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Ranking Decision Making Units with the Ideal and Anti-Ideal Points

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Abstract

This paper introduces two virtual Decision Making Units (DMUs) called ideal point and anti-ideal point, then calculates distances of each DMU to the ideal and anti-ideal point. The two distinctive distances are combined to form a comprehensive index called the relative closeness (RC) just like the TOPSIS approach. The RC index is used as an overall ranking for all the DMUs. Then, this method compares with AP [1], Wang et al. [2], and Wu [3] methods. The proposed method is more simple and better than other methods and also it doesn't have drawbacks of the previous ranking methods.

Keywords: Ideal point, Anti-ideal point, TOPSIS, the relative closeness (RC).

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1. Introduction

Data Envelopment Analysis (DEA) proposed by Charnes et al. [4] (CCR model) and developed by Banker et al. [5] (BCC model) is an approach for evaluating the efficiencies of Decision Making Unit (DMU). The results of DEA models are an efficiency score equal to one to efficient DMUs and less than one to inefficient DMUs.

DEA efficient DMUs are always believed to perform better than DEA inefficient DMUs. If a DEA efficient DMU, however, also has a poorer relative efficiency than a DEA inefficient DMU when they are both evaluated from the position of the worst possible relative efficiency, can we still say that the DEA efficient DMU performs better than the DEA inefficient DMU? In this situation, the conclusion is clearly uncertain. Therefore, there is a clear need to combine the best and the worst possible relative efficiencies to give a global evaluation of each DMU.

Entani et al. [6] considered DEA efficiencies from both the optimistic and the pessimistic viewpoints. In their DEA models, the worst and the best possible relative efficiencies were utilized to constitute an interval. Their model for the computation of the worst possible relative efficiency, however, has a deadly drawback that it lost some information on inputs and outputs because only one input and one output data of the DMU under evaluation were effectively utilized and all the other input and output data did not work.

Doyle et al. [7] and Entani et al. [6] considered DEA efficiencies from both the optimistic and the pessimistic points of view. Their models have similar structures and the computation of the worst possible relative efficiency have some significant drawbacks. Wang et al. [2] reconsider their models and have been found three main drawbacks. They conclude that the upper bound model of Doyle et al. [7] and Entani et al. [6] cannot reasonably measure the

worst relative efficiencies of DMUs and cannot determine the inefficiency frontier. So, they developed a new DEA model with the constraint of the upper and lower bounds on efficiency. Their Bounded DEA model measures the performances of DMUs within the range of an interval and thus can effectively make the most of all the input and output data to measure both the best and the worst relative efficiencies of DMUs. It can therefore identify both the efficiency and inefficiency frontiers.

Wang et al. [8] evaluated DEA efficiency problems by a different way. They introduced two virtual DMUs, the ideal DMU (IDMU) and the anti-ideal DMU (ADMU) into DEA model. The two virtual DMUs are used to create two DEA models for the calculation of the best possible and the worst possible relative efficiencies. The two distinctive efficiencies are integrated using the well-known TOPSIS approach in multiple attribute decision making (MADM) to generate a composite index called the relative closeness (RC) to the IDMU. The RC index will be used as the overall assessment of each DMU, based on which a complete ranking for all the DMUs can be produced very simply.

Wu [3] focused on the efficiencies and ranking method of Wang and Luo [8]. Then, he discovered that their method is problematic in employing the negative ideal point (NIP) for DEA computation. TOPSIS is based on the idea that alternatives should be selected that have the shortest distance from the positive ideal solution (PIS) and the farthest distance from the negative ideal solution (NIS). The PIS has the best measures over all points, while the NIS has the worst measures over all points. Wang and Luo [8] used the TOPSIS idea for a full ranking of DMUs. But the two DEA models using the positive ideal point (PIP) and the negative ideal point (NIP) are logically conflicted because the PIP measures rely on an input orientation while the NIP measures rely on an output orientation.

Hence, this is logically inconsistent if two different efficiency results are aggregated into the relative closeness for ranking, as proposed in [8]. Recognizing this problem, Wu [3] slightly revised the approach to determine the worst efficiency of the NIP so that it was logically acceptable for using both the PIP and the NIP.

In this paper, two virtual DMUs, IDMU (an ideal DMU) and ADMU (an anti-ideal DMU) will be introduced into our model, and then we will propose a ranking method for all DMUs by IDMU, ADMU and TOPSIS approach. We try that the proposed ranking technique doesn't have problems from the above articles' drawbacks.

We begin in the following section with an explanation about how to measure the best and worst performance of each DMU. Section 3 provides a proposed ranking method. Section 4 gives an example for compare our method with AP [8], Wang et al. [6] and Wu [7] methods. Finally, our conclusion is presented in Section 5.

2. Background

Presume that there are n DMUs to be evaluated, indexed by $j = 1, \dots, n$ and each DMU is assumed to produce s different outputs from m different inputs. Let the observed input and output vectors of DMU_o be $X_o = (x_{1o}, \dots, x_{mo})$ and $Y_o = (y_{1o}, \dots, y_{so})$, respectively, provided that all components of vectors X_o and Y_o for all DMUs are non-negative and each DMU has at least one strictly positive input and output. The efficiency of DMU_o i.e., DMU under consideration, and one comparative set of weights of inputs and outputs, can be calculated by the following CCR model associated to constant return to scale:

$$\theta_o^* = \max \sum_{r=1}^s u_r y_{ro}$$

$$\begin{aligned} s.t \quad & \sum_{i=1}^m v_i x_{io} = 1, \\ & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0, \quad j = 1, \dots, n, \\ & v_i \geq \varepsilon > 0, \quad i = 1, \dots, m, \\ & u_r \geq \varepsilon > 0, \quad r = 1, \dots, s, \end{aligned} \tag{1}$$

where the symbol ε is a positive Archimedean infinitesimal constant, which is employed to prevent the appearance of zero weights.

DMU_o is DEA efficient in the model (1) iff $\theta_o^* = 1$, otherwise ($\theta_o^* < 1$), it is non-DEA efficient. DEA efficient DMUs are usually thought to perform better than non-DEA efficient DMUs.

As mentioned before, efficiency is a relative measure. It can be measured within different ranges. If the efficiencies of DMUs are measured within the range of greater than or equal to one, then the following linear programming model can be constructed to measure the worst performance of each DMU [9]:

$$\begin{aligned} \Phi_o^* = \min \quad & \sum_{r=1}^s u_r y_{ro} \\ s.t \quad & \sum_{i=1}^m v_i x_{io} = 1, \\ & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \geq 0, \quad j = 1, \dots, n, \\ & v_i \geq \varepsilon > 0, \quad i = 1, \dots, m, \\ & u_r \geq \varepsilon > 0, \quad r = 1, \dots, s, \end{aligned} \tag{2}$$

DMU_o is DEA inefficient in the model (1) if $\Phi_o^* = 1$, otherwise ($\Phi_o^* > 1$), it is non-DEA inefficient. DEA inefficient DMUs are usually thought to perform worse than non-DEA inefficient DMUs.

From the above analyses we can see that efficiency is a relative measure. It can be measured either within the range of less than or equal to one, or within the range of greater than or equal to one. When

measured within different ranges, it has different meanings. The resultant assessment conclusions are usually different. Any assessment using only one type of efficiency is obviously one-sided. Ideally, both types of efficiencies should be used at the same time to assess the performances of DMUs.

In this article, we are going to propose a method for ranking DMUs that uses both type of efficiencies at the same time and doesn't have drawbacks of previous methods.

3. A proposed ranking method

In this section we want to rank all of DMUs (efficient DMUs and non-efficient DMUs) by using two virtual DMUs (an ideal DMU and an anti-ideal DMU) and TOPSIS criteria. Therefore, first we add two virtual points to set of DMUs. They are an ideal DMU and an anti-ideal DMU that are defined as following:

Definition 3.1: The ideal point is a virtual point, which can use the least inputs to generate the most outputs, i.e., if we show the ideal point with $\overline{DMU} = (\overline{X}, \overline{Y})$ then $\overline{x}_i = \min\{x_{ij} | j = 1, \dots, n\}, (i = 1, \dots, m)$, and $\overline{y}_r = \max\{y_{rj} | j = 1, \dots, n\}, (r = 1, \dots, s)$.

Definition 3.2: The negative ideal point is a virtual point which consumes the most inputs only to produce the least outputs. That is if we show the positive ideal point with $\overline{DMU} = (\overline{X}, \overline{Y})$ then $\tilde{x}_i = \max\{x_{ij} | j = 1, \dots, n\}, (i = 1, \dots, m)$, and $\tilde{y}_r = \min\{y_{rj} | j = 1, \dots, n\}, (r = 1, \dots, s)$.

Note that the ideal DMU may not exist in practical production activity at least at the current technical level, whereas some anti-ideal DMU may exist in practical production activity since theoretically waste of resources in production has always been a permissible possibility set.

Then, for ranking DMUs we should do the following three stages:

- I. We calculate a distance of each DMU ($DMU_j, j = 1, \dots, n$) to the ideal point ($d_j^+, j = 1, \dots, n$).
- II. We calculate a distance of each DMU ($DMU_j, j = 1, \dots, n$) to the anti-ideal point ($d_j^-, j = 1, \dots, n$).
- III. We calculate a TOPSIS criteria of each DMU ($RC_j, j = 1, \dots, n$).

We will explain in detail how to calculate each stage.

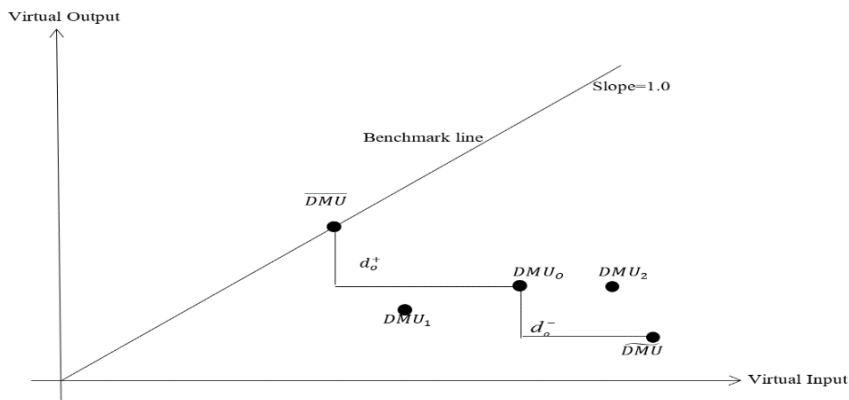


Figure 1: Gap analysis showing point below the virtual benchmark line

Stage I:

In Fig. 1 the vertical and horizontal axes are set to be the virtual output (weighted sum

of s outputs) and virtual input (weighted sum of m inputs), respectively. By the definition of the efficiency score, the common benchmark level is one straight line that passes through the origin, with slope 1.0 in the coordinate. If one set of weights $v_i (i = 1, \dots, m)$ and $u_r (r = 1, \dots, s)$ are given such that a coordinate of the ideal DMU would be the best DMU i.e. $\overline{DMU} = (\sum_{i=1}^m v_i \bar{x}_i, \sum_{r=1}^s u_r \bar{y}_r)$ and it locates on the benchmark line, also a coordinate of the anti-ideal DMU would be the worst DMU i.e. $\widehat{DMU} = (\sum_{i=1}^m v_i \hat{x}_i, \sum_{r=1}^s u_r \hat{y}_r)$, therefore, a coordinate of all DMUs are on or under the benchmark line in Fig.1.

For example, we are going to measure the distance of $DMU_o, (o \in \{1, \dots, n\})$ to the ideal point. So, we should determine an optimal set of weights U^* and V^* , such that the ideal point is the best point and be on the benchmark line, and also the anti-ideal point is the worst point and all DMUs be on or under the benchmark line and above all, the distance of $DMU_o, (o \in \{1, \dots, n\})$ to the positive point (d_o^+) is the shortest. The distance of $DMU_o, (o \in \{1, \dots, n\})$ to the positive point is calculated by using L1 - norm. In fact, we consider the following model:

$$\begin{aligned}
 d_o^+ &= \min d_o^+ \\
 \text{s.t.} \quad & \frac{U\bar{Y}}{V\bar{X}} = 1, \\
 & \frac{UY_j}{VX_j} \leq 1, \quad j = 1, \dots, n, \\
 & |U\bar{Y} - UY_o| + |V\bar{X} - VX_o| = d_o^+, \\
 & V\bar{X} \leq VX_o, \quad U\bar{Y} \geq UY_o, \\
 & \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 & v_i \geq \varepsilon > 0, \quad i = 1, \dots, m, \\
 & u_r \geq \varepsilon > 0, \quad r = 1, \dots, s.
 \end{aligned} \tag{3}$$

Here, the constraint $\sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1$ is added for normalization purpose. It is

obvious that model (3) is non-linear, but this model converts to a linear model easily, because $U\bar{Y} \geq UY_o$ and $V\bar{X} \leq VX_o$, so we have:

$$\begin{aligned}
 d_o^+ &= |U\bar{Y} - UY_o| + |V\bar{X} - VX_o| \\
 &= U\bar{Y} - UY_o + VX_o - V\bar{X}
 \end{aligned}$$

Therefore, we will have:

$$\begin{aligned}
 d_o^+ &= \min d_o^+ \\
 \text{s.t.} \quad & U\bar{Y} - V\bar{X} = 0, \\
 & UY_j - VX_j \leq 0, \quad j = 1, \dots, n, \\
 & U\bar{Y} - UY_o + VX_o - V\bar{X} = d_o^+, \\
 & \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 & v_i \geq \varepsilon > 0, \quad i = 1, \dots, m, \\
 & u_r \geq \varepsilon > 0, \quad r = 1, \dots, s.
 \end{aligned} \tag{4}$$

Stage II:

In this stage, we are going to measure the distance of each DMU to the anti-ideal point. As we see in Fig.1, for example we want to measure the distance of $DMU_o, (o \in \{1, \dots, n\})$ to the anti-ideal point. So, we should determine an optimal set of weights U^* and V^* , such that the ideal point is the best point and be on the benchmark line, and also the anti-ideal point is the worst point and all DMUs be on or under the benchmark line and above all, the distance of $DMU_o, (o \in \{1, \dots, n\})$ to the anti-point (d_o^-) is the furthest. The distance of $DMU_o, (o \in \{1, \dots, n\})$ to the anti-point is calculated by using L1 - norm. In fact, we consider the following model:

$$\begin{aligned}
 d_o^- &= \min d_o^- \\
 \text{s.t.} \quad & \frac{U\bar{Y}}{V\bar{X}} = 1, \\
 & \frac{UY_j}{VX_j} \leq 1, \quad j = 1, \dots, n, \\
 & |U\bar{Y} - UY_o| + |V\bar{X} - VX_o| = d_o^-, \\
 & V\bar{X} \geq VX_o,
 \end{aligned} \tag{5}$$

$$\begin{aligned}
 U\tilde{Y} &\leq UY_o, \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1 \\
 v_i &\geq \varepsilon > 0, \quad i = 1, \dots, m, \\
 u_r &\geq \varepsilon > 0, \quad r = 1, \dots, s.
 \end{aligned}$$

Here, the constraint $\sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1$ is added for normalization purpose. It is obvious that model (5) is non-linear, but this model converts to a linear model easily, because $U\tilde{Y} \leq UY_o$ and $V\tilde{X} \geq VX_o$, so we have:

$$\begin{aligned}
 d_o^- &= |U\tilde{Y} - UY_o| + |V\tilde{X} - VX_o| \\
 &= UY_o - U\tilde{Y} + V\tilde{X} \\
 &\quad - VX_o
 \end{aligned}$$

Therefore, we will have:

$$\begin{aligned}
 d_o^- &= \min d_o^- \\
 \text{s.t.} \quad &U\tilde{Y} - V\tilde{X} = 0, \\
 &UY_j - VX_j \leq 0, \quad j = 1, \dots, n, \\
 &UY_o - U\tilde{Y} + V\tilde{X} - VX_o = d_o^-, \\
 &\sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 &v_i \geq \varepsilon > 0, \quad i = 1, \dots, m, \\
 &u_r \geq \varepsilon > 0, \quad r = 1, \dots, s.
 \end{aligned} \tag{6}$$

Stage III:

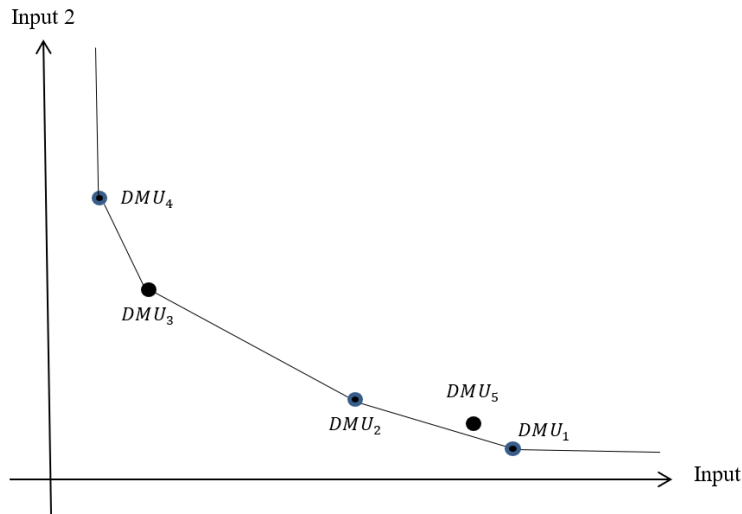


Figure 2: The CCR frontier of five DMUs

In the last stage we are going to calculate TOPSIS criteria of each DMU. An overall performance index for each DMU ($DMU_j, j = 1, \dots, n$) is thus computed by

$$RC_j = \frac{d_j^-}{d_j^- + d_j^+}, \quad j = 1, \dots, n.$$

Since $d_j^- \geq 0$ and $d_j^+ \geq 0$, then clearly $RC_j \in [0,1]$ and the larger the index value, the better the DMU performance.

To elaborate on our proposed ranking method, we present the following simple example. We consider 5 DMUs with two inputs and one output. The data of the DMUs are shown in Table 1. According to Figure 2, DMU_1, DMU_2, DMU_3 and DMU_4 are efficient and DMU_5 is non-efficient. The CCR efficiency scores of them are given in the fourth column of Table 1. Therefore, DMU_5 has the latest rank in most ranking method. Now, we are going to rank DMUs by our proposed method. As stated in method, in stage I, we obtain the ideal and the anti-ideal points with the following coordinates:

$$\overline{DMU} = (1,1,1) \text{ and } \overline{DMU} = (8,6,1).$$

After obtaining d_j^-, d_j^+ , ($j = 1, \dots, 5$), we calculate RC of five DMUs. Then we rank DMUs by the RC scores. The results of them are given in the two last columns. As we see, in the proposed method, DMU_5 has

a third rank. In the other words, DMU_5 's performance is better than the performance of DMU_2 , and DMU_3 . This result is closer to reality.

Table 1: Input and Output and ranking DMUs by proposed method

DMU	Input 1	Input	Output	CCR efficiency	RC	Ranking
1	8	1	1	1	0.9997	2
2	5	2	1	1	0.7999	5
3	2	4	1	1	0.8571	4
4	1	6	1	1	1	1
5	7.5	1.5	1	0.9166	0.8975	3

4. Example

In this section, we are going to compare the ranking results of our proposed method with the ranking results of AP model [1], Wang et al. [8] method and Wu [3] method. Therefore, we consider a DEA

efficiency evaluation problem with five DMUs, each DMU with two inputs and one output. The data set is taken from Andersen and Petersen [1] and is shown in Table 2. The CCR efficiency of each DMU is presented in the last column of Table 2.

Table 2: Data for five DMUs with two inputs and one output

DMU	Input 1	Input 2	Output	CCR efficiency
1	2	12	1	1
2	2	8	1	1
3	5	5	1	1
4	10	4	1	1
5	10	6	1	0.75
IDMU	2	4	1	
ADMU	10	12	1	

As can be seen from the rating results of Table 2 that the conventional CCR model identifies DMU_1 through DMU_4 as DEA efficient units, but in fact, the performance of DMU_2 is better than DMU_1 because DMU_2 consumes less resource of input 2 to generate the same output as DMU_1 . In order to rank the four DEA efficient units, Andersen and Petersen [1] suggested a ranking approach that compares the DMU under evaluation with a linear combination of all the other DMUs, i.e., the DMU itself is excluded. Based on their approach, the following ranking order was obtained: $DMU_2 > DMU_4 > DMU_3 > DMU_1 > DMU_5$, where the symbol “>” means “performs better than”. This ranking

model considers only the best possible relative efficiency of each DMU. Therefore, it is somewhat one-sided.

Wang et al. [8] proposed DEA models with IDMU and ADMU to reconsider these five DMUs. The virtual IDMU and ADMU are defined in the last two rows of Table 2. The final overall ranking was gained: $DMU_3 > DMU_2 > DMU_4 > DMU_1 > DMU_5$. This ranking achieved by using the systematic RC index. Their resulting of ranking are presented in Table 3.

Wang et al. [2] believed that the ranking results are different from the ranking obtained by Andersen and Petersen [1] because the overall ranking considers both

the best and the worst possible relative efficiencies of each DMU.

Wu [3] focused on the Wang et al. [2] of method. He showed the method of Wang et al. [6] has some mistakes. He elaborated these mistakes by the above example. θ_{PIP}^* and φ_{NIP}^* denote the maximal efficiency of the PIP (the IDMU) and the minimal efficiency of the NIP (the ADMU) respectively. One of the important mistakes of their method was: θ_{PIP}^* measures rely on an input orientation while the φ_{NIP}^* measures rely on an output orientation. Wu [3] believed that the method of Wang et al. [2] isn't convincing since their overall ranking considers both the best and the worst possible relative efficiencies of each DMU. This is not true since they employ conflicted efficiency concepts in their method and due to this reason; their result is almost completely different from that by Andersen and Petersen [1]. Then he proposed a model for computing the worst possible relative

efficiency φ_{NIP}^* . After that he performed his method by Andersen and Petersen [8] example. The result earned: $DMU2 > DMU1 > DMU3 > DMU4 > DMU5$.

DMU_2 and DMU_5 are selected as the best and worst performers respectively, which is consistent with result of Andersen and Petersen [1].

Now, we use the proposed DEA models with IDMU and ADMU to reevaluate these five DMUs. The RC values and the ranking results are presented in the last two columns of Table 3.

As we see, our results are similar to the ranking results of Wu [3], but our proposed method is better than the proposed method of Wu [3], because we can rank DMUs by two LP and the RC relation but his ranking method need to calculate four LP and the RC relation. This demonstrates the simplicity and superiority of our method to Wu's method.

Table 3: The results of AP method, Wang method, Wu method and the proposed method

DMU	AP method	Wang method	Wu method	RC	Ranking
1	4	4	2	0.9997	2
2	1	2	1	0.9998	1
3	3	1	3	0.8927	4
4	2	3	4	0.9995	3
5	5	5	5	0.7496	5

5. Conclusion

In this paper we have proposed one ranking method, which is based on two virtual points (the ideal and anti-ideal points). First, we have computed the distance of each DMU to the ideal point and then calculated the distance of DMUs to the anti-ideal point; The two distinctive distances are integrated using a relative closeness index, which can thus be used as the basis of ranking the DMUs. We have compared this ranking method with the existing DEA ranking methods as AP [1] method, Wang et al. [2] method and Wu [3] method. As we have seen our method is more simple and better than other

methods. Also, it doesn't have drawbacks of the previous ranking methods.

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