\ & Y\lambda \ge (y_o + \Delta y_o) \\ & 1^T \lambda = 1 \\ & \lambda \ge 0 \end{aligned}$$

Where θ_o^{D*} is the optimal value of model (1). Note that $\Delta x \in R^p$ is a p-dimensional variable, thus the above model vector is a MOLP. Different solving techniques of MOLP can be applied to solve the MOLP model (3).

Definition. Suppose $(\lambda, \Delta x)$ is a feasible solution of model (3), if there is no feasible solution $(\bar{\lambda}, \overline{\Delta x})$ of this model such that $\overline{\Delta x} < \Delta x$, then the $(\lambda, \Delta x)$ is a weak efficient solution of model (3).

4.6. Inverse DEA model dealing with discretionary and non-discretionary inputs

In contrast to the previous subsection, this subsection assumes that both discretionary and non-discretionary inputs are allowed to be considered. Having p non-discretionary inputs, assume DMUo changes its output from y_o to $y_o + \Delta y_o$, the InvDEA model which is capable of dealing with non-discretionary inputs is developed follows:

$$\begin{aligned} Min \quad & \Delta x \\ s.t. \quad & X\lambda \le \theta_o^{ND*}(x_o + \Delta x) \qquad (4) \\ & Z\lambda \le z_o \\ & Y\lambda \ge (y_o + \Delta y_o) \\ & 1^T \lambda = 1 \\ & \lambda \ge 0 \end{aligned}$$

Where θ_o^{ND*} is the optimal value of model (2), namely, efficiency of DMUo in presence of non-discretionary inputs. In fact, the above model finds the minimum required discretionary inputs to produce the new output value of $y_o + \Delta y_o$ and the non-discretionary input is kept to be fixed. Note that these inputs are exogenously fixed and cannot be reduced. Therefore, they are not considering them in the objective function of the model.

Now we show that the efficiency scores of DMUo do not alter following the change of its input-output level, using model (4). Moreover, we will show that not only the efficiency score of DMUo stays unchanged, but also the efficiency score of other DMUs will not change.

Theorem 1. Assume that DMUo changes its output form y_o to $y_o+\Delta y_o$. If $(\bar{\lambda},\overline{\Delta x})$ is a weak efficient solution of model (4), then the efficiency of all DMUs including DMUo will change, that is $(x_o+\overline{\Delta x}, y_o+\Delta y_o)$ stays unchanged.

Proof. In the presence of non-discretionary inputs, we use model (2) as a testing model to check the efficiency of DMUo after changing its input-output levels, that is, $(x_o+\overline{\Delta x}, y_o+\Delta y_o)$:

$$\begin{aligned} Min \quad & \theta \\ s.t. \quad & X\lambda \le \theta(x_o+\overline{\Delta x}) \\ & Z\lambda \le z_o \\ & Y\lambda \ge y_o+\Delta y_o \\ & 1^T\lambda = 1 \\ & \lambda \ge 0 \end{aligned}$$

Now, let $(\bar{\lambda},\overline{\Delta x})$ be a weak efficient solution of model (4), thus just by the feasibility of this solution we have:

$$\begin{aligned} & X\bar{\lambda} \le \theta_o^{ND*}(x_o+\overline{\Delta x}) \\ & Z\bar{\lambda} \le z_o \\ & Y\bar{\lambda} \ge (y_o+\Delta y_o) \\ & 1^T\bar{\lambda} = 1 \\ & \bar{\lambda} \ge 0 \end{aligned}$$

The above set of constraints show that $(\bar{\lambda},\theta_o^{ND*})$ is a feasible solution of testing model. Thus, $\tilde{\theta}_o^{ND*} \le \theta_o^{ND*}$, where $\tilde{\theta}_o^{ND*}$ is the optimal value of testing model which is equal to the efficiency of DMUo after changes. It is claimed that $\tilde{\theta}_o^{ND*} = \theta_o^{ND*}$. In other words, the efficiency of DMUo does not alter after changes. To provide this claimant, it is required to show that $\tilde{\theta}_o^{ND*} = \theta_o^{ND*}$. Assume by contradiction that $\tilde{\theta}_o^{ND*} = \theta_o^{ND*}$. Thus we have:

$$\begin{aligned} & X\tilde{\lambda} \le \tilde{\theta}_o^{ND*}(x_o+\overline{\Delta x}) = \theta_o^{ND*}k(x_o+\overline{\Delta x}) \\ & Z\tilde{\lambda} \le z_o \\ & Y\tilde{\lambda} \ge (y_o+\Delta y_o) \\ & 1^T\tilde{\lambda} = 1 \\ & \tilde{\lambda} \ge 0 \end{aligned}$$

Where $0<k<1$, thus $(\tilde{\lambda},\overline{\Delta x})$ is a feasible solution of model (4) and this contradicts the weak efficiency of $(\bar{\lambda},\overline{\Delta x})$ since $0<k<1$ and therefore $k\overline{\Delta x} < \overline{\Delta x}$.

Now, it is shown that the efficiency of DMUj $1\le j\le n,\ j\ne o$ is not also changed following the alteration of DMUo from (x_o, y_o) to $(x_o+\overline{\Delta x}, y_o+\Delta y_o)$. Assume an arbitrary DMUt where $1\le t\le n,\ t\ne o$, using testing model to check the efficiency of this DMU after changing DMUo , the following model is developed:

$$\begin{aligned} Min \quad & \theta \\ s.t. \quad & X\lambda \le \theta x_t \\ & Z\lambda \le z_t \\ & Y\lambda \ge y_t \\ & 1^T\lambda = 1 \\ & \lambda \ge 0 \end{aligned}$$

This model is identical to model (2) to evaluate the DMUt that is used for efficiency measurement of DMUt before changing the input-output of DMUo. Therefore, the efficiency of DMUt is same even by changing DMUo.

The above theorem proves that the proposed model, namely, model (4) guarantees that the efficiency score of DMU under evaluation does not change as a consequence of changing it input and output levels, in the presence of non-discretionary factors. This fact is due to the nature of DEA models than can be consider as a multi-objective programming. Please note that in the DEA we aim to minimize multiple inputs, in input orientation or maximize multiple outputs in the output orientation. Thus, in both cases we face with a multi-objective problem. This case a mutual relationship between efficiency concept in the DEA models and weak efficient solutions of the multi-objective programming.

Also, the proposed models maybe extended for the cases with non-discretionary output. We consider two cases in presence of non-discretionary output. In the first case, non-discretionary output and input are assumed to be discretionary. Assume that decision maker changes the output vector of DMUo from (y_o, w_0) to $(y_o + \Delta y_o, w_o)$, where y_o is discretionary and w_o is non-discretionary. For the first case, where the only outputs are non-discretionary, after adding the constraint of $W\lambda \geq w_o$, model (1) can be used for efficiency evaluation as follows:

$$\begin{aligned} Min \quad & \theta \\ s.t. \quad & X\lambda \leq \theta x_o \\ & \begin{pmatrix} Y \\ W \end{pmatrix} \lambda \geq \begin{pmatrix} y_o \\ w_o \end{pmatrix} \qquad (5) \\ & 1^T \lambda = 1 \\ & \lambda \geq 0 \end{aligned}$$

For InvDEA model of this case, model (3) can be extended by adding the later constraint and updating the efficiency parameter of θ_o^{D**} as follows:

$$\begin{aligned} Min \quad & \Delta x \\ s.t. \quad & X\lambda \leq \theta_o^{D**}(x_o + \Delta x) \\ & \begin{pmatrix} Y \\ W \end{pmatrix} \lambda \geq \begin{pmatrix} y_o + \Delta y_o \\ w_o \end{pmatrix} \qquad (6) \\ & 1^T \lambda = 1 \\ & \lambda \geq 0 \end{aligned}$$

Where θ_o^{D**} is the optimal value of model (5).

Theorem 2. Assume only non-discretionary output and suppose that DMUo changes its output form (y_o, w_0) to $(y_o + \Delta y_o, w_o)$. If $(\overline{\lambda}, \overline{\Delta x})$ is a weak efficient solution of model (6), then the efficiency of all DMUs including DMUo , after the change of $(x_o + \overline{\Delta x}, y_o + \Delta y_o)$, stay unchanged.

Proof. It is straightforward by following proof of theorem 1 and some minor modifications which are omitted.

The presence of both non-discretionary inputs and non-discretionary output is assumed for the second case. Therefore, the following models may be used for efficiency evaluation of DMUo:

$$\begin{aligned} Min \quad & \theta \\ s.t. \quad & X\lambda \leq \theta x_o \\ & Z\lambda \leq z_o \\ & \begin{pmatrix} Y \\ W \end{pmatrix} \lambda \geq \begin{pmatrix} y_o \\ w_o \end{pmatrix} \qquad (7) \\ & 1^T \lambda = 1 \\ & \lambda \geq 0 \end{aligned}$$

Assuming θ_o^{ND**} as the optimal value of the above model, the associated InvDEA model to estimate the required inputs to preserve the efficiency scores can be developed as follows:

$$\begin{aligned} Min \quad & \Delta x \\ s.t. \quad & X\lambda \leq \theta_o^{ND**}(x_o + \Delta x) \\ & Z\lambda \leq z_o \\ & \begin{pmatrix} Y \\ W \end{pmatrix} \lambda \geq \begin{pmatrix} y_o + \Delta y_o \\ w_o \end{pmatrix} \qquad (8) \\ & 1^T \lambda = 1 \\ & \lambda \geq 0 \end{aligned}$$

Theorem 3. Assume both non-discretionary output and non-discretionary input and suppose that DMUo changes its output form (y_o, w_0) to $(y_o + \Delta y_o, w_o)$. If $(\bar{\lambda}, \overline{\Delta x})$ is a weak efficient solution of model (6), then the efficiency of all DMUs including DMUo after changes, that is $(x_o + \overline{\Delta x}, y_o + \Delta y_o)$, stays unchanged.

Proof. It is straightforward by following proof of theorem 1 and some minor modifications which are omitted.

Similar with the case that we have only non-discretionary inputs, we use the mutual relationship between DEA and multi-objective programming for dealing with the inverse models when we have non-discretionary factor in both input and output sides. Proposed models in the current article are based on the input oriented and cab be simply modifying to the output orientation and we do not mention associated models for the sake of briefing.

It is important to point out that in input-oriented models of (5) and (7) and their associated inverse models, constraints of discretionary and non-discretionary outputs are not separated. This is due to the fact that an input oriented model does not contract the output vector. Another extension of our proposed models is perturbs the input vector and estimates the expected output to preserve the efficiency score of DMUs. This is a straightforward task by changing the model from input to output orientation and perform the associated inverse models in presence of non-discretionary data.

The current section dealt with different types of non-discretionary factors in the process of input estimation while keeping efficiency scores unchanged in the InDEA models. The next section illustrates all proposed models in the current paper. Thus, regarding with the condition of the case study and existence of any type non-discretionary factors one may choose proper model to the analysis.

5. Numerical Example

The applicability of the proposed model is shown in this section. The following example in this section illustrates proposed models. Figure 2 summarize the computational procedure of the proposed models. 15 DMUs with 2 inputs and 2 outputs are chosen to be investigated. Table 1 presents the input and output values. It is aimed to change the value of inputs and outputs of DMUs to keep the efficiency of an investigated DMU to be unchanged. Four cases are considered to show the ability of the proposed models.

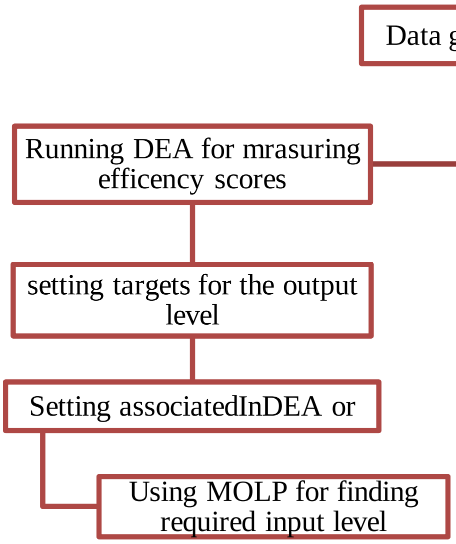

Figure 2: Research framework

Table 1 The input and output values of DMUs

DMUs	I_1	I_2	O_1	O_2
1	4	76	3	12
2	6	24	8	20
3	2	11	10	24
4	2	65	9	3
5	7	30	10	42
6	5.4	78	5	32
7	3.6	25	20	45
8	9	94	12.3	15
9	2.5	58	13.7	67.7
10	3	34	14.7	98
11	8.5	76	8.9	43
12	9	32	9.6	17
13	3	55	14	53.9
14	8	66	7.3	65
15	4.6	91	12	83

The first case includes the discretionary inputs and outputs. For this purpose, the first step is to calculate the efficiency of each DMU (θ^*) using model (1). The results are presented in the second column of Table 2. According to the definition of efficient DMU, only DMUs {3, 4, 7, 9, and 10} are efficient. In the next step, the outputs of the efficient DMU4 are changed to 12 and 8, respectively. Then we compute Δx, that is, the minimum required input level for producing the new level of output, while staying at the same level of efficiency. Please note that in this case that we have no non- discretionary factor, we use the traditional DEA for efficiency measurement and traditional InEA for input estimation given a perturbed output and unchanged efficiency score. Let's denote DMU4 after changing its input-output level by DMU_4'. Therefore, the inputs of DMU_4' corresponding to the achieved Δx converts to 2.32 and 13.8, respectively. Checking the efficiency score of new DMU shows that DMU_4' remains efficient. Therefore, we reevaluate the efficiencies of other DMUs ($\theta^{*\prime}$) in the presence of DMU_4' instead of DMU4 using model (1). The results are reported in the fourth column of Table 2. DMUs {3, 4, 7, 9, and 10} are still efficient and the efficiency score of other DMUs remain unchanged after perturbing DMU_4'. Let's assume the second input of this example is non-discretionary as a second case shown by Z. Model (2) is applied to compute the efficiencies of DMUs, θ^*_{ND}. The results are represented in the third column of Table 2. According to the definition of efficient DMU, only DMUs {3, 4, 7, 9, and 10} are efficient. Then, the outputs of DMU4 are changed to 12 and 8, respectively. Now the question is finding the required input level for DMUs that can produce new output level without changing the efficiency level. it is important to point out that this aim is not reachable neither by the classical DEA models nor by the existing inverse DEA models in the literature. However, yield to this aim by the proposed models in the previous section.

This time, the calculated Δx which only includes Δx_1 is zero since the DMU4 is efficient enough. Therefore, it is not necessary to increase or decrease its input by changing its outputs. We reevaluate the efficiencies of other DMUs in the presence of DMU_4' instead of DMU4 using model (2), $\theta^{*}_{ND}{}'$. The results are reported in the fifth column of Table 2. DMUs {3, 4, 7, 9, and 10} which are still efficient are shown in the fifth column of Table 2 and the efficiency score of other DMUs stay unchanged.

Table 2 Optimal efficiencies of DMUs

DMUs	θ^*	θ^*_{ND}	$\theta^{*\prime}$	$\theta^*_{ND}{}'$
1	0.50	0.50	0.50	0.50
2	0.46	0.33	0.46	0.33
3	1	1	1	1
4	1	1	1	1
5	0.55	0.32	0.55	0.32
6	0.39	0.39	0.39	0.39
7	1	1	1	1
8	0.26	0.26	0.26	0.26
9	1	1	1	1
10	1	1	1	1
11	0.26	0.26	0.26	0.26
12	0.34	0.22	0.34	0.22
13	0.86	0.85	0.86	0.85
14	0.36	0.31	0.36	0.31
15	0.60	0.60	0.60	0.60

We further assume that inputs are discretionary and the second outputs are non-discretionary. In the first place we need to choose a proper classical DEA model to find the efficiency score of DMUs in this case, that is, model (7). Secondly, we find required input level for producing a desired output level that may be non-discretionary with unchanged efficiency score that is gained in the previous step. As it was explained, the efficiency score of DMUs are obtained and shown in the second column of Table 3, $\theta^*_{ND-output}$. It is shown that only DMUs {3, 4, 7, 9, and 10} are efficient. The results are as same as the second column of Table 2. However, at least an optimal solution is available for the proposed model if and only if the new output vector is in the set of current PPS.

We increase the first output of DMU_4 to 14 and achieve its Δx using model (6). Therefore, the inputs of DMU_4' corresponding to the achieved Δx convert to 2.64 and 16.6, respectively.

Table 3 Optimal efficiencies of DMUs

DMUs	$\theta^{*}_{ND-output}$	$\theta^{*-new}_{ND-output}$	$\theta^{*}_{ND-input/output}$	$\theta^{*-new}_{ND-input/output}$
1	0.50	0.50	0.50	0.50
2	0.46	0.46	0.33	0.33
3	1	1	1	1
4	1	1	1	1
5	0.55	0.55	0.32	0.32
6	0.39	0.39	0.39	0.39
7	1	1	1	1
8	0.26	0.26	0.26	0.26
9	1	1	1	1
10	1	1	1	1
11	0.26	0.26	0.26	0.26
12	0.34	0.34	0.22	0.22
13	0.86	0.86	0.85	0.85
14	0.36	0.36	0.31	0.31
15	0.60	0.60	0.60	0.60

Checking the efficiency score of the new DMU shows that DMU_4' remains efficient. The results are obtained using model (5) by considering perturbing DMU as shown in the third column of Table 3, $\theta^{*-new}_{ND-output}$.

In the final case, we assume the second input and output of Table 1 are non-discretionary. Model (7) is applied to measure their efficiencies. The result of Model (7) are reported in the fourth column of Table 3. As it is shown, DMUs {3, 4, 7, 9, and 10} are efficient again. The First output of DMU4 in Table 1 is converted to 14. Model (8) is applied to find the change quantity of inputs to keep the efficiency of DMU4 unchanged. The corresponding Δx for both inputs is zero. Therefore, the inputs are as same as before equal to 2 and 65, respectively. As the result, this DMU is efficient enough and it is not necessary to change the inputs of the investigated DMU. Model 7 calculates the efficiencies of the DMUs when DMU4 is replaced with DMU_4'. (See the last column of Table 3). The results of the proposed models demonstrate that the efficiencies of the DMUs remain unchanged.

6. Obtained Results of a Real Case Study

The developed model is applied to the real world data of 23 public libraries in Tokyo to show the strength and applicability of the proposed methods. The L1-L23 symbols stand as the representative of the considered libraries adopted from Cooper (2004). In total, 4 inputs and 2 outputs are used. The inputs are floor area (unit = 1000 m2) [Area], the number of books (unit = 1000) [Book], staffs (unit = 1000) [Staff], and the population of the area (unit = 1000) [Population]. The outputs are the number of registered residents (unit = 1000) [Register] and the number of borrowed books (unit = 1000) [Borrow]. Table 4 shows the applied data set. The normal efficiency evaluation of library is reported in the second column of table 5. In this case, all inputs and outputs are considered to be controllable (discretionary). However, in evaluating the efficiency of a library, the population of the area is an important (input) factor which might be non-controllable. Model 2 is applied to calculate the efficiencies, θ^*_{ND} . The results are shown in the third column of Table 5. As it is shown, Libraries {1, 2, 5, 6, 9, 15, 17, 19 and 23} are efficient.

Table 4 Public library data

Library	I1: Area	I2: Book	I3: Staff	I4: Population	O1: Register	O2: Borrow
L1	2249	163,523	26	49,196	5561	105,321
L2	4617	338,671	30	78,599	18,106	314,682
L3	3873	281,655	51	176,381	16,498	542,349
L4	5541	400,993	78	189,397	30,810	847,872
L5	11,381	363,116	69	192,235	57,279	758,704
L6	10,086	541,658	114	194,091	66,137	1,438,746
L7	5435	508,141	61	228,535	35,295	839,597
L8	7524	338,804	74	238,691	33,188	540,821
L9	5077	511,467	84	267,385	65,391	1,562,274
L10	7029	393,815	68	277,402	41,197	978,117
L11	11121	509,682	96	330,609	47,032	930,437
L12	7072	527,457	92	332,609	56,064	1,345,185
L13	9348	601,594	127	356,504	69,536	1,164,801
L14	7781	528,799	96	365,844	37,467	1,348,588
L15	6235	394,158	77	389,894	57,727	1,100,779
L16	10,593	515,624	101	417,513	46,160	1,070,488
L17	10,866	566,708	118	503,914	102,967	1,707,645
L18	6500	467,617	74	517,318	47,236	1,223,026
L19	11,469	768,484	103	537,746	84,510	2,299,694
L20	10,868	669,996	107	590,601	69,576	1,901,465
L21	10,717	844,949	120	622,550	89,401	1,909,698
L22	19,716	1,258,981	242	660,164	97,941	3,055,193
L23	10,888	1,148,863	202	808,369	191,166	4,096,300

If the manager of library decides to expand the services of the library, that is, expanding the number of registered residents and the number of borrowed books, it is important to estimate the required resources for this expansion plan. The proposed models help the manager for this expansion planning. In mathematical terms, we investigate if the level of a certain output need to be increased and the efficiency of the considered DMU remain unchanged. Therefore, it is necessary to address the questions of how much additional resources (inputs) are required for the considered DMU.

The results provide useful information for managers for a proper decision making. For this purpose, assume managers expect to increase the level of the first and the second outputs (number of registered residents and number of borrowed books, respectively) of L2 to 18323 and 323573, respectively. Now, bearing in mind that the fourth input (population of the area) is not under control, it is necessary to address the question of how much input is required to make the new level of output while keeping the DMU's efficiency unchanged. The required Δx of inputs are calculated using model (4). Therefore, inputs of L2', that is, L2 after perturbing corresponding to achieved Δx converts to 4617, 338671, 30 and 78599, respectively. First of all, notice that L2 remains efficient.

Table 5 The efficiencies scores of libraries before and after changing inputs and outputs

DMUs	θ^*	θ^*_{ND}	$\theta^*_{ND}{}'$
L1	1	1	1
L2	1	1	1
L3	0.94	0.94	0.94
L4	0.85	0.85	0.85
L5	1	1	1
L6	1	1	1
L7	0.87	0.87	0.87
L8	0.82	0.82	0.82
L9	1	1	1
L10	0.94	0.94	0.94
L11	0.71	0.70	0.70
L12	0.87	0.87	0.87
L13	0.80	0.75	0.75
L14	0.87	0.86	0.86
L15	1	1	1
L16	0.76	0.75	0.75
L17	1	1	1
L18	0.94	0.94	0.94
L19	1	1	1
L20	0.91	0.91	0.91
L21	0.84	0.84	0.84
L22	0.86	0.70	0.70
L23	1	1	1

In order to track the consequence of perturbing L2 on the efficiency score of other libraries, we reevaluate the efficiencies ($\theta_{ND}^{*}{}'$) of other libraries in the presence of L2' instead of L2 using model (2) by applying the Δx which is achieved by model (4). The results are reported in the fourth column of Table 5. DMUs {L1, L2, L5, L6, L9, L15, L17, L19 and L23} are still efficient, as shown in fourth column of Table 2. Checking the efficiency score of other DMUs shows that their efficiency score also stays unchanged after perturbing L2.

7. Concluding Remarks

Performance measurement topic has been vastly investigated in previous literature. The topic is properly developed varying from cost-based performance assessment to global concerns of environment. As an important topic of performance measurement, there are many circumstances when the managers prefer to set the performance of their business to a specific score and determine the optimized input/outputs. In these cases, the classic DEA cannot be applied as their function is not designed to handle this problem. However, InvDEA which is a developed version of classic DEA is applicable. This study was completed in three linked phases to discuss the application of an integrated MOLP-DEA model in investigating the non-discretionary factors of performance measurement. To do so, firstly, an InvDEA model was developed to investigate the non-discretionary data of the considered problems. Following, to enrich the problem and make it more applicable for real world problems, the developed InvDEA model was linked with a MOLP. The integrated model was more investigated to check its applicability and validation. In this regard, a numerical example and a real case study data were applied to run the developed model. The obtained results were satisfactory with regard to both applicability and validation. There are numerous implications which are obtained from the outputs of this study. As an important implication for academic purposes, the researchers can apply the developed model of this research to assess the performance of any operating system with non-discretionary data. In addition to academic implications, the major implication of this research can be applied in practical and real world problems as the managers and practitioners can set the performance of their company or business to a fix value and determine the optimum values of input and outputs. Finally, the developed model and the research methodology can be beneficial for society as there are many areas where the researchers are interested in assessing the performance from social viewpoints. However, although the developed model is unique, there are some limitation which should be addressed in future research. For example, there are not enough data sets to be applied in these types of models which makes it hard to check their applicability from different perspectives. In addition, the application of fuzzy data can be further investigated in future research. In other words, other scholars can apply fuzzy data to develop the model of this research and compare the obtained results.

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