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Decomposition of efficiency in a network

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Abstract
Today, the network has an important role in a system. Determining the efficiency of the system as well as the efficiency of the sub-sections of a network helps to effectively manage the network. If the optimal value of the sub-sections is not unique, an efficiency range can be obtained for them. In this paper, the performance of a network is analysis with the help of data envelopment analysis and performance intervals are determined for each of the sub- sections of the network. With help decomposition the performance of a network structure, the performance of its components was obtained. The relationship between component performance and interval value was calculated with the overall efficiency network.

Keywords: Data Envelopment Analysis, Efficiency Decomposition, Efficiency Interval, Network.

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1. Introduction

Today, the network plays a very important role in a system. Network management coordinates and controls relevant activities so that customers can receive fast and decent services and products or quality at the lowest cost. Network management should manage the flow between different stages and in each stage only in the network so that you get the most profit.

Data Envelopment Analysis (DEA) is a method for measuring the relative efficiency of decision-making units (DMUs) consistent with multiple inputs and outputs. In classical DEA, the network was considered a black box, with an initial input and a final output to measure its performance, and the internal structure of the DMU is not taken into consideration. In network management, all possible efficiency plays an important role in achieving the dual goal of reducing costs and increasing profits.

Note that an independent decision maker at each stage of the network maximizes its technical efficiency regardless of other components and the network as a whole. For this reason, network models are useful models for modeling the overall network.

The non-parametric DEA method was first proposed by Charans for performance estimation, who introduced the CCR model for measuring multi-input and multi-output performance [1]. This model has under constant returns to scale (CRS) assumption. The BCC model was proposed by Banker and has variable returns to scale (VRS) assumption [2]. Over the years, the DEA method has become an efficient method for evaluating performance (see [3]). As discussed in many DEA studies, DMUs can have a two-stage structure that uses first-stage inputs to generate outputs, which become second-stage inputs. In the second stage, these inputs are used to product output. Seiford presented a two-step process for measuring the performance of US commercial banks, including 55 commercial banks, that aimed to measure the profitability and marketability of these banks [3]. In this paper, profitability was measured using manpower and assets as input and profit and income as output. In the second stage, the ability to supply the profit market and income as input and market value, returns and earnings per share were selected as output. However, they did not make any assumptions about the sequential relationship between the two stages. Kao used a two-step process for 24 non-life insurance companies in Taiwan. The usual method for two-step problems is to apply the standard DEA model separately for each step [4]. Kao applied the same method and considered each step in the two-step process as an operation independent of the other [4]. They modified the standard DEA model and considered sequential two-step relationships throughout the process. They used the following assumptions in their model:

a) is a constant return to scale (CRS),

b) The weights of the intermediate measures are the same for both stages.

Assumption (b) means that the weights of the outputs in the first stage are equal to the weights of the inputs in the second stage. The reason for using this assumption is to convert the original nonlinear programing to a linear programing.

Chen mentioned a model that can also be used for variable return to scale [5]. They mentioned the efficiency of the whole two-step process as the weighted average of the two separate steps. Wang [6] extended the model of Chen [5] by introducing relative weights for two separate steps.

There is a fact that the network usually has more than two components. Three-stage network or more. For example, the supplier-manufacturer-distributor network, which cannot be evaluated using a two-stage DEA. Tavana used the two-stage DEA model to measure performance in a three levels network including supplier-manufacturer-distributor and extended the algorithm to more than three stages [7]. Their model can be used under the assumption of constant returns to scale as well as variable scale. It can be used for comprehensive analysis of multilevel networks. They evaluated the performance of DMUs in the three-level network in three steps. In the first step, two-level sections were evaluated. They used the two-stage model of [6]. In the second step, they calculated the efficiency of the three-level network, and in the third step, they decided on the comprehensive performance of the entire three-level network. Because two-stage DEA models or linear programming models usually have multiple optimal solutions, in this paper we obtain the efficiency interval for each of the subsections based on the performance of the whole system. The structure of the paper is as follows: Section 2 describes the two-step DEA models, Section 3 presents the model, Section 4 provides numerical example, and finally, Section 5 contains the conclusion.

2. Description of two-stage DEA models

Suppose there are n decision-making units (DMUs) that each consume m inputs to product **s** output. We assign weights to each of the inputs and outputs. The following symbols are used for formulation.

DMU_j: Represents the j unit of the decision-making unit, $j=1,\ldots,n$

x_{ij}: Indicates the i-th input of the j-th decision-making unit in the first step, $i = 1,\ldots, m$, $j = 1,\ldots, n$

z_{dj}: represents the dth output of the j-th decision-making unit in the first stage and the d-th input of the j-th decision-making unit in the second stage, $d = 1,\ldots, D$, $j = 1,\ldots, n$

y_{rj}: represents the output of the rth of j the second decision-making unit in the second stage, $r = 1,\ldots, s$, $j = 1,\ldots, n$

v_i: Indicates the input weight of the i in the first stage, $i = 1,\ldots, m$

μ_d^1 : Indicates the output of d in the first stage $d = 1,\ldots, D$,

μ_d^2 : Indicates the input of d in the second stage $d = 1,\ldots, D$,

u_r: represents the output of r in the second step $r = 1,\ldots, s$,

In a two-step process, there are intermediate values between the two steps. In the first stage, inputs are used to product outputs that are considered as intermediate values. In the second stage, these intermediate values are used to product the final outputs. The basic assumption is that the outputs of the first stage are only the inputs of the second stage. Figure 1 shows a two-stage DEA model.

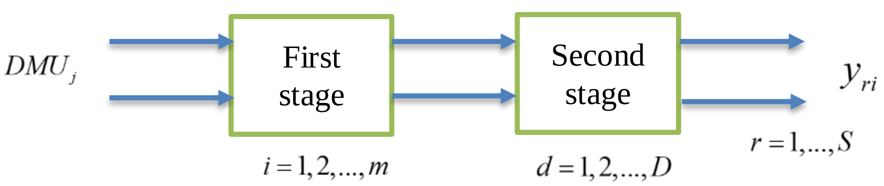

Figure 1. Two-stage DEA model

According to the serial relationship between the two phases, Kao expressed the total efficiency with constant returns to scale for DMUs as a product of the efficiency of the first and second phases, ie, as mentioned earlier, the assumption that they used for [4]. Their total performance model is as follows:

$$\begin{aligned} \theta_o^* &= \max \sum_{r=1}^{s} u_r y_{ro} \\ s.t. \quad & \sum_{i=1}^{m} v_i x_{io} = 1 \\ & \sum_{d=1}^{D} \mu_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} \mu_d z_{dj} \le 0 \\ & v_i \ge 0, u_r \ge 0, \mu_d \ge 0 \end{aligned} \quad (1)$$

If the total efficiency is calculated, the efficiency of one stage can be calculated and the efficiency of the other stage can be found using the above relation. For example $\theta_o^{2*} = \frac{\theta_o^*}{\theta_o^{1*}}$, the following LP models calculate the efficiency of stages one and two, respectively:

$$\begin{aligned} \theta_o^{1*} &= \max \sum_{d=1}^{D} \mu_d z_{dj} \\ s.t. \quad & \sum_{i=1}^{m} v_i x_{io} = 1 \\ & \theta_o^* = \sum_{r=1}^{s} u_r y_{ro} \\ & \sum_{d=1}^{D} \mu_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} \mu_d z_{dj} \le 0 \\ & v_i \ge 0, u_r \ge 0, \mu_d \ge 0 \end{aligned} \quad (2)$$

And

$$\begin{aligned} \theta_o^{2*} &= \max \sum_{r=1}^{s} u_r y_{ro} \\ s.t. \quad & \sum_{d=1}^{D} \mu_d z_{dj} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \theta_o^* \sum_{i=1}^{m} v_i x_{ij} = 0 \\ & \sum_{d=1}^{D} \mu_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} \mu_d z_{dj} \le 0 \\ & v_i \ge 0, u_r \ge 0, \mu_d \ge 0 \end{aligned} \quad (3)$$

Chen assigned a weight to each step to obtain total efficiency and showed that the two methods are equivalent under the assumption of constant returns to scale [5]. The advantage of Chen model is that it can also be used to variable returns to scale (VRS) assume [5].

Wang [6] developed the model of Chen [5] by assigning weights $\lambda_1 \ge 0, \lambda_1 \ge 0$ with $\lambda_1 + \lambda_2 = 1$ two stages one and two, respectively, and defining total efficiency as $\theta_o^* = \lambda_1 \theta_o^{1*} + \lambda_2 \theta_o^{2*}$. Their model can be used for both constant and variable returns to scale. Their models for calculating total efficiency are the efficiency of the first and second stages, assuming CRS as follows:

$$\begin{aligned} \theta_o^* &= \max \lambda_1 \sum_{d=1}^{D} \mu_d z_{do} + \lambda_2 \sum_{r=1}^{s} u_r y_{ro} \\ s.t. \quad & \lambda_1 \sum_{i=1}^{m} v_i x_{io} + \lambda_2 \sum_{d=1}^{D} \mu_d z_{do} = 1 \\ & \sum_{d=1}^{D} \mu_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} \mu_d z_{dj} \le 0 \\ & v_i \ge 0, u_r \ge 0, \mu_d \ge 0 \end{aligned} \quad (4)$$

And

$$\begin{aligned} \theta_o^{1*} &= \max \sum_{d=1}^{D} \mu_d z_{do} \\ s.t. \quad & \sum_{i=1}^{m} v_i x_{io} = 1 \\ & (\lambda_1 - \lambda_2 \theta_o^*) \sum_{d=1}^{D} \mu_d z_{do} + \lambda_2 \sum_{r=1}^{s} u_r y_{ro} = \lambda_1 \theta_o^* \\ & \sum_{d=1}^{D} \mu_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} \mu_d z_{dj} \le 0 \\ & v_i \ge 0, u_r \ge 0, \mu_d \ge 0 \end{aligned} \quad (5)$$

$$\begin{aligned}
\theta_o^{2*} &= \max \sum_{r=1}^{s} u_r y_{ro} \\
s.t.\quad & \sum_{d=1}^{D} \mu_d z_{do} = 1 \\
& \lambda_1 \sum_{r=1}^{s} u_r y_{ro} - \lambda_1 \theta_o^* \sum_{i=1}^{m} v_i x_{io} = \lambda_2 \theta_o^* - \lambda_1 \\
& \sum_{d=1}^{D} \mu_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} \mu_d z_{dj} \le 0 \\
& v_i \ge 0, u_r \ge 0, \mu_d \ge 0
\end{aligned} \tag{6}$$

To the variable returns to scale assume, the models are converted as follows:

$$\begin{aligned}
\theta_o^* &= \max \lambda_1 (\sum_{d=1}^{D} \mu_d z_{do} + \sigma_1) + \lambda_2 (\sum_{r=1}^{s} u_r y_{ro} + \sigma_2) \\
s.t.\quad & \lambda_1 \sum_{i=1}^{m} v_i x_{io} + \lambda_2 \sum_{d=1}^{D} \mu_d z_{do} = 1 \\
& \sum_{d=1}^{D} \mu_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} \mu_d z_{dj} \le 0 \\
& v_i \ge 0, u_r \ge 0, \mu_d \ge 0
\end{aligned} \tag{7}$$

And

$$\begin{aligned}
\theta_o^{1*} &= \max \sum_{d=1}^{D} \mu_d z_{do} + \sigma_1 \\
s.t.\quad & \sum_{i=1}^{m} v_i x_{io} = 1 \\
& (\lambda_1 - \lambda_2 \theta_o^*) \sum_{d=1}^{D} \mu_d z_{do} + \lambda_1 \sigma_1 \\
& + \lambda_2 \sum_{r=1}^{s} u_r y_{ro} + \lambda_2 \sigma_2 = \lambda_1 \theta_o^* \\
& \sum_{d=1}^{D} \mu_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} \mu_d z_{dj} \le 0 \\
& v_i \ge 0, u_r \ge 0, \mu_d \ge 0
\end{aligned} \tag{8}$$

$$\begin{aligned}
\theta_o^{2*} &= \max \sum_{r=1}^{s} u_r y_{ro} + \sigma_2 \\
s.t.\quad & \sum_{d=1}^{D} \mu_d z_{do} = 1 \\
& \lambda_1 \sum_{r=1}^{s} u_r y_{ro} + \lambda_2 \sigma_2 - \lambda_1 \theta_o^* \sum_{i=1}^{m} v_i x_{io} \\
& + \lambda_1 \sigma_1 = \lambda_2 \theta_o^* - \lambda_1 \\
& \sum_{d=1}^{D} \mu_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} \mu_d z_{dj} \le 0 \\
& v_i \ge 0, u_r \ge 0, \mu_d \ge 0
\end{aligned} \tag{9}$$

Tavana [7] measured the three-levels chain operation using the two-step DEA idea. Based on Wang and Chen (2010) model, they presented the following model for total efficiency for CRS.

$$\begin{aligned}
(\theta_o^*)_{S_2} &= \max \left[\lambda_1 \left(\sum_{d=1}^{D} \mu_d z_{do} + \sum_{t=1}^{T} \alpha_t w_{to} \right) + \lambda_2 \left(\sum_{r=1}^{s} u_r y_{ro} + \sum_{t=1}^{T} \alpha_t w_{to} \right) \right] \\
s.t.\quad & \lambda_1 \left(\sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} \mu_d z_{do} \right) + \lambda_2 \left(\sum_{r=1}^{s} u_r y_{ro} + \sum_{t=1}^{T} \alpha_t w_{to} \right) = 1 \\
& \left(\sum_{d=1}^{D} \mu_d z_{dj} + \sum_{t=1}^{T} \alpha_t w_{tj} \right) - \left(\sum_{i=1}^{m} v_i x_{ij} + \sum_{d=1}^{D} \mu_d z_{dj} \right) \le 0 \\
& \left(\sum_{r=1}^{s} u_r y_{rj} + \sum_{t=1}^{T} \alpha_t w_{tj} \right) - \left(\sum_{d=1}^{D} \mu_d z_{dj} + \sum_{t=1}^{T} \alpha_t w_{tj} \right) \le 0 \\
& \alpha_t \ge 0, v_i \ge 0, \mu_d \ge 0
\end{aligned} \tag{10}$$

They used other Wang and Chen equations to obtain the efficiency of each step. For VRS assumption, they used the following model to calculate the performance of the whole system.

$$\begin{aligned}
(\theta_o^*)_{S_2} &= \max \left[\begin{array}{l} \lambda_1 \left(\sum_{d=1}^{D} \mu_d z_{do} + \sigma_1 + \sum_{t=1}^{T} \alpha_t w_{to} + \sigma_2 \right) \\ + \lambda_2 \left(\sum_{r=1}^{s} u_r y_{ro} + \sigma_3 + \sum_{t=1}^{T} \alpha_t w_{to} + \sigma_4 \right) \end{array} \right] \\
s.t.\quad & \lambda_1 \left(\sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} \mu_d z_{do} \right) + \lambda_2 \left(\sum_{r=1}^{s} u_r y_{ro} + \sum_{t=1}^{T} \alpha_t w_{to} \right) = 1 \\
& \left(\sum_{d=1}^{D} \mu_d z_{dj} + \sum_{t=1}^{T} \alpha_t w_{tj} \right) - \left(\sum_{i=1}^{m} v_i x_{ij} + \sum_{d=1}^{D} \mu_d z_{dj} \right) \le 0 \\
& \left(\sum_{r=1}^{s} u_r y_{rj} + \sum_{t=1}^{T} \alpha_t w_{tj} \right) - \left(\sum_{d=1}^{D} \mu_d z_{dj} + \sum_{t=1}^{T} \alpha_t w_{tj} \right) \le 0 \\
& \alpha_t \ge 0, v_i \ge 0, \mu_d \ge 0
\end{aligned}$$

3. The model presented

Consider a three-step network presented in Figure 2.

Figure 2. A three-stage network

First, we state and prove the following theorem.

Theorem: If there are two fractions $\frac{a}{b}$ and $\frac{c}{d}$ where $a,b,c,d>0$, then $\frac{a+c}{b+d}=\lambda\frac{a}{b}+(1-\lambda)\frac{c}{d}$ where $0<\lambda<1$.

Proof: From $\frac{a+c}{b+d}=\lambda\frac{a}{b}+(1-\lambda)\frac{c}{d}$ we have $\frac{a+c}{b+d}=\frac{\lambda ab+(1-\lambda)bc}{bd}$ We get from this

$$abd+cbd=\lambda abd+\lambda ad^2+$$

$$cbd+cb^2-\lambda cb^2-\lambda cbd$$

So with assume $ad\neq bc$ we have

$$\lambda=\frac{b(ad-bc)}{(ad-bc)(b+d)}=\frac{b}{b+d} \qquad (*)$$

Obvious $\lambda\leq 1$

According to Figure 1:

$$\theta^{12}=\frac{\eta z_p+\alpha w_p}{vx_p+\eta z_p}\ ,\ \theta^{23}=\frac{uy_p+\alpha w_p}{\alpha w_p+\eta z_p}$$

$$\theta^{1}=\frac{\eta z_p}{vx_p}\ ,\ \theta^{2}=\frac{\alpha w_p}{\eta z_p}\ ,\ \theta^{3}=\frac{uy_p}{\alpha w_p}$$

θ^{12} is the efficiency of the first and second stages and θ^{23} shows the efficiency of the second and third stages together. θ^{12} and θ^{23} are convex θ^1,θ^2 and θ^2,θ^3 , respectively. that's mean

$\theta^{12}=\varsigma\theta^1+(1-\varsigma)\theta^2$ that $0\leq\varsigma\leq 1$,

$\theta^{23}=\delta\theta^2+(1-\delta)\theta^3$ that $0\leq\delta\leq 1$

If θ^{12} and θ^{23} are unique, then the desired answer is in hand and

$$\theta^{a}=\mu\theta^{12}+(1-\mu)\theta^{23},\quad 0\leq\mu\leq 1 \qquad (1-1)$$

If θ^{12} and θ^{23} are not unique, so they can all be changed in one interval. That is

$$\theta^{23}\in[\theta^{23l},\theta^{23u}],\theta^{12}\in[\theta^{12l},\theta^{12u}]$$

we put $\theta_o^{\alpha^*}=\alpha^*$ which is obtained from the following formula:

$$\alpha_o^* = \max\theta^a = \max\left[\lambda_1\left(\sum_{d=1}^{D}\mu_d z_{do} + \sum_{t=1}^{T}\alpha_t w_{to}\right) + \lambda_2\left(\sum_{r=1}^{s}u_r y_{ro} + \sum_{t=1}^{T}\alpha_t w_{to}\right)\right]$$

$$\begin{aligned}
s.t.\quad & \lambda_1\left(\sum_{i=1}^{m}v_i x_{io} + \sum_{d=1}^{D}\mu_d z_{do}\right) + \lambda_2\left(\sum_{d=1}^{D}\mu_d z_{do} + \sum_{t=1}^{T}\alpha_t w_{to}\right) = 1 \\
& \left(\sum_{d=1}^{D}\mu_d z_{dj} + \sum_{t=1}^{T}\alpha_t w_{tj}\right) - \left(\sum_{i=1}^{m}v_i x_{ij} + \sum_{d=1}^{D}\mu_d z_{dj}\right) \le 0 \qquad (12) \\
& \left(\sum_{r=1}^{s}u_r y_{rj} + \sum_{t=1}^{T}\alpha_t w_{tj}\right) - \left(\sum_{d=1}^{D}\mu_d z_{dj} + \sum_{t=1}^{T}\alpha_t w_{tj}\right) \le 0 \\
& \sum_{d=1}^{D}\mu_d z_{dj} - \sum_{i=1}^{m}v_i x_{ij} \le 0 \\
& \sum_{r=1}^{s}u_r y_{rj} - \sum_{d=1}^{D}\mu_d z_{dj} \le 0 \\
& \alpha_t \ge 0, v_i \ge 0, u_r \ge 0, \mu_d \ge 0 \ \forall t,i,r,d
\end{aligned}$$

θ^a is unique, so θ^{12} when is minimum, where θ^{23} is maximum, and also θ^{23} when is minimum, where θ^{12} is maximum. In this case, we find the maximum θ^{12u} with the following model.

$$\theta^{12u} = \max\frac{\eta z_o + \alpha w_o}{v x_o + \eta z_o}$$

$$\begin{aligned}
s.t.\quad & \lambda_1\left(\sum_{i=1}^{m}v_i x_{io} + \sum_{d=1}^{D}\mu_d z_{do}\right) + \lambda_2\left(\sum_{d=1}^{D}\mu_d z_{do} + \sum_{t=1}^{T}\alpha_t w_{to}\right) = 1 \\
& \lambda_1\left(\sum_{t=1}^{T}\alpha_t w_{to} + \sum_{d=1}^{D}\mu_d z_{do}\right) + \lambda_2\left(\sum_{r=1}^{s}u_r y_{rj} + \sum_{t=1}^{T}\alpha_t w_{to}\right) = \alpha_o^* \\
& \left(\sum_{d=1}^{D}\mu_d z_{dj} + \sum_{t=1}^{T}\alpha_t w_{tj}\right) - \left(\sum_{i=1}^{m}v_i x_{ij} + \sum_{d=1}^{D}\mu_d z_{dj}\right) \le 0 \qquad (13) \\
& \left(\sum_{r=1}^{s}u_r y_{rj} + \sum_{t=1}^{T}\alpha_t w_{tj}\right) - \left(\sum_{d=1}^{D}\mu_d z_{dj} + \sum_{t=1}^{T}\alpha_t w_{tj}\right) \le 0 \\
& \sum_{d=1}^{D}\mu_d z_{dj} - \sum_{i=1}^{m}v_i x_{ij} \le 0 \\
& \sum_{r=1}^{s}u_r y_{rj} - \sum_{d=1}^{D}\mu_d z_{dj} \le 0 \\
& \alpha_t \ge 0, v_i \ge 0, u_r \ge 0, \mu_d \ge 0 \ \forall t,i,r,d
\end{aligned}$$

Having the optimal answer of model (2), $(\eta^*, \lambda^*, u^*, v^*, \alpha^*)$ and with the help of the stated theorem, the value μ^* in relation (1-1) is obtained. So we have

$$\theta^a = \mu\theta^{12u} + (1-\mu)\theta^{23l}$$

So

$$\theta^{23l} = \frac{\theta^a - \mu^*\theta^{12u}}{1-\mu^*}$$

Using the following model, we find θ^{23u}.

$$\theta^{23u} = \max\frac{u y_o + \alpha w_o}{\alpha w_o + \eta z_o}$$

$$\begin{aligned}
s.t.\quad & \lambda_1\left(\sum_{i=1}^{m}v_i x_{io} + \sum_{d=1}^{D}\mu_d z_{do}\right) + \lambda_2\left(\sum_{r=1}^{s}u_r y_{ro} + \sum_{t=1}^{T}\alpha_t w_{to}\right) = 1 \\
& \lambda_1\left(\sum_{t=1}^{T}\alpha_t w_{to} + \sum_{d=1}^{D}\mu_d z_{do}\right) + \lambda_2\left(\sum_{r=1}^{s}u_r y_{rj} + \sum_{t=1}^{T}\alpha_t w_{to}\right) = \alpha_o^* \\
& \left(\sum_{d=1}^{D}\mu_d z_{dj} + \sum_{t=1}^{T}\alpha_t w_{tj}\right) - \left(\sum_{i=1}^{m}v_i x_{ij} + \sum_{d=1}^{D}\mu_d z_{dj}\right) \le 0 \qquad (14) \\
& \left(\sum_{r=1}^{s}u_r y_{rj} + \sum_{t=1}^{T}\alpha_t w_{tj}\right) - \left(\sum_{d=1}^{D}\mu_d z_{dj} + \sum_{t=1}^{T}\alpha_t w_{tj}\right) \le 0 \\
& \sum_{d=1}^{D}\mu_d z_{dj} - \sum_{i=1}^{m}v_i x_{ij} \le 0 \\
& \sum_{r=1}^{s}u_r y_{rj} - \sum_{d=1}^{D}\mu_d z_{dj} \le 0 \\
& \alpha_t \ge 0, v_i \ge 0, u_r \ge 0, \mu_d \ge 0 \ \forall t,i,r,d
\end{aligned}$$

By solving model (3), the optimal answer $(\eta^*, \lambda^*, u^*, v^*, \alpha^*)$ is obtained. Again, with the help of the stated theorem, the value μ^* in relation (1-1) is obtained. So we have

$$\theta^a = \mu\theta^{12l} + (1-\mu)\theta^{23u}$$

Therefore

$$\theta^{12l} = \frac{\theta^a - \mu^*\theta^{23u}}{1-\mu^*}$$

$$\theta^{23} \in [\theta^{23l}, \theta^{23u}], \theta^{12} \in [\theta^{12l}, \theta^{12u}]$$

.Therefore intervals $[\theta^{12l}, \theta^{12u}]$ and $[\theta^{23l}, \theta^{23u}]$ are obtained. But because $\theta^{12} = \varsigma\theta^1 + (1-\varsigma)\theta^2$ where $0 \le \varsigma \le 1$,

$\theta^{23} = \delta\theta^2 + (1-\delta)\theta^3$ where $0 \le \delta \le 1$. We need to find the intervals $[\theta^{11}, \theta^{1u}]$, $[\theta^{21}, \theta^{2u}]$ and $[\theta^{31}, \theta^{3u}]$. For this purpose, we consider the following two models:

$$\begin{aligned}
&\beta_1^* = \max \frac{\eta z_o}{v x_o} \\
&s.t.\ \frac{\eta z_o + \alpha w_o}{v x_o + \eta z_o} = \theta^{12l^*} \\
&\quad \lambda_1\left(\sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} \mu_d z_{do}\right) + \lambda_2\left(\sum_{r=1}^{s} u_r y_{ro} + \sum_{t=1}^{T} \alpha_t w_{to}\right) = 1 \\
&\quad \lambda_1\left(\sum_{t=1}^{T} \alpha_t w_{to} + \sum_{d=1}^{D} \mu_d z_{do}\right) + \lambda_2\left(\sum_{r=1}^{s} u_r y_{rj} + \sum_{t=1}^{T} \alpha_t w_{to}\right) = \alpha_o^* \\
&\quad \left(\sum_{d=1}^{D} \mu_d z_{dj} + \sum_{t=1}^{T} \alpha_t w_{tj}\right) - \left(\sum_{i=1}^{m} v_i x_{ij} + \sum_{d=1}^{D} \mu_d z_{dj}\right) \le 0 \\
&\quad \left(\sum_{r=1}^{s} u_r y_{rj} + \sum_{t=1}^{T} \alpha_t w_{tj}\right) - \left(\sum_{d=1}^{D} \mu_d z_{dj} + \sum_{t=1}^{T} \alpha_t w_{tj}\right) \le 0 \\
&\quad \sum_{d=1}^{D} \mu_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \\
&\quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} \mu_d z_{dj} \le 0 \\
&\quad \alpha_t \ge 0, v_i \ge 0, u_r \ge 0, \mu_d \ge 0\ \forall t,i,r,d
\end{aligned} \tag{15}$$

And

$$\begin{aligned}
&\beta_2^* = \max \frac{\eta z_o}{v x_o} \\
&s.t.\ \frac{\eta z_o + \alpha w_o}{v x_o + \eta z_o} = \theta^{12u^*} \\
&\quad \lambda_1\left(\sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} \mu_d z_{do}\right) + \lambda_2\left(\sum_{r=1}^{s} u_r y_{ro} + \sum_{t=1}^{T} \alpha_t w_{to}\right) = 1 \\
&\quad \lambda_1\left(\sum_{t=1}^{T} \alpha_t w_{to} + \sum_{d=1}^{D} \mu_d z_{do}\right) + \lambda_2\left(\sum_{r=1}^{s} u_r y_{rj} + \sum_{t=1}^{T} \alpha_t w_{to}\right) = \alpha_o^* \\
&\quad \left(\sum_{d=1}^{D} \mu_d z_{dj} + \sum_{t=1}^{T} \alpha_t w_{tj}\right) - \left(\sum_{i=1}^{m} v_i x_{ij} + \sum_{d=1}^{D} \mu_d z_{dj}\right) \le 0 \\
&\quad \left(\sum_{r=1}^{s} u_r y_{rj} + \sum_{t=1}^{T} \alpha_t w_{tj}\right) - \left(\sum_{d=1}^{D} \mu_d z_{dj} + \sum_{t=1}^{T} \alpha_t w_{tj}\right) \le 0 \\
&\quad \sum_{d=1}^{D} \mu_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \\
&\quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} \mu_d z_{dj} \le 0 \\
&\quad \alpha_t \ge 0, v_i \ge 0, u_r \ge 0, \mu_d \ge 0\ \forall t,i,r,d
\end{aligned} \tag{16}$$

we put $\theta^{1u} = \beta = \max\{\beta_1^*, \beta_2^*\}$. By placing the optimal answer $(\eta^*, \lambda^*, u^*, v^*, \alpha^*)$ related to θ^{1u} and with the help of the stated theorem, the value γ^* in relation (1-1) is obtained. So we have $\theta^{12} = \gamma^*\beta + (1-\gamma^*)\theta^2$. Because θ^{12} is interval, that is $\theta^{12} = [a, b]$, therefore

$$[a,b] = \gamma^*\beta + (1-\gamma^*)\theta^2$$

then

$$\theta^2 = \frac{[a,b] - \gamma^*\beta}{1-\gamma^*} = [a', b'].$$

Because the difference of an interval from a fixed number is an interval, and also the division of an interval by a fixed number is an interval. Then

$$\begin{aligned}
&P_1^* = \max \frac{\alpha w_o}{\eta z_o} \\
&s.t.\ \frac{\eta z_o + \alpha w_o}{v x_o + \eta z_o} = \theta^{12l^*} \\
&\quad \lambda_1\left(\sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} \mu_d z_{do}\right) + \lambda_2\left(\sum_{r=1}^{s} u_r y_{ro} + \sum_{t=1}^{T} \alpha_t w_{to}\right) = 1 \\
&\quad \lambda_1\left(\sum_{t=1}^{T} \alpha_t w_{to} + \sum_{d=1}^{D} \mu_d z_{do}\right) + \lambda_2\left(\sum_{r=1}^{s} u_r y_{rj} + \sum_{t=1}^{T} \alpha_t w_{to}\right) = \alpha_o^* \\
&\quad \left(\sum_{d=1}^{D} \mu_d z_{dj} + \sum_{t=1}^{T} \alpha_t w_{tj}\right) - \left(\sum_{i=1}^{m} v_i x_{ij} + \sum_{d=1}^{D} \mu_d z_{dj}\right) \le 0 \\
&\quad \left(\sum_{r=1}^{s} u_r y_{rj} + \sum_{t=1}^{T} \alpha_t w_{tj}\right) - \left(\sum_{d=1}^{D} \mu_d z_{dj} + \sum_{t=1}^{T} \alpha_t w_{tj}\right) \le 0 \\
&\quad \sum_{d=1}^{D} \mu_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \\
&\quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} \mu_d z_{dj} \le 0 \\
&\quad \alpha_t \ge 0, v_i \ge 0, u_r \ge 0, \mu_d \ge 0\ \forall t,i,r,d
\end{aligned} \tag{17}$$

$$P_2^* = \max \frac{\alpha w_o}{\eta z_o}$$

$$s.t.\ \frac{\eta z_o + \alpha w_o}{v x_o + \eta z_o} = \theta^{12u*} \qquad (18)$$

$$\lambda_1\left(\sum_{i=1}^{m} v_i x_{io} + \sum_{d=1}^{D} \mu_d z_{do}\right) + \lambda_2\left(\sum_{r=1}^{s} u_r y_{ro} + \sum_{t=1}^{T} \alpha_t w_{to}\right) = 1$$

$$\lambda_1\left(\sum_{t=1}^{T} \alpha_t w_{to} + \sum_{d=1}^{D} \mu_d z_{do}\right) + \lambda_2\left(\sum_{r=1}^{s} u_r y_{rj} + \sum_{t=1}^{T} \alpha_t w_{to}\right) = \alpha_o^*$$

$$\left(\sum_{d=1}^{D} \mu_d z_{dj} + \sum_{t=1}^{T} \alpha_t w_{tj}\right) - \left(\sum_{i=1}^{m} v_i x_{ij} + \sum_{d=1}^{D} \mu_d z_{dj}\right) \le 0$$

$$\left(\sum_{r=1}^{s} u_r y_{rj} + \sum_{t=1}^{T} \alpha_t w_{tj}\right) - \left(\sum_{d=1}^{D} \mu_d z_{dj} + \sum_{t=1}^{T} \alpha_t w_{tj}\right) \le 0$$

$$\sum_{d=1}^{D} \mu_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \le 0$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} \mu_d z_{dj} \le 0$$

$$\alpha_t \ge 0, v_i \ge 0, u_r \ge 0, \mu_d \ge 0 \ \forall t, i, r, d$$

We put $\theta^{2u} = \rho^* = \max\{\rho_1^*, \rho_2^*\}$. By placing the optimal answer $(\eta^*, \lambda^*, u^*, v^*, \alpha^*)$ related to θ^{2u} and with the help of the stated theorem, the value $\gamma^{*\prime}$ in relation (1-1) is obtained. So we have $\theta^{12} = \gamma^{*\prime}\theta^{2u} + (1-\gamma^{*\prime})\theta^1$. Similar to the above mode for θ^1 we have $\theta^1 = [a'', b'']$. Because there is no guarantee that a' is less than ρ_1^* and ρ_2^* , so we put $\theta^{21} = \min\{a', \rho_1^*, \rho_2^*\}$. The same reason, we have $\theta^{11} = \min\{a'', \beta_1^*, \beta_2^*\}$. The rest of the Intended intervals can be found in the same way.

4. Practical example

An example is taken from the paper [7]. Assume that the inputs and outputs of each network component in a cement company are as shown in Figure 2 below.

Suppliers' inputs include capital (million dollars), cooperation experience (years) and shipping costs (one hundred thousand dollars). Suppliers' output, which is considered as producer input, includes on-time delivery (٪) and technology level (٪). Finally, it is assumed that manufacturers deliver the amount of order and inventory to distributors for profit. The data are presented in the following tables.

Table 1. Input values

DMU	capital	Collaboration experience	shipping costs	on-time delivery	technology level
1	14	3	9	90	86
2	12	2	14	63	75
3	10	3	32	86	73
4	1.6	2	15	75	83
5	10	2	25	69	90
6	7	3	52	78	84
7	3	3	37	82	90

Table2. output values

DMU	profit	on-time delivery	technology level	inventory	amount of order
1	14	3	9	90	86
2	12	2	14	63	75
3	10	3	32	86	73
4	1.6	2	15	75	83
5	10	2	25	69	90
6	7	3	52	78	84
7	3	3	37	82	90

The example for $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{1}{2}$ was solved using GAMS software, the results of which are shown in Table 3.

Table 3. Efficiency results

DMUs	θ^a	θ_{12}^u	θ_{23}^u	θ_{12}^l	θ_{23}^l	θ_1^u	θ_1^l	θ_{122}^u	θ_{122}^l	θ_{232}^u	θ_{232}^l	θ_3^u	θ_3^l
DMU1	0.9999	1.0000	0.9997	0.9999	0.9999	0.5000	0.5000	0.9980	0.9980	0.9980	0.9980	0.5151	0.5150
DMU2	1.0000	1.0000	1.0000	1.0000	1.0000	0.5000	0.5000	0.9985	0.9985	0.9985	0.9985	0.5400	0.5400
DMU3	0.9781	0.9752	0.9779	0.9781	0.9788	0.4253	0.4253	0.9557	0.9557	0.9765	0.9765	0.3924	0.3920
DMU4	0.9093	0.8385	0.9086	0.9093	0.9113	0.0248	0.0248	0.8330	0.8330	0.9082	0.9082	0.0309	0.0299
DMU5	0.9998	1.0000	0.9997	0.9998	0.9998	0.5000	0.5000	0.9982	0.9982	0.9982	0.9982	0.4357	0.4356
DMU6	0.9997	1.0000	0.9994	0.9997	0.9997	0.4535	0.4535	0.9982	0.9982	0.9982	0.9982	0.1830	0.1823
DMU7	0.9995	1.0000	0.9989	0.9995	0.9995	0.2998	0.2998	0.9981	0.9981	0.9981	0.9981	0.0698	0.0694

Note that in order to obtain θ_{12}^l based on the relation (*), first μ is found and then calculated. Based on the above table, the desired intervals can be obtained.

5. Conclusion

The network plays an important role in life today. Determining the efficiency range of the whole system and the efficiency of its sub-components will make the network management less likely to make decisions and determine the organization's policy. When there is no unique answer for the sub-divisions of a network, it is important to determine the efficiency interval in which the value of the objective function of that step is located. In this paper, using data envelopment analysis models for each sub- divisions of a network, we obtain efficiency intervals, which in fact, the efficiency of the whole system and the efficiency of each sub-division are broken down into efficiency based intervals. In this paper, a three-stage network was considered. The logic of this article can be extended to networks with any number of stages.

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