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Technically Efficient Targets for the Groups by Using the Centralized Scenario and Enhanced Russell Measure

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Abstract
Cooper et al. [Efficiency aggregation with enhanced Russell measures in data envelopment analysis, Socio-Economic Planning Sciences, 41 (2007) 1–21] presented a method for measuring aggregate efficiency, using the enhanced Russell measure. In that paper, they posed questions and opened the way for new opportunities for studying and extending the proposed method and some other related fields. One of these issues is the extension of the proposed method in the case where there is the possibility of reallocation among the units in order to improve group efficiency. In this paper, we propose a model for evaluating the group efficiency, and employ the centralized scenario to set targets for each unit as well as for the group.

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1. Introduction

Data envelopment analysis (DEA) is a mathematical programming approach for evaluating the performance of decision making units (DMUs) especially in obtaining the efficiency score of a DMU in comparison with others. This can be performed only through using the inputs and outputs consumed or produced by DMUs. The set of all possible operating points is called the production possibility set (PPS) which is defined as follows:

T={(x,y)| x can produce y}

The two most common technologies used in DEA are the constant returns to scale (CRS) and the variable returns to scale (VRS). Each DMU on the frontier of T is "technically" efficient, but others are inefficient. Determining whether or not a DMU is on the frontier is possible by using various DEA models. One of these models is the enhanced Russell measure, which gives a score of 1 to each efficient unit and a score between 0 and 1 to each inefficient unit.

While analyzing the efficiency or productivity of DMUs, it is usually of interest to consider not only the efficiency of each individual but also (and often more importantly) an aggregate measure summarizing the efficiency of a group [23]. There is a close connection between aggregate efficiency and group efficiency concepts. Usually, in DEA, whenever we want to measure the performance of a set or group of units based on their units, the concept of aggregation is used. Sometimes resource aggregation is first performed, and then a method for evaluating the virtual unit that represents the system or industry or group of units being evaluated is introduced. In some cases, the overall performance and efficiency are calculated as a function of the performance or efficiency of the subsystems or units within that set or group, and the efficiency of the group is the aggregation of the group members'

efficiency. So, aggregate efficiency measures, measure the efficiency of a group of DMUs.

Cooper et al. [11] introduced a method for evaluating the technical efficiency of a group consisting of a number of similar subsets. They have only dealt with technical efficiency in that paper, addressing the aggregate technical efficiency, rather than group efficiency. Previous works had used either input efficiency measures or output efficiency measure to calculate the aggregate technical efficiency, while their method employed both input and output efficiency measures, simultaneously, to compute the aggregate technical efficiency. Moreover, since previous methods employed radial methods, the measures obtained by them were incomplete because they did not take into account the non-zero input and output slacks, and the resulting models cannot therefore find all the inefficiencies. In order to remove such problems of previous methods, Cooper et al. [11] used the enhanced Russell measure.

Blackorby and Russell [10] derived conditions on the firm technologies that are required to aggregate technical efficiency indices. They showed that, under reasonable assumptions about the efficiency indices, output (input) aggregation is possible only if the efficiency indices are ratios of linear functions of input and output quantities and the aggregate index (or its inverse) is a convex combination—a linearly weighted average—of the individual indices (or their inverses). They focused either on the input or on the output efficiency measures alone, and ignored the case where the efficiency measure consists simultaneously of both input and output efficiency. Regarding this case, Cooper et al. [11] showed that their proposed method, which consists of both input and output efficiencies, satisfies the conditions put forward by Blackorby and

Russell [10].

Before proceeding, we note that others have also studied the aggregation problem. For instance, Fare and Zelenyuk [13] established the fact that industry maximal revenue is the sum of its firms' maximal revenues. This fact enables us to discover conditions for the aggregation of Farrell efficiency. Making use of this fact, they derived the industry efficiency measure from the firms' measures of both technical and overall efficiencies. Fare and Zelenyuk [14] showed that the weighted geometric mean is required in order to aggregate individual efficiencies into group efficiency, in such a way that the multiplicative structure of further decompositions is preserved with equal weights across components. Zelenyuk [29] extended the work of Fare and Zelenyuk [13] to obtain a theoretically justified method for aggregating Malmquist productivity indices and their decompositions. Färe and Karagiannis [12] derived alternative weighting schemes that complement those of Färe and Zelenyuk [13] for consistent aggregation of Farrell when the technology exhibits (global) constant returns to scale. Zelenyuk [30] extended the aggregation theory in efficiency and productivity analysis by providing a practical way of estimating scale efficiency of a group (e.g., industry). Walheer [28] made a distinction between two types of inputs in the aggregation framework: private and public inputs, and considered the aggregation of Farrell efficiencies with private and public inputs.

Now, we proceed by mentioning some points regarding [11]:

1- The concept of consistency is used in aggregation, which means that the results obtained from evaluating the group and evaluating individual units should be the same. Especially, if the group is evaluated as efficient, then all the units should be evaluated as efficient.

2- As is known, the evaluation of DMUs in DEA is relative and is based on comparison with other units. This is a shortcoming when it comes to evaluating groups, since there is not another group consisting of all the units, so that the group under evaluation can be compared against it. Therefore, the group must be compared against itself and, by DEA, it will certainly be evaluated as efficient. So, in [11] it is tried to perform evaluation against others, rather than self-evaluation of a group. However, the group is compared against the frontier of the PPS containing the group itself (obtained as putting all the units together) and the other units that make up the group. Nevertheless, the point is that this is the case only for constant returns to scale, but not for variable returns to scale. This is because in the CRS case, adding or removing such a group that consists of all the units to the PPS with constant returns to scale (T_c) will not change the frontier and consequently the shape of the PPS (See Fig. 1. Unit G in the figure is the group made up of units A,B,C). However, in variable returns to scale, if the group is compared against the frontier produced by the individual units, it will lie outside the region; and if we consider the PPS resulting from the addition of the group to the other units, the group will certainly be efficient because it will have the maximum output in all output indices. This is incompatible with the consistency introduced in [11], since the group is certainly efficient but all the units are not necessarily efficient, while in the CRS case the efficiency of the group leads to the efficiency of all the units, and vice versa. Although many real world problems involve VRS, their proposed method does not provide a means to avoid self-evaluation in VRS.
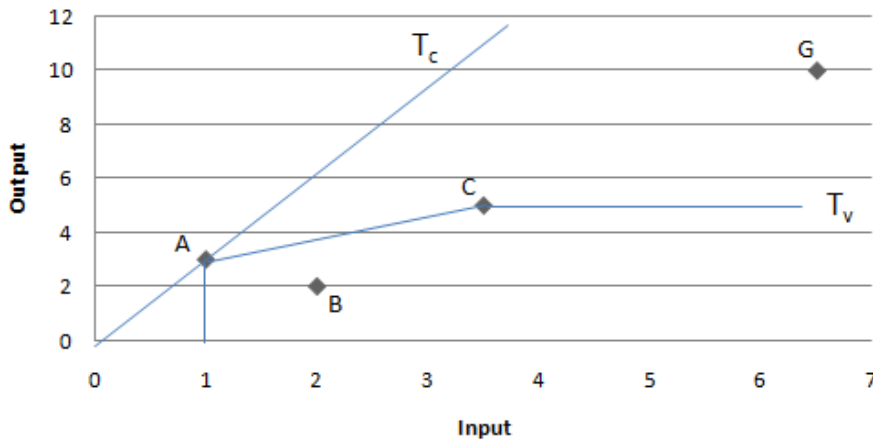
Fig.1. Group G lies in T_c but not in T_v.

3- It is assumed in [11] that the units are separate from each other, i.e., trade-off is not possible between units. This means that it is not possible to reallocate the resources in order to improve the performance of some units and, consequently, the whole group. Such an improvement is primarily more obvious in the case of VRS; for instance, if sufficient resources are allocated to a unit with increasing returns to scale (IRS), not only its efficiency but also that of the group will improve. Similar comments could be made about the units in the case of decreasing returns to scale (DRS). For problems in which resource reallocation and trade-off are possible, however, group efficiency can be improved. As is pointed out in [11], the efficiency of the units is the necessary condition for group efficiency. In other words, if reallocation is possible, then the group might be inefficient although all the units are efficient. Therefore, potential resource reallocation from one unit to another is introduced for [11].

Considering the above-mentioned shortcomings, the present paper proposes a method to evaluate group technical efficiency in which the technical efficiency of all the DMUs comprising the group can be inferred from group technical efficiency, i.e., it has consistency. Furthermore, the method can be applied to DMUs with VRS. In addition, there is the possibility of reallocation between the units to improve group technical efficiency. This is basically possible when the units are interrelated; for instance, the units belong to a larger unit or they are controlled by the same decision maker. It is then obvious that the decision maker is willing to improve group performance. Clearly, the inefficiency of one of the group members will lead to group inefficiency, while the efficiency of all the units is not sufficient for group efficiency, as was mentioned earlier. The decision maker, who provides the resources, would like the group to consume less input and produce more output. One could seek such an improvement in resource reallocation, which can be the "potential resource reallocation" as pointed out by Cooper et al. [11]. Improvement in resource consumption and, consequently, group efficiency enhancement requires simultaneous attention to the whole input and the whole output. This means that all the inputs and all the outputs of all the DMUs should be considered simultaneously. In other words, to evaluate group performance, we should

consider the performance of units as they are related to each other. Although ordinary DEA models examine the DMUs individually, simultaneous attention to all inputs and outputs implies that the units cannot perform independently.

There have been some previous approaches in the literature that handle the DMUs in a joint manner. A DEA-based model with aggregated, non-radial input orientation was presented by Golany et al. [15], which jointly determines the inputs of each DMU subject to a total input availability constraint and bounds on the changes of the solution for each DMU. The objective function used is heuristic and difficult to interpret. Golany and Tamir [16] proposed an output-oriented resource allocation model which includes constraints on the total input consumption. The possible objective functions proposed include total output (for the case of a single output) and a weighted sum of the total outputs or an unweighted sum of deviations from pre-established output thresholds (for the multiple, commensurable output case).

Athanassopoulos [5] introduced a goal programming DEA model (GoDEA) for centralized planning. In this model, in order to set global targets for the total consumption of each input and the total production of each output, a series of independent DEA models, one for each input and each output, was proposed.

Beasley [7] also presented a non-linear resource allocation model based on the ratio form that aimed at jointly computing inputs and outputs for each DMU for the next period, which seeks to maximize the average efficiency.

Recently, in some papers, Lozano et al. considered the units simultaneously, and called it the centralized scenario. They believe that the units cannot be independently evaluated when they are not independent from each other, but are managed by the same decision maker, who is interested in minimizing the total input consumption or maximizing the total output production, together with maximizing the efficiency of each individual unit. But they considered the simultaneous projection of the units onto the efficient frontier, and regarding the centralized theory they proposed centralized target setting [18]. They used this concept to study the performance of some Spanish city councils and tried to set appropriate targets for the recycling plants affiliated to these councils. All these councils were managed by the same decision maker, whose objective was to maximize the total output, i.e., increasing the amount of recycled matter. An output-oriented model was employed for target setting. Since one of the inputs considered in the evaluation was the number of trucks of each council and considering the point that the decision maker would allow the allocation of the trucks of one council to another, the centralized scenario was used for appropriate target setting. Although the existing models had been proposed for the specific problem, the idea could be extended to other problems. Therefore, Lozano and Villa [18] proposed a general approach for centralized resource allocation. In their paper, two new DEA models have been presented for this purpose. Both of them project all DMUs onto the efficient frontier. One model seeks radial reductions of the total consumption of all the inputs, while the other tries to find separate reductions for each total input according to a preference structure. Consistent with the centralized point of view, total output conservation constraints are imposed. Lozano et al. extended the centralized scenario again and introduced "Centralized DEA models with the possibility of downsizing." They

believe that the intra-organizational perspective opens up the possibility that greater technical efficiency for the organization as a whole might be achieved by closing down some of the existing DMUs.

The main idea in our approach is based on Lozano's centralized scenario. It should be noted that Lozano dealt with either the input orientation or the output orientation alone, and did not consider the inputs and outputs simultaneously. Moreover, he used the centralized scenario for the optimal allocation of resources or the optimal acquisition of outputs, but never used it to measure group efficiency and to obtain an efficiency index. This is what we intend to achieve in this paper.

The rest of this paper is structured as follows. Our proposed model is introduced in section 2. Some properties of the model are pointed out and their validity is established. Furthermore, target setting for the group and its comprising units is carried out and the proposed model, which is a fractional model, is converted to a linear model by appropriate transformations. Section 3 exhibits how the model works, through a numerical example and appropriate diagrams. In section 4, we apply our model to an example provided in several previous papers. Section 5 contains the conclusions.

2. Proposed model for evaluating the group, and target setting
2.1. Discussion
We assume n DMUs, each of which consumes m inputs to produce s outputs. Let $X \in R^m$ and $Y \in R^s$ be matrices containing the observed inputs and outputs, respectively, for n DMUs. We denote by X_j (the j th column of X) the vector of inputs consumed by DMU_j, and by Y_j (the j th column of Y) the vector of outputs produced by DMU_j.

The production possibility set T_v, with variable returns to scale technology is defined as follows:

$$T_V = \left\{ (X,Y) \, \middle| \, X \geq \sum_{j=1}^{n} \lambda_j X_j \, , Y \leq \sum_{j=1}^{n} \lambda_j Y_j \, , \right.$$
$$\left. \sum_{j=1}^{n} \lambda_j = 1 \, , \, \lambda_j \geq 0 \, ; j = 1,...,n \right\}$$

The Enhanced Russell Measure (ERM) model for evaluating DMU_o in Tv is:

$$\min \quad \frac{\sum_{i=1}^{m} \theta_i}{m} \bigg/ \frac{\sum_{k=1}^{s} \varphi_k}{s}$$

$$s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_i \, x_{io}, \quad i = 1,...,m$$

$$\sum_{j=1}^{n} \lambda_j y_{kj} \geq \varphi_k \, y_{ko}, \quad k = 1,...,s$$

$$\sum_{j=1}^{n} \lambda_j = 1, \qquad\qquad (1)$$

$$\lambda_j \geq 0, \qquad j = 1,...,n$$
$$\theta_i \leq 1, \qquad i = 1,...,m$$
$$\varphi_k \geq 1, \qquad k = 1,...,s$$

See [22] for more about this measure.

It is reasonable to always try to produce more output by using less input. So, we would like to increase group output production by using as little group input as possible. Considering this, the group is efficient if the same output level cannot be produced by consuming less input, or more output cannot be produced by consuming the same input level. However, if a set of n DMUs cannot be found in T_v such that at least one of the components of their total inputs is smaller than the group input or at least one of the components of their total outputs is larger than the group output, then the group under evaluation is efficient. It is noteworthy, however, that the efficiency based on this definition is technical efficiency, and it does not necessarily mean that the group is efficient in all

respects. In this paper, by efficiency we mean technical efficiency. In order to evaluate the group under consideration, we propose the following non-radial model, as a combination of Lozano's centralized method and the enhanced Russell model.

$$\min R = \frac{\sum_{i=1}^{m} \theta_i}{m} \bigg/ \frac{\sum_{k=1}^{s} \varphi_k}{s}$$

$$s.t. \quad \sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr} x_{ij} \le \theta_i \sum_{j=1}^{n} x_{ij}, \quad i=1,...,m$$

$$\sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr} y_{kj} \ge \varphi_k \sum_{j=1}^{n} y_{kj}, \quad k=1,...,s$$

$$\sum_{j=1}^{n} \lambda_{jr} = 1, \quad r=1,...,n \qquad (2)$$

$$\lambda_{jr} \ge 0, \quad j,r=1,...,n$$

$$\theta_i \le 1, \quad i=1,...,m$$

$$\varphi_k \ge 1, \quad k=1,...,s$$

Notice that $\left(\sum_{j=1}^{n} \lambda_{jr} x_{ij}, \sum_{j=1}^{n} \lambda_{jr} y_{kj} \right) \in T_v$, and therefore $\left(\sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr} x_{ij}, \sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr} y_{kj} \right)$ is the summation of n DMU belonging to the T_v, which it may some of them be the same. We call the set containing the elements similar to the $\left(\sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr} x_{ij}, \sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr} y_{kj} \right)$ as T_v^g and its frontier as ∂T_v^g. Therefore, it can be inferred that the model (2) evaluate the group with compare to the elements on the ∂T_v^g.

In what follows, we state some properties related to model (2).

Cooper et al. in [11] write about the enhanced Russell measure: "R is 'complete' and therefore differs from the commonly employed radial measures which (a) fail to reflect the non-zero

slacks and (b) are either output-oriented or input-oriented. They thus fail to reflect performances of the outputs or inputs represented in those constraints not covered by the radial measure".

Regarding the criteria stated above for the completeness of the enhanced Russell measure, it could be inferred that the index R obtained from model (2) has the same property, as well. Obviously model (2) is feasible and its optimal value is between 0 and 1.

Definition. A group is technically efficient if the optimal value of model (2) is 1. Otherwise it is inefficient.

We now provide some properties of our proposed model.

Theorem 1. R is units invariant.

Proof. Notice that θ_i and φ_k are unit-invariant so that R is also unit-invariant. To see that this is the case, we note that these inequalities may be formulated as equations without affecting the optimal value of R^*. Then, writing

$$\theta_i = \sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr} x_{ij} \bigg/ \sum_{r=1}^{n} x_{ir}, i=1,...,m \quad \text{and}$$

$$\varphi_k = \sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr} y_{kj} \bigg/ \sum_{r=1}^{n} y_{kr}, k=1,...,s \quad \text{we}$$

see that we can multiply numerators and denominators by $p_i, q_k > 0, i=1,...,m, k=1,..s$ respectively, without affecting the values of θ_i and φ_k.

Theorem 2. Each unit $(\hat{x}_{ir}, \hat{y}_{kr}) = \left(\sum_{j=1}^{n} \lambda_{jr}^* x_{ij} \; \forall i, \sum_{j=1}^{n} \lambda_{jr}^* y_{kj} \; \forall k \right)$ is Pareto efficient in T_v.

Proof: By contradiction, suppose that $(\hat{x}_{ir'}, \hat{y}_{kr'}), i=1,...,m, k=1,...,s$, is not technically efficient, and others are

efficient. Then vectors $(\lambda_{1r}, ..., \lambda_{1r})$ can be found satisfying $\sum_{j=1}^{n}\lambda_{jr}=1$ and defining a virtual operating point

$$
\begin{cases}
\hat{\hat{x}}_{ir} = \sum_{j=1}^{n}\lambda_{jr}x_{ij} \leq \hat{x}_{ir} & \forall i \\
\hat{\hat{y}}_{ir} = \sum_{j=1}^{n}\lambda_{jr}y_{kj} \geq \hat{y}_{ir} & \forall k
\end{cases}
$$

such that the above inequality is strict for at least one input or output index of the rth unit. Without loss of generality, assume that strict inequality occurs in the i'th component of the units (the proof procedure is similar for the case where inequality occurs in the output index or both input and output indices). Then

$$
\begin{cases}
\hat{\hat{x}}_{i'r'} = \sum_{j=1}^{n}\lambda_{jr'}x_{i'j} < \hat{x}_{i'r'} \\
\hat{\hat{x}}_{ir'} = \sum_{j=1}^{n}\lambda_{jr'}x_{ij} \leq \hat{x}_{ir'} & \forall i \neq i' \\
\hat{\hat{y}}_{ir'} = \sum_{j=1}^{n}\lambda_{jr'}y_{kj} \geq \hat{y}_{ir'} & \forall k
\end{cases}
$$

By summation on index r, we will have:

$$
\begin{cases}
\text{for index } i' : \sum_{r=1}^{n}\hat{\hat{x}}_{i'r} < \sum_{r=1}^{n}\hat{x}_{i'r} & (a) \\
\text{for others}: \sum_{r=1}^{n}\hat{\hat{x}}_{ir} \leq \sum_{r=1}^{n}\hat{x}_{ir} & (b) \quad (7) \\
\sum_{r=1}^{n}\hat{\hat{y}}_{kr} \geq \sum_{r=1}^{n}\hat{y}_{kr} & (c)
\end{cases}
$$

Dividing the two sides of (7-a) by $\sum_{r=1}^{n}x_{i'r}$, those of (7-b) by $\sum_{r=1}^{n}x_{ir}$, and those of (7-c) by $\sum_{r=1}^{n}y_{kr}$ we get:

$$\overline{\theta}_{i'} = \sum_{r=1}^{n}\hat{\hat{x}}_{i'r} \Big/ \sum_{r=1}^{n}x_{i'r} < \sum_{r=1}^{n}\hat{x}_{i'r} \Big/ \sum_{r=1}^{n}x_{i'r} = \theta_{i'}^{*}$$

$$\overline{\theta}_{i} = \sum_{r=1}^{n}\hat{\hat{x}}_{ir} \Big/ \sum_{r=1}^{n}x_{ir} \leq \sum_{r=1}^{n}\hat{x}_{ir} \Big/ \sum_{r=1}^{n}x_{ir} = \theta_{i}^{*}$$

$$\overline{\varphi}_{k} = \sum_{r=1}^{n}\hat{\hat{y}}_{kr} \Big/ \sum_{r=1}^{n}y_{kr} \geq \sum_{r=1}^{n}\hat{y}_{kr} \Big/ \sum_{r=1}^{n}y_{kr} = \varphi_{k}^{*}$$

Therefore, $\left(\overline{\lambda}_{jr}, \overline{\theta}_{i}, \overline{\varphi}_{k}\right)$ is a feasible solution for (2) and its objective function value is smaller than R^{*}, which contradicts the optimality of $\left(\lambda_{jr}^{*}, \theta_{i}^{*}, \varphi_{k}^{*}\right)$. The contradiction assumption is thus invalid, and hence $\left(\hat{x}_{ir'}, \hat{y}_{kr'}\right), i = 1, ..., m, k = 1, ..., s$ is Pareto efficient for every index r.

Corollary 1. $\left(\hat{x}_{ir}, \hat{y}_{kr}\right)$ is a target for unit r

Theorem 3. $\left(\sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}^{*}x_{ij} \;\; \forall i, \sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}^{*}y_{kj} \;\; \forall k\right) \in T_{v}^{g}$ is Pareto efficient.

Proof. Suppose that $\left(\lambda_{jr}^{*}, \theta_{i}^{*}, \varphi_{k}^{*}\right)$ is the optimal solution, R^{*}, of model (2), and $\left(\sum_{r=1}^{n}\hat{x}_{ir}, \sum_{r=1}^{n}\hat{y}_{kr}\right) = \left(\sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}^{*}x_{ij}, \sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}^{*}y_{kj}\right)$ Since the constraints are expressed as equality at optimality, by Theorem (2), we have:

$$\theta_{i}^{*} = \sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}^{*}x_{ij} \Big/ \sum_{r=1}^{n}x_{ir} = \sum_{r=1}^{n}\hat{x}_{ir} \Big/ \sum_{r=1}^{n}x_{ir}$$

$$\varphi_{k}^{*} = \sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}^{*}y_{kj} \Big/ \sum_{r=1}^{n}y_{kr} = \sum_{r=1}^{n}\hat{y}_{kr} \Big/ \sum_{r=1}^{n}y_{kr}$$

Now, by contradiction suppose that $\left(\sum_{r=1}^{n}\hat{x}_{ir}, \sum_{r=1}^{n}\hat{y}_{kr}\right)$ is not Pareto efficient. Then, there exist $\left(\sum_{r=1}^{n}\widetilde{x}_{ir}, \sum_{r=1}^{n}\widetilde{y}_{kr}\right) \in T_{v}^{g}$

such that $\left(\widetilde{x}_{ir},\widetilde{y}_{kr}\right)\in T_v$ and

$$\left(\sum_{r=1}^{n}\widetilde{x}_{ir},-\sum_{r=1}^{n}\widetilde{y}_{kr}\right)\underset{\neq}{\leq}\left(\sum_{r=1}^{n}\hat{x}_{ir},-\sum_{r=1}^{n}\hat{y}_{kr}\right) \quad (8)$$

Without loss of generality, assume that strict inequality occurs in the i'th component of the inputs (the proof procedure is similar for the case where inequality occurs in the output index or both input and output indices). Since $\left(\widetilde{x}_{ir},\widetilde{y}_{kr}\right)\in T_v$, there exists a $\lambda_{jr}\geq 0$ for every r such that $\sum_{j=1}^{n}\lambda_{jr}=1,\ \forall r$ and

$$\begin{cases}\widetilde{\widetilde{x}}_{ir}=\sum_{j=1}^{n}\lambda_{jr}x_{ij}\leq\widetilde{x}_{ir}\\[2mm]\widetilde{\widetilde{y}}_{ir}=\sum_{j=1}^{n}\lambda_{jr}y_{kj}\geq\widetilde{y}_{ir}\end{cases}$$

By summation on index r, we will have:

$$\begin{cases}\widetilde{\widetilde{x}}_{ir}=\sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}x_{ij}\leq\sum_{r=1}^{n}\widetilde{x}_{ir}\\[2mm]\widetilde{\widetilde{y}}_{ir}=\sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}y_{kj}\geq\sum_{r=1}^{n}\widetilde{y}_{kr}\end{cases}$$

Now, by (8) we will have:

$$\begin{cases}for\,index\,i':\ \sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}x_{i'j}\leq\sum_{r=1}^{n}\widetilde{x}_{i'r}<\sum_{r=1}^{n}\hat{x}_{i'r}\quad(a)\\[2mm]for\,others\ \ \sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}x_{ij}\leq\sum_{r=1}^{n}\widetilde{x}_{ir}\leq\sum_{r=1}^{n}\hat{x}_{ir}\quad(b)\\[2mm]\qquad\qquad\sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}y_{kj}\geq\sum_{r=1}^{n}\widetilde{y}_{kr}\geq\sum_{r=1}^{n}\hat{y}_{kr}\quad(c)\end{cases} \quad (9)$$

By dividing the two sides of (9-a) by $\sum_{r=1}^{n}x_{i'r}$, those of (9-b) by $\sum_{r=1}^{n}x_{ir}$, and those of (9-c) by $\sum_{r=1}^{n}y_{kr}$, we get:

$$\overline{\theta}_{i'}=\sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}x_{i'j}\Big/\sum_{r=1}^{n}x_{i'r}<\sum_{r=1}^{n}\hat{x}_{i'r}\Big/\sum_{r=1}^{n}x_{i'r}=\theta_{i'}^{*}$$

$$\overline{\theta}_{i}=\sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}x_{ij}\Big/\sum_{r=1}^{n}x_{ir}\leq\sum_{r=1}^{n}\hat{x}_{ir}\Big/\sum_{r=1}^{n}x_{ir}=\theta_{i}^{*}$$

$$\overline{\varphi}_{k}=\sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}y_{kj}\Big/\sum_{r=1}^{n}y_{kr}\geq\sum_{r=1}^{n}\hat{y}_{kr}\Big/\sum_{r=1}^{n}y_{kr}=\varphi_{k}^{*}$$

Therefore, $\left(\overline{\lambda}_{jr},\overline{\theta}_{i},\overline{\varphi}_{k}\right)$ is a feasible solution for (2) and its objective function value is smaller than R^{*}, which contradicts the optimality of $\left(\lambda_{jr}^{*},\theta_{i}^{*},\varphi_{k}^{*}\right)$. The contradiction assumption is thus invalid and $\left(\sum_{r=1}^{n}\hat{x}_{ir},\sum_{r=1}^{n}\hat{y}_{kr}\right)$ is therefore Pareto efficient.

Corollary 1. $\left(\sum_{r=1}^{n}\hat{x}_{ir},\sum_{r=1}^{n}\hat{y}_{kr}\right)$ is a target for the group.

This is realized when each of the units reaches its targets, which are set by $\left(\hat{x}_{ir},\hat{y}_{kr}\right)$. In fact, the efficiency of all the units is the necessary condition for group efficiency.

Theorem 4. Suppose that $\left(\sum_{r=1}^{n}x_r^A,\sum_{r=1}^{n}y_r^A\right)\in T_v^g$ and $\left(\sum_{r=1}^{n}x_r^B,\sum_{r=1}^{n}y_r^B\right)\in T_v^g$, and that R^A and R^B are the optimal values of model (2) in evaluating them, respectively. So, if $\left(\sum_{r=1}^{n}x_r^A,-\sum_{r=1}^{n}y_r^A\right)\underset{\neq}{\leq}\left(\sum_{r=1}^{n}x_r^B,-\sum_{r=1}^{n}y_r^B\right)$, then $R^A<R^B$.

Proof. Obvious.

Suppose that $\left(\lambda_{jr}',\theta_{ir}',\varphi_{kr}'\right)$ is the optimal solution obtained by evaluating unit r by the enhanced Russell model and

$\left(\sum_{j=1}^{n}\lambda'_{jr}x_{ij}, \sum_{j=1}^{n}\lambda'_{jr}y_{kj}\right)$ is the projection of

unit r onto the frontier T_v.

Theorem 5. Suppose that R' is the optimal value of model (2) in evaluating $\left(\sum_{r=1}^{n}\sum_{j=1}^{n}\lambda'_{jr}x_{ij}, \sum_{r=1}^{n}\sum_{j=1}^{n}\lambda'_{jr}y_{kj}\right)$. Then $R' \le R^*$.

Proof. Obvious.

The efficiency of each individual unit is the sufficient condition for efficiency of the group, but this is not a sufficient condition, by the corollary of Theorem 5, that is the efficiency of the units does not necessarily mean the efficiency of the group.

Now, using Fig. 2, we present the relationship between the projections obtained from the enhanced Russell model and those obtained from our proposed model, and study some properties of the latter. The PPS with VRS containing group A is as follows, with ∂T_v^g as the frontier of this PPS. We would like to evaluate how the proposed method works on group A. As was shown in Theorem 3, the projection point obtained from the proposed method is Pareto efficient, but lies on ∂T_v^g. Moreover, by the constraints of model (2), it can be inferred that the projection point dominates the group under assessment. Therefore, by Fig. 2, it can be said that the projection point will lie on arc BC of ∂T_v^g. This projection is obviously technically efficient. Now suppose we want to obtain the group members by direct projection onto the efficient frontier T_v^g by the enhanced Russell model in the first place, and then obtain a pattern for evaluating group performance by summing these projections. We know that the projection

points corresponding to each unit certainly dominate the units, and finally the sum of these projections dominates the group. The pattern obtained from the summation of the projections lies in the hatched area. Also, it can be located on the frontier of this area, i.e., arc BC. But as will be shown in the numerical example in section three, it will not necessarily lie on ∂T_v^g and would therefore be a point F, which is technically inefficient. Now we deal with the relationship between the group projection by the proposed method and the projection obtained from direct projection methods such as the enhanced Russell method. It should be mentioned that the projection point of our propose method is certainly efficient, but this is not the case for the other method (point F). This does not mean that the point obtained from our method definitely dominates that of the other method. We will face such a case in the numerical example. For instance, considering Fig. 2, this point can lie in the BD or EC region of the frontier, which does not dominate F. However, as is shown in the figure, there exists a point on ∂T_v^g which dominates F. Such points can be obtained by adding appropriate constraints to model (2). This can also be achieved in another way. It is sufficient to apply the proposed method to F. Considering what has been stated so far, the projection point of F will lie on arc DE by using the proposed method. As can be observed, we can use this method to obtain points with higher efficiency compared to those obtained from ordinary methods which do not consider the possibility of trade-off among units. Such ordinary methods assign a higher efficiency score to the group when evaluating it and, in fact, they do not consider the inefficiency resulting from the lack of trade-off.
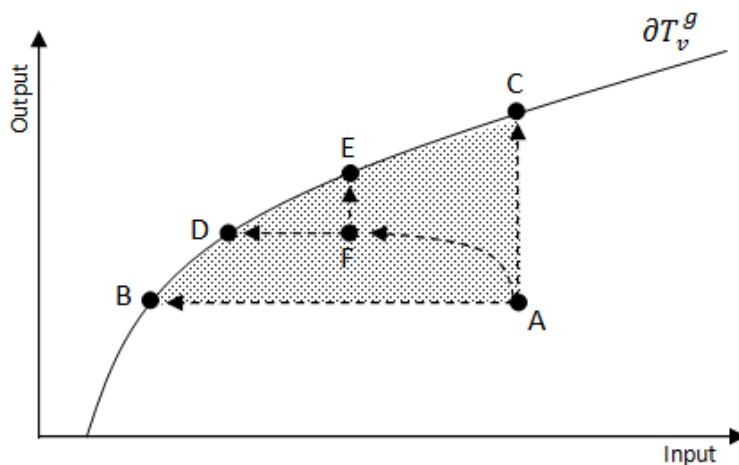
Fig. 2. Comparison between the projection obtained from our proposed method and the sum of projections obtained from ERM model.

2.2 Transformation of proposed model into a linear form

Model (2) has a fractional programming structure. We can therefore utilize the approach in Charnes and Cooper [10] to transform the nonlinear (non-convex) model (2) into an ordinary linear programming formulation, which is readily solved by the many computer codes now available. For this purpose, we introduce new variables:

$$\beta = \left(\frac{\sum \varphi_k}{s}\right)^{-1}, \qquad 0 < \beta \leq 1$$

$$u_i = \beta \, \theta_i, \quad i = 1,...,m$$
$$v_k = \beta \, \varphi_k, \quad k = 1,...,s \qquad (5)$$
$$t_{jr} = \beta \, \lambda_{jr}, \quad j,r = 1,...,n$$

We can then multiply the numerator and denominator in the objective function of (2) by β without changing its value. Also, since $\beta > 0$, we can multiply both sides of the constraints in (2) by β without changing any of the orientations. We then write the thus transformed model as follows:

$$\min \; \frac{\sum_{i=1}^{m} u_i}{m}$$

$$s.t. \; \sum_{r=1}^{n}\sum_{j=1}^{n} t_{jr} x_{ij} \leq u_i \sum_{r=1}^{n} x_{ir} \quad i = 1,...,m$$

$$\sum_{r=1}^{n}\sum_{j=1}^{n} t_{jr} y_{kj} \geq v_k \sum_{r=1}^{n} y_{kr} \quad k = 1,...,s$$

$$\sum_{j=1}^{n} t_{jr} = \beta, \qquad r = 1,...,n$$

$$t_{jr} \geq 0, \qquad j,r = 1,...,n \qquad (6)$$

$$u_i \leq \beta, \qquad i = 1,...,m$$

$$v_k \geq \beta, \qquad k = 1,...,s$$

$$\sum_{k=1}^{s} v_k = s$$

$$0 \leq \beta \leq 1$$

Notice that, in the formulation of (6), we do not need to state explicitly that $\beta > 0$. It is sufficient to state that $\beta \geq 0$ since any feasible solution of (6) necessarily satisfies this inequality strictly. We now have an ordinary linear programming formulation that can be readily solved.

Suppose, therefore, that we have a solution to (6). If we divide both sides of the constraints in (6) by factor β, we can reproduce the constraints in (2). Factoring β from the objective function in (6), and utilizing the definition in (5), we recover the objective function in (2). Also by using

$$\theta_i = u_i / \beta, \qquad i = 1,...,m$$

$$\varphi_k = v_k / \beta, \qquad k = 1,...,s$$

$$\lambda_{jr} = t_{jr} / \beta, \quad j,r = 1,...,n$$

We can obtain the optimal amount of the variables of model (2).

3. Numerical Example

In this section, we try to elaborate on our propose method through a simple numerical example. Consider 7 units with a single input and a single output in T_v. We intend to evaluate the group consisting of these 7 units. The data of the inputs and outputs of these units is provided in the second and third columns of Table 1. As can be observed, the input and output of the group are 21.5 and 30, respectively. First, we evaluate the group by our proposed method. The projection points of each unit by the proposed method can be seen in Fig. 3. Points F,E,A are projected onto point C, and points B,G,D,C are projected onto point G. Numerical values of these projection points can be seen in the fourth and fifth columns of Table 1. The total input and output levels of these two projection points are 11.5 and 30, respectively. Since the projection points lie in T_v, by the definition of T_v^g, $(\hat{x}, \hat{y}) = (11.5, 30)$ is also in T_v^g and dominates the group $(21.5, 30)$. It should be mentioned that by Theorem 2, $(\hat{x}, \hat{y}) = (11.5, 30)$ is Pareto efficient. Therefore, if it is evaluated by the proposed method, it should obtain an efficiency score of 1, as seen in column 10 of Table 1. You can also see in column 10 the efficiency score given by our proposed model when evaluating the group. Thus, the score assigned to the group under evaluation by this method is 0.534884. Regarding the projection points, it is obvious that the inefficiency of the group lies in incorrect use of its inputs.

Table 1: Data for the Numerical Example

DMU	Input	Output	$\hat{x}$	$\hat{y}$	X'	Y'	X"	Y"
A	5.5	7	2.5	6	5.5	7	2.5	6
B	2	2	1	3	1	3	2.5	6
C	2.5	6	1	3	2.5	6	2.5	6
D	3	5	1	3	2	5	1.5	4
E	4	4	2.5	6	1.5	4	1	3
F	3.5	3	2.5	6	1	3	1	3
G	1	3	1	3	1	3	1	3
SUM	21.5	30	11.5	30	14.5	31	12	31
T-Eff	0.534884		1		0.827586		1	

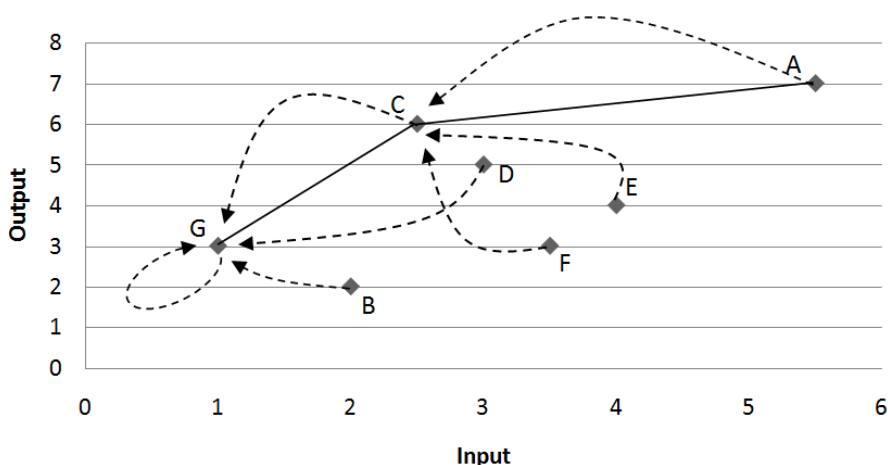T-Eff ~Technical Efficiency Obtained from the Proposed Model

Fig. 3. The projection of units using the proposed model.

Now we evaluate each unit by the enhanced Russell method. In Fig. 4, the projection points of each unit by ERM can be seen. Also, the numerical values of these projection points are given in columns 6 and 7 of Table 1. The total input and output levels of these projection points are 14.5 and 31, respectively. Because these projection points also lie in T_v, by the definition of T_v^g, $(x^{'},y^{'})=(14.5,31)$ is in T_v^g, as well, and dominates the group (21.5,30). But $(x^{'},y^{'})$

does not necessarily lie on the ∂T_v^g. In order to evaluate the position of $(x^{'},y^{'})$ in T_v^g, we evaluate it by the proposed method. The score given to $(x^{'},y^{'})$ by this method is 0.827586, which implies the inefficiency of $(x^{'},y^{'})$ in T_v^g. However, as was mentioned with respect to Fig. 2, the efficiency of $(\hat{x},\hat{y})$ and the inefficiency of $(x^{'},y^{'})$ does not imply that $(\hat{x},\hat{y})$ dominates $(x^{'},y^{'})$, as can be seen from their respective numerical values.
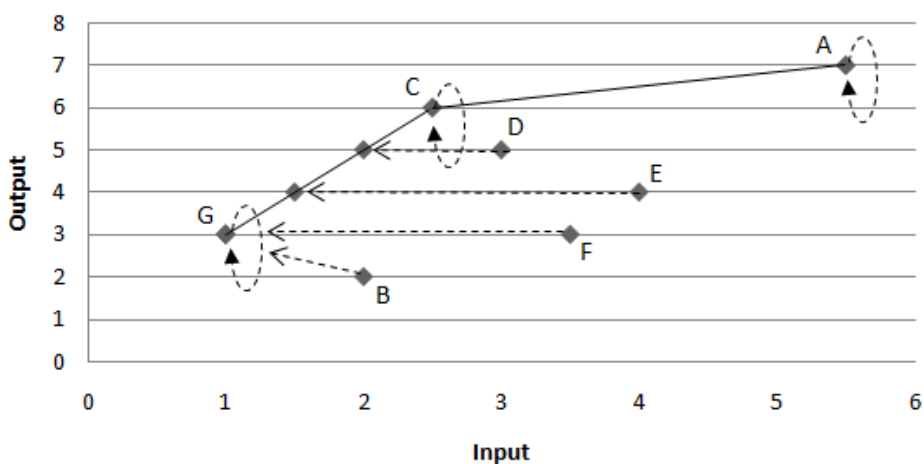

Fig. 4. The projection of units by using ERM model.

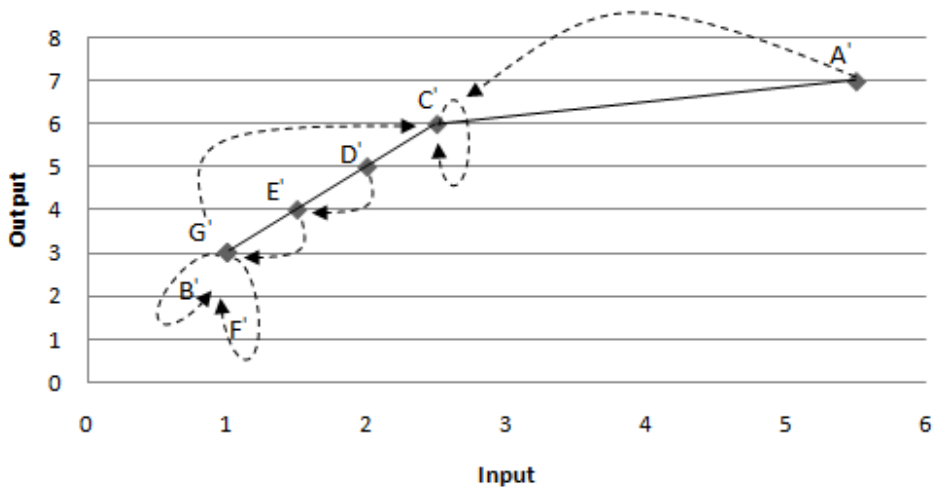**Fig. 5. The projections obtained from ERM model using the proposed model.**

In Fig. 5, points A',B',C',D',E',F',G' are the projection points resulting from ERM model, and their sum makes up (x',y'). The projection given by the proposed model after evaluating (x',y') can be seen in this figure. Point C' is the projection point of A', C', G', point G' is the projection of B', E', F', and point G' is the projection of D'. The numerical values for these projected points are given in columns 8 and 9 of Table 1. As can be seen, the total input and output levels of these projection points are $(x'',y'')=(12,31)$. By evaluating (x'',y'') using the proposed model, an efficiency score of 1 is obtained. Now we have two efficient points (x'',y'') and $(\hat{x},\hat{y})$. Obviously neither of them dominates the other, and either of them can be considered as a target for the group under study.

A few points are worth mentioning with respect to the proposed method and example.

1. First, we should pay attention to the targets that the group sets for itself and its members. In Fig. 2, any point of arc BC can be a target for group A. For instance, in the numerical example above, it was observed that the two extreme points (11.5,30) and (12,31) are targets for the

group (21.5,30). In setting these targets, none of the inputs or outputs was given priority over others. In other words, in model (2) a weight of 1 was assigned to all inputs and outputs. However, other points on the frontier can be selected by considering weights that reflect the decision maker's preferences. Also we might be able to employ other methods which are used in target setting. See [3,6,16,19,22,25] for examples. Moreover, different targets can be selected for each target point corresponding to the group. One of these methods is to employ the alternative optimal solutions of model (2). This is because as can be seen in the numerical example and Figures 2 and 4, the projected points by the propose model for the units are not necessarily closest to them, and there might be alternative optimal solutions that are nearer to the units and seem more reasonable for the units to select as their targets. There are different approaches to reach such targets, which can be found in the literature. For instance, some of the approaches for finding the nearest target, pointed out by Aparicio et al. [2], can be mentioned.

2. Another point to be taken into account is the possibility of braking up some very

large units. With regard to the results obtained and shown in the figures, it can be reasoned that some larger units can be broken up into smaller units to improve group efficiency. The reduction in the size of some units can be attributed to this issue. For instance, in Fig. 3, if the input of unit A is very large but its output is not much larger than that of C, our proposed method suggests breaking up the unit. This is while ERM does not do so, and it will demonstrate a lot of inefficiency in the group. Such a situation can be more readily seen in Fig. 6, where unit A consumes 20 units of input but produces only one more unit of output than unit C whose input is 2.5. If unit A is broken up into smaller units, group output production will certainly increase. In fact, this unit is said to be size inefficient (A unit is ''too large'' if breaking it up into a number of smaller units would result in a larger output bundle than what could be produced from the same input by a single unit. When this is the case, the unit is not size efficient.) For more information on size efficiency, see [25]. We do not claim to be able to remove all size inefficiencies of units by our proposed method but, considering the above example, one of the

differences between this method and others can be the removing of a lot of inefficiencies due to the large size of some units, which would then lead to an improvement in group efficiency. In other way, it seems that the proposed method for large-size DMUs, with decreasing return to scale and small-size DMUs, with increasing return to scale, just with solving one LP, introduces targets around the MPSS range.

4. Empirical Example
Let us consider the following example from Ali et al. [1], Seiford and Zhu [26], and Valter Boljuncic [8]. There are eleven DMUs, each using two inputs to produce two outputs, see Table 2. We intend to evaluate the group consisting of these 11 DMUs with inputs (818,592) and outputs (1660,929), using our proposed method. In columns 2-5 of Table 2, the inputs and outputs of the 11 units are provided. In columns 6-9, the projections obtained for each unit using our proposed method can be seen. The last row shows that group efficiency with the proposed method is 0.488047, and the model suggests using input (440,330) and output (1760,1100).
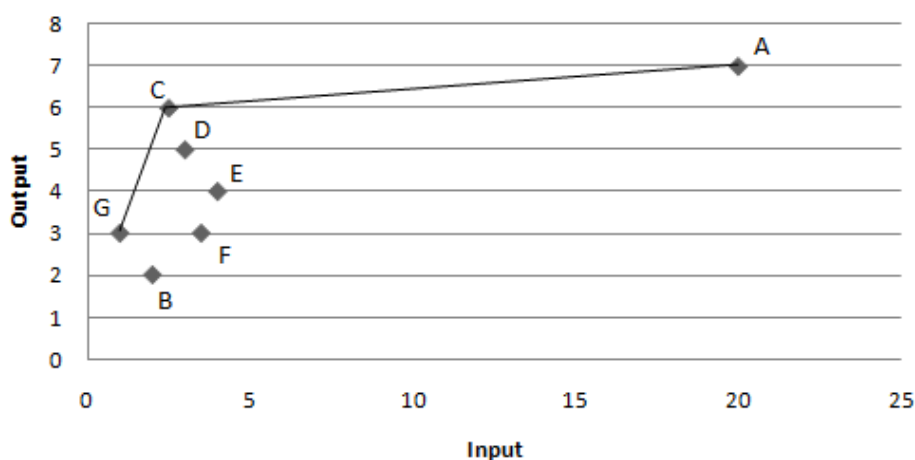

Fig. 6. Unit A is size inefficient and it should be broken up to improve group efficiency.

Table 2: Data for the Empirical Example

DMU	Input 1	Input 2	Output 1	Output 2	X_1'	X_2'	Y_1'	Y_2'
1	40	30	160	100	40	30	160	100
2	30	60	180	70	40	30	160	100
3	93	40	170	60	40	30	160	100
4	50	70	190	130	40	30	160	100
5	80	30	180	120	40	30	160	100
6	35	45	140	82	40	30	160	100
7	105	75	120	90	40	30	160	100
8	97	67	100	82	40	30	160	100
9	100	50	140	40	40	30	160	100
10	90	60	140	105	40	30	160	100
11	98	65	140	50	40	30	160	100
SUM	818	592	1660	929	440	330	1760	1100
T-Eff	0.488047							

5. Conclusion

In this paper, using Lozano et al.'s centralized scenario and the enhanced Russell measure, we introduced an index to measure group efficiency. To do so, by assuming the possibility of reallocation among units, we set targets for each unit as well as for the group. In previous works only the group has been evaluated, but there is no suggestion for better operation of each DMU. In this paper we not only set target for the group but also set targets for each unit. Using model (2), and by taking into account the potential of the units, it can be found out how much inefficient group needs to modify its inputs and outputs in order to achieve efficiency. In this work, we considered the case when efficient operation of individual economic units does not necessarily imply efficiency for a group of these units. But we showed that technical efficiency of the group implies the efficiency of individual units, and so the defined efficiency index was consistent. We tried to establish the validity of the proposed method by a numerical as well as an empirical example.

A number of points seem necessary to be mentioned regarding the extension of this work. In this paper we considered the case where reallocation is possible for all inputs, while this might not be the case and reallocation might be impossible owing to some reasons. For instance, the cost of transferring resources from one unit to another might be too high as a result of remoteness of the geographical location of units from each other, and consequently reallocation will be possible only among subunits of a unit. In such a case, the centralized scenario should be employed such that only a subunit of inputs in which reallocation is possible is utilized. Moreover, as can be seen in Figures 3 and 5 in the empirical example, when the centralized scenario is used, some efficient units also move on the frontier, a case which might be of little interest to the decision maker. This problem can be removed by some minor revisions in the model. Asmild [4] transformed the centralized resource allocation model in a way that it only modified inefficient units. Also, there is the possibility that it may be more efficient for the organization as a whole

to permit some existing DMUs to be eliminated entirely. As was mentioned earlier, Lozano et al. [19] considered such a case. We attempted to consider the least complicated case in this paper.

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