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An Algorithm for the Anchor Points of the PPS of the CCR Model

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Abstract
Anchor DMUs are a new class in the general classification of Decision Making Units (DMUs) in Data Envelopment Analysis (DEA). An anchor DMU in DEA is an extreme-efficient DMU that defines the transition from the efficient frontier to the free-disposability part of the boundary of the Production Possibility Set (PPS). In this paper, the anchor points of the PPS of the CCR model are investigated. A basic definition of anchor point based on the supporting hyperplanes of the PPS of CCR model is provided. Then, by using a variant of super-efficiency models, the necessary and sufficient conditions for a DMU to be an anchor DMU are provided via some theorems. To illustrate the applicability of the proposed model, some numerical examples are finally given.

Keywords: Data Envelopment Analysis (DEA); Production Possibility Set (PPS), Efficient and inefficient frontier.

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1. Introduction

An anchor point in DEA is an extreme-efficient DMU lying on the intersection of some strongly and weakly efficient frontiers of the PPS. An anchor point is, therefore, an extreme-efficient DMU in which some inputs can increase and/or some outputs can decrease without passing through the interior of the PPS. Anchor points play a significant role in DEA theory and applications. The concept of anchor point was used in Thanassoulis and Allen [11] (1998) for the generation of unobserved DMUs in order to reduce appropriately the DEA-inefficient boundary of the PPS. Anchor points were first named and identified by Allen and Thanassoulis [2] (2004). They proposed a method for detecting anchor points of the constant returns to scale production possibility set (CRS-PPS) with one input and multiple outputs. However, their method is not applicable to multiple inputs and outputs. Thanassoulis et al. [11](2012) proposed another approach to identify anchor points, using the radial efficiency scores and slack variables at the optimal solution of envelopment models. They extended the proposed approach in Allen and Thanassoulis [12] (2004) to the multiple inputs and outputs case in variable returns to scale production possibility set (VRS-PPS) in order to improve envelopment by means of unobserved DMUs. Bougnol and Dulá [3] (2009) defined the anchor point for the VRS-PPS. They provided a specialized procedure to identify anchor points based on their geometrical properties. Rouse [10](2004) employed this idea in identifying prices for health care services. For more detail about the notion and applications of the anchor DMUs, see Bougnol [4] (2001) and Allen and Thanassoulis [2] (2004). Since the set of anchor DMUs is a subset of the set of extreme DMUs, the set of extreme DMUs must be obtained. For this aim, one can use the proposed algorithms in Charnes, Cooper and Thrall [6] (1991) as well as Dulá and López [7] (2006) among others. This paper provides a definition for the anchor point of the CRS-PPS. Subsequently, it utilizes a novel approach to identify the anchor points of the PPS of the CCR model in the multiple inputs and outputs case; through testing all CCR-efficient DMUs by a variant of super-efficiency models (see models (3) and (4), after eliminating the inefficient-CCR DMUs from the PPS). In fact, extreme and non-extreme CCR-efficient DMUs and anchor DMUs can be obtained, using models (3) and (4). An advantage of the proposed approach is in determining which inputs (outputs) of anchor DMUs can increase (decrease) without penetrating into the interior of the production possibility set. Another advantage of the proposed approach is in discovering the edges of the PPS on which the anchor DMUs lie; whereas the aforementioned methods are unable to do these two advantages. Some useful facts related to the properties of models (3) and (4) and the necessary and sufficient conditions for a DMU to be an anchor DMU are stated and proved. In addition, three numerical examples are provided.

2. Background

Consider a set of *n* DMUs which is associated with *m* inputs and *s* outputs. Particularly, each $DMU_j = (X_j;\ Y_j)$ $(j \in J = \{1,\dots,n\})$ consumes amount x_{ij} (> 0) of input $\boldsymbol{i}$ and produces amount y_{rj} (> 0) of output $\boldsymbol{r}$. The production possibility set *T*, $T \subset \left\{ (X;\ Y) \mid X \in E^m, Y \in E^s, X \geq 0, Y \geq 0 \right\}$ is based on postulate sets which are presented with a brief explanation (see Banker et al. [13] (1984)). One of the DEA models to evaluate the relative efficiency of a set of DMUs is the CCR model, which is, proposed by Charnes et al. [5](1978). The (PPS) of the CCR model can be defined as follows (Charnes et al. [5] (1978)):

$$T=\left\{(X,Y)\,\middle|\,X\geq\sum_{j\in J}\lambda_j X_j,\ Y\leq\sum_{j\in J}\lambda_j Y_j,\ \lambda_j\geq 0,\ j\in J\right\}$$

in which X_j and Y_j are vectors of inputs and outputs of DMU_j, respectively.

We will employ a DMU classification based on the categories (i) CCR-inefficient (weak efficient and interior), (ii) non-extreme CCR-efficient, and (iii) extreme CCR-efficient. The three categories define the subsets `$F$ ', `E', and `E ', respectively. These three subsets partition the set J. Any DMU in E; lies on the boundary (non-extreme) ray and any DMU in E; lies on the extreme ray of the PPS of the CCR model and named as *extreme* DMU.

The set $E^*$is also called the *frame* of J. The frames are important in DEA because the PPS of the DEA models are constructed by them and the exclusion each of them alters the shape of the PPS. The PPS of the CCR model is depicted in Figures (1) and (2). In Figure (1), $J=\{D_1,D_2,D_3,D_4\}$, $F=\{D_4\}$, $E=\{D_3\}$ and $E^*=\{D_1,D_2\}$. Also D_1 and D_2 are anchor DMUs.

The input-oriented CCR model, corresponds to $DMU_k, k\in J$, is given by:

$$\begin{aligned}
&\min\ \theta-\varepsilon\left(\sum_{i=1}^{m}s_i^-+\sum_{r=1}^{s}s_r^+\right)\\
&s.t.\ \sum_{j\in J}\lambda_j y_{rj}-s_r^+=y_{rk},\quad r=1,\ \dots,\ s\\
&\sum_{j\in J}\lambda_j x_{ij}+s_i^-=\theta x_{ik},\qquad i=1,\ \dots,\ m\\
&\lambda_j\geq 0, \qquad\qquad\qquad\qquad (1)\\
&s_r^+\geq 0,\\
&s_i^-\geq 0,\\
&\theta \qquad\qquad\qquad\qquad free
\end{aligned}$$

Also, the output-oriented CCR model, corresponds to $DMU_k, k\in J$, is as follows:

$$\begin{aligned}
&\max\ \varphi+\varepsilon\left(\sum_{i=1}^{m}t_i^-+\sum_{r=1}^{s}t_r^+\right)\\
&s.t.\ \sum_{j\in J}\lambda_j y_{rj}-t_r^+=\varphi y_{rk},\ r=1,\ \dots,\ s\\
&\sum_{j\in J}\lambda_j x_{ij}+t_i^-=x_{ik},\quad i=1,\ \dots,\ m\\
&\lambda_j\geq 0, \qquad\qquad\qquad (2)\\
&t_r^+\geq 0,\\
&t_i^-\geq 0,\\
&\varphi \qquad\qquad\qquad free
\end{aligned}$$

Where ϵ is non-Archimedean small and positive number and s_i^-, s_r^+, t_i^-, and t_i^-, $i=1,\ \dots,\ m,$ $r=1,\ \dots.,s$ are called slack variables belonging to $R^{\geq 0}$. Note that s_i^- and t_i^- represent input excesses; also s_r^+ and t_r^+ represent output shortfalls. In here, θ, φ and $\lambda_j(\geq 0),\ j\in J,$ are real numbers. Models (1) and (2) are called envelopment forms (with non-Archimedean number).

DMU_k is said to be *strongly efficient* (*CCR-efficient*) if and only if for each optimal solutions, either (*i*) or (*ii*) happen:

(i) $\theta^*=1$ and $(s^{+*},\ s^{-*})=(0,\ 0)$

(ii) $\varphi^*=1$ and $(t^{+*},\ t^{-*})=(0,\ 0)$

DMU_k is said to be *weak efficient* if and only if for some optimal solutions, either (*v*) or (*iv*) happen:

(v) $\theta^*=1$ and $(s^{+*},\ s^{-*})\neq(0,\ 0)$

(iv) $\varphi^*=1$ and $(t^{+*},\ t^{-*})\neq(0,\ 0)$

Note that if $\theta^*<1$ and $\varphi^*>1$ then DMU_k is an *interior point of the PPS*[1].

Each inefficient and weak efficient DMU in the CCR model is said to be a *CCR-inefficient DMU*.

Efficient Frontier is the set of all points (real or virtual DMUs) with efficiency score is equal to unity. Efficient frontier is divided into two categories:

i) *Strong efficient frontier* is the set of all (real or virtual) strong efficient (CCR efficient) DMUs.

ii) *Weak efficient frontier* in which all its relative interior points (real or virtual DMUs) are weak efficient DMUs.

$DMU_k = (X_k, Y_k)$ is said to be *non-dominated* if and only if there is not any $DMU = (X, Y)$ (real or virtual) such that: $(-X_k, Y_k) \geq (-X, Y)$ and $(-X_k, Y_k) \neq (-X, Y)$

We use the following theorem in the next section.

Theorem 1: *There does not exist any virtual DMU (a member of the PPS) that dominates an DEA-efficient DMU.*

Proof. See H. Fukuyama et al. (2012).

In this paper, corresponding to each strong efficient DMU $DMU_j = (x_{1j},...,x_{mj}, y_{1j},..., y_{sj})$, the virtual DMUs $DMU_j^l = \left(x_{1j},...,x_{lj}+\alpha,...,x_{mj}, y_{1j},..., y_{sj}\right)$ and $DMU_j^q = \left(x_{1j},...,x_{mj}, y_{1j},...y_{qj}-\gamma,..., y_{sj}\right)$ in which $\alpha, \gamma > 0$, are called "*Dominated Input Virtual*" DIV_k^l and "*Dominated Output Virtual*" DOV_k^q DMUs, respectively. These virtual DMUs are either interior points of the PPS of the CCR model or lie on the some weak efficient frontiers (see theorems 3 and 6 and the proof of theorem 4). In the latter case we call these virtual DMUs as "*weak efficient virtual DMUs*" or *WEV* DMUs, hereafter. It is important to note that the *WEV* DMUs play an important role in identifying anchor points. Accordingly, this paper tries to find them. In figure 1, DMU $D_1 = \left(x_{12}, x_{22}, y_{12}\right)$ is strong efficient DMU and DIV_1^2, $D_1' = \left(x_{12}, x_{22}+\alpha, y_{12}\right)$ is a WEV DMU. In addition, in figure 2, DOV_1^2 D_1' and DOV_1^1 D_1''', corresponding to DMU D_1, are WEV DMUs.

The following definition introduces the anchor points of the PPS of the CCR model.

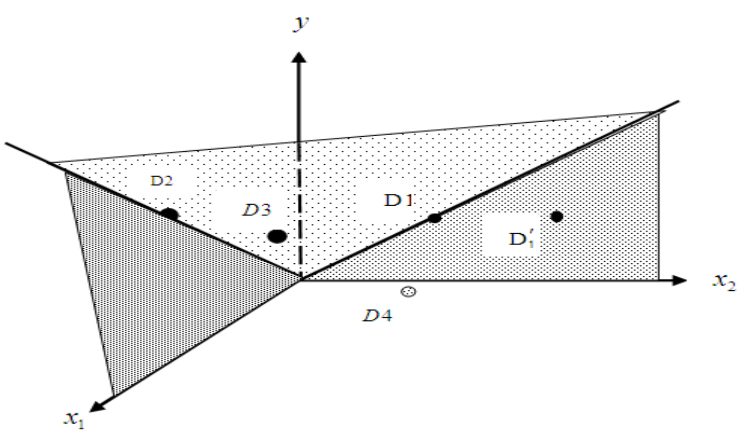

Figure 1: DIV_2^1 DMU D_2 is a WEV DMU and also model 3 corresponding to D_2 with l=1 is infeasible therefore, D_2 is an anchor point.

1. (*) is used for optimal solution

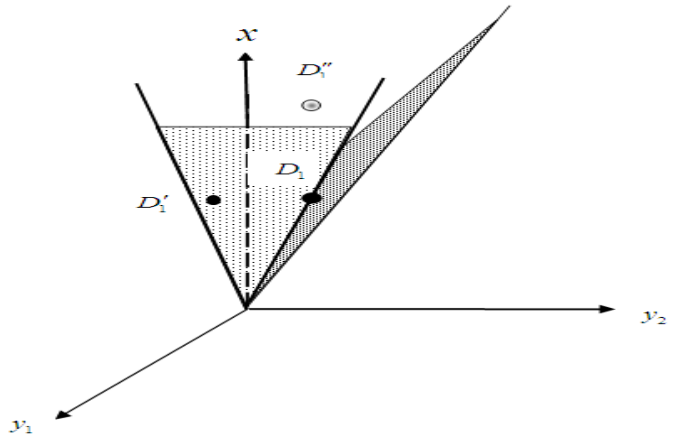

Figure 2: Model 4 corresponding to DMU D_1 with *q* = 2 is infeasible therefore, *D*1 is an anchor point. Also, DIV_1^1 D_1'' is an interior point 7.

Definition. *$DMU_k \in E^*$ is an anchor DMU if it belongs to an unbounded face of the PPS of the CCR model.*

Remark1. *By the above definition; $DMU_k \in E^*$ is an anchor DMU if there exist some l (or q) so that DIV_k^l (or DOV_k^q) DMUs are WEV DMUs.*

In figure 1, DIV_2^1 DMU D_2' is a WEV DMU and so, D_2' is an anchor point.

Throughout this paper, we must assume that there are not any two strong efficient DMUs as (x,y) and (tx,ty) for all $t > 0$ and $t \neq 1$. Otherwise, one of them must be deleted.

3. Identifying the anchor DMUs of the PPS of the CCR model

In this section, the anchor DMUs of the PPS of the CCR model are defined in the following way.

First, each DMU_k; $(k \in J)$, is evaluated by models (1) or (2). Then, we hold all CCR-efficient DMUs, and remove other DMUs. Suppose that the set of all CCR-efficient DMUs is denoted by $E' (= E \cup E^*)$. Corresponding to each $DMU_k = (x_{1k},\dots,x_{mk},y_{1k},\dots,y_{sk})$, $(k \in E')$, the following models are solved:

$$
\begin{aligned}
\min\ & \theta_l^k \\
s.t.\ & \sum_{j \in E' - \{k\}} \lambda_j^k y_{rj} \geq y_{rk}, \\
& r = 1,\ \dots,\ s \\
& \sum_{j \in E' - \{k\}} \lambda_j^k x_{ij} \leq x_{ik}, \\
& i = 1,\ \dots,\ m,\ i \neq l \\
& \sum_{j \in E' - \{k\}} \lambda_j^k x_{lj} \leq \theta_l^k x_{lk}, \\
& \lambda_j^k \geq 0, \qquad j \in E' - \{k\} \\
& \theta_l^k \qquad \text{free} \quad l = 1,\ \dots,\ m
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
\max\ & \varphi_q^k \\
s.t.\ & \sum_{j \in E' - \{k\}} \mu_j^k y_{rj} \geq y_{rk}, \\
& r = 1,\ \dots,\ s,\ r \neq q \\
& \sum_{j \in E' - \{k\}} \mu_j^k y_{qj} \geq \varphi_q^k y_{qk}, \\
& \sum_{j \in E' - \{k\}} \mu_j^k x_{ij} \leq x_{ik}, \\
& i = 1,\ \dots,\ m,
\end{aligned}
\tag{4}
$$

$\mu_j^k \geq 0, \qquad j \in E' - \{k\}$

$\varphi_q^k \qquad free \quad q = 1, \ldots, s$

The following Theorems are held for models (3) and (4). The theorems 3-8 provide the necessary and sufficient conditions for a DMU to be an anchor DMU.

Theorem 2: *In model (3) (or (4)), if for some l (or q), $\theta_l^{k*} > 1$ (or, $\varphi_q^{k*} < 1$) or if for some l (or q), model (3) (or model (4)) is infeasible, then, DMU_k is an extreme DMU and vice versa.*

Proof. Suppose that $\theta_l^{k*} > 1$. First, we show that DMU_k is CCR-efficient. By contradiction, suppose that DMU_k is CCR-inefficient. Let $(\theta^*, S^{+*}, S^{-*})$ be the optimal solution of model (1).
Two cases can occur:

(i) $\theta^* = 1$ and $(S^{+*}, S^{-*}) \neq (0, 0)$

(ii) $\theta^* < 1$

in each case it can be shown that $\theta_l^{k*} \leq 1$, a contradiction.

Now we show that DMU_k is, in fact, an extreme CCR-efficient DMU. By contradiction suppose that DMU_k is a non-extreme CCR-efficient. So, the following system has solution:

$$\sum_{j \in E'} \lambda_j y_j = y_k,$$
$$\sum_{j \in E'} \lambda_j x_j = x_k, \tag{5}$$
$$\lambda_j \geq 0, \quad j \in E'$$

Suppose that $\bar{\lambda}_j, \quad j \in E'$ is a solution of the above system. If $\bar{\lambda}_k = 0$ then, $(\theta_l^k = 1, \ \lambda_j = \bar{\lambda}_j, \ j \in E - \{k\})$ is a solution of model (3). Therefore, $\theta_l^{k*} \leq 1$, a contradiction. On the other hand if $\bar{\lambda}_k \neq 0$ system (5) can be rewritten as follows:

$$\sum_{j \in E' - \{k\}} \bar{\lambda}_j y_j = (1 - \bar{\lambda}_k) y_k,$$
$$\sum_{j \in E' - \{k\}} \bar{\lambda}_j x_j = (1 - \bar{\lambda}_k) x_k$$

By dividing both sides of the above equations by $(1 - \bar{\lambda}_k > 0)$; a solution of model (3) is obtained as $\left(\theta_l^k = 1, \ \lambda_j = \dfrac{\bar{\lambda}_j}{1 - \bar{\lambda}_j}, \ j \in E' \text{-} \{k\}\right)$.

Therefore, $\theta_l^{k*} \leq 1$, a contradiction. Thus, DMU_k is an extreme CCR-efficient DMU. Now, suppose that for some *l*, model (3) is infeasible. In the similar manner, it can be shown that DMU_k is an extreme CCR-efficient DMU. Conversely, suppose that DMU_k is extreme DMU and model (3) is feasible. We show that $\theta_l^{k*} \geq 1$. Consider the following problem corresponding to DMU_k:

$$\begin{aligned}
\min \quad & \theta_l^{'k} \\
s.t. \quad & \sum_{j \in E'} \lambda_j^k y_{rj} - s_r^+ = y_{rk}, \\
& r = 1, \ldots, s \\
& \sum_{j \in E'} \lambda_j^k x_{ij} + s_i^- = x_{ik}, \\
& i = 1, \ldots, m, \ i \neq l \\
& \sum_{j \in E'} \lambda_j^k x_{lj} + s_l^- = \theta_l^{'k} x_{lk}, \\
& \lambda_j^k \geq 0, \qquad j \in E' \\
& s_r^+, s_i^- \geq 0, \quad r = 1, \ldots, s \quad i = 1, \ldots, m, \\
& \theta_l^{'k} \qquad free \quad l = 1, \ldots, m
\end{aligned} \tag{6}$$

Now suppose that $\theta^* (=1)$, $\theta_l^{'k*}$ and θ_l^{k*} are the optimal objective functions of the models (1), (6) and (3) with respect to DMU_k, respectively. It is not difficult to show that $\theta^* \le \theta_l^{'k*} \le \theta_l^{k*}$ Therefore, $\theta_l^{k*} \ge 1$. This completes the proof.

Corollary: *In models (3) and (4), for each l and q* $\theta_l^{k*} = \varphi_q^{k*} = 1$ *if and only if DM U_k is a non-extreme CCR-efficient DMU.*

Proof. Omitted.

Theorem 3: *In a single input case, corresponding to each* $DMU_k = (x_{1k}, y_{1k}, ..., y_{sk})$ *the* DIV_k^1 $DMU_k^{'} = (x_{1k} + \alpha, y_{1k}, ..., y_{sk})$ *in which* $\alpha > 0$, *is an interior point of the PPS of the CCR model.*

Proof. First, we add $DMU_k^{'}$ to the PPS and then, evaluate its performance by the input and output-oriented CCR models (see models (1) and (2)). It is enough to show that $\theta^* < 1$ and $\varphi^* > 1$. Consider the input-oriented CCR model corresponding to virtual DMU DMU_k' as follows:

$$
\begin{aligned}
&\min\ \theta \\
&s.t.\ \ \sum_{j \in E'} \lambda_j y_{rj} + \mu_k y_{rk} \ge y_{rk},\ \ r = 1,\ ...,\ s \\
&\sum_{j \in E'} \lambda_j x_{1j} + \mu_k (x_{1k} + \alpha) \le \theta (x_{1k} + \alpha), \qquad (7) \\
&\lambda_j \ge 0, \qquad\qquad j \in E' \\
&\theta \qquad\qquad\quad free
\end{aligned}
$$

$$\left(\bar{\lambda}_j = 0\ (j \ne k),\ \bar{\lambda}_k = 1,\ \bar{\mu}_k = 0,\ \theta = \frac{x_{1k}}{x_{1k} + \alpha} (<1) \right)$$

is a feasible solution of (7). Since model (7) has a minimization-type objective function, $\theta^* < 1$; where "*" is used to indicate optimality. In a similar manner, it can be shown that in output-oriented maximization problem, $\varphi^* > 1$ Therefore, DMU_k is an interior point of the PPS. This completes the proof.

In Figure 2, corresponding to DMU $D_1 = (x_{11}, y_{11}, y_{21})$, the DIV_1^1 $D_1^{"} = (x_{11} + \alpha, y_{11}, y_{21})$ is an interior point of the PPS.

Theorem 4: *In a multiple inputs case, if for some l, model (3) is infeasible, then extreme CCR-efficient* DMU_k *is an anchor DMU.*

Proof. In view of Remark 1, we show that if for some *l*, model (3) is infeasible, then, the DIV_k^l

$$DMU_k' = \begin{pmatrix} x_{1j}, ..., x_{(l-1)j}, x_{lj} + \alpha, \\ x_{(l+1)j}, ..., x_{mj}, y_{1j}, ..., y_{sj} \end{pmatrix} \alpha > 0,$$

is a WEV DMU. For this aim, it can be shown that in the performance evaluation of DMU_k', using model (2); $\varphi^* = 1$. Consider model (2); corresponding to virtual DMU DMU_k' as follows (without ϵ):

$$
\begin{aligned}
&\max\ \varphi \\
&s.t.\ \ \sum_{j \in E'} \lambda_j y_{rj} + \mu_k y_{rk} \ge \varphi y_{rk}, \\
&\qquad r = 1,\ ...,\ s \\
&\qquad \sum_{j \in E'} \lambda_j x_{ij} + \mu_k x_{ik} \le x_{ik}, \\
&\qquad i = 1,\ ...,\ m,\ i \ne l \qquad\qquad (8) \\
&\qquad \sum_{j \in E'} \lambda_j x_{lj} + \mu_k (x_{lk} + \alpha) \le x_{lk} + \alpha, \\
&\qquad \lambda_j \ge 0, \qquad j \in E' \\
&\qquad \mu_k \ge 0, \\
&\qquad \varphi \qquad\qquad free
\end{aligned}
$$

By contradiction, suppose that $(\lambda_j^* (j \in E'), \mu_k^*,\ \varphi(>1))$ is the optimal solution of (8). The constraints of model (8) can be written as follows:

$$\sum_{j\in E'-\{k\}} \lambda_j^* y_{rj} > \left(1-\lambda_k^* - \mu_k^*\right) y_{rk},$$
$$r = 1, \ldots, s$$
$$\sum_{j\in E'-\{k\}} \lambda_j^* x_{ij} \le \left(1-\lambda_k^* - \mu_k^*\right) x_{ik}, \qquad (9)$$
$$i = 1, \ldots, m,\ i \ne l$$
$$\sum_{j\in E'-\{k\}} \lambda_j x_{lj} \le \left(1-\lambda_k^* - \mu_k^*\right) x_{lk} + \left(1-\mu_k^*\right)\alpha,$$

From model (9), it is easy to show that $1-\lambda_k^* - \mu_k^* > 0$. Divide both sides of model (9) by $1-\lambda_k^* - \mu_k^* > 0$ and define $\mu_j^* = \dfrac{\lambda_j^*}{1-\lambda_k^* - \mu_k^*}$, $j \in E'-\{k\}$; so, model (9) becomes as follows:

$$\sum_{j\in E'-\{k\}} \bar{\mu}_j y_{rj} > y_{rk}, \quad r = 1, \ldots, s$$
$$\sum_{j\in E'-\{k\}} \bar{\mu}_j x_{ij} \le x_{ik}, \quad i = 1, \ldots, m,\ i \ne l \ (10)$$
$$\sum_{j\in E'-\{k\}} \bar{\mu}_j x_{lj} \le x_{lk} + \beta$$

in which $\beta = \left(\dfrac{1-\mu_k^*}{1-\lambda_k^* - \mu_k^*}\right)\alpha$. Since $\beta > 0$, there is $\hat{\theta} > 0$; so that $x_{lk} + \beta = \hat{\theta} x_{lk}$; therefore, the constraints of model (10) can be rewritten as follows:

$$\sum_{j\in E'-\{k\}} \bar{\mu}_j y_{rj} > y_{rk}, \quad r = 1, \ldots, s$$
$$\sum_{j\in E'-\{k\}} \bar{\mu}_j x_{ij} \le x_{ik}, \quad i = 1, \ldots, m,\ i \ne l$$
$$\sum_{j\in E'-\{k\}} \bar{\mu}_j x_{lj} \le \hat{\theta} x_{lk}$$

So, $\left(\bar{\mu}_j\ \left(j \in E'-\{k\}\right),\ \hat{\theta}\right)$ is a feasible solution for model (3); a contradiction.

This implies that $\varphi^* = 1$ i.e. DMU'_k lies on the efficient frontier. Now, since DMU'_k is dominated by CCR-efficient DMU_k, so, the DIV_k^l DMU'_k is a WEV DMU. Therefore, in view of Remark 1 DMU_k is an anchor DMU. This completes the proof.

In Figure 1, model (3) corresponding to DMU $D_2 = (x_{12}, x_{22}, y_{12})$, with l = 1, is infeasible; so, DMU D_2 is an anchor DMU. The following theorem is, in fact, the converse of Theorem 4.

Theorem 5: *In multiple inputs case, if extreme CCR-efficiency DMU* $DMU_k = \left(x_{1k}, \ldots, x_{lk}, \ldots, x_{mk}, \mathrm{y}_{1k}, \ldots, \mathrm{y}_{sk}\right)$ *is an anchor DMU and the* DIV_k^1 *DMU is a WEV DMU; then model (3) is infeasible.*

Proof. By contradiction, suppose that model (3) is feasible. The first constraint of the model (3) implies that the optimal solution of model (3) is bounded. Suppose that, $(\theta_l^{k*},\ \lambda_j^*\ (j \ne k))$ is an optimal solution of it. Note that the first constraint of model (3) is tight at optimality.

We first show that $\theta_l^{k*} > 1$. By contradiction suppose $\theta_l^{k*} \le 1$. If $\theta_l^{k*} < 1$ we have:

$$\sum_{j\in E'-\{k\}} \lambda_j x_{lj} = \theta_l^{k*} x_{lk} < x_{lk}$$
$$\sum_{j\in E'-\{k\}} \lambda_j^{k*} x_{ij} \le x_{ik},$$
$$i = 1, \ldots, m,\ i \ne l \qquad (11)$$
$$\sum_{j\in E'-\{k\}} \lambda_j^{k*} y_{rj} \ge y_{rk},$$
$$r = 1, \ldots, s$$

It shows that virtual DMU

$$\begin{pmatrix} \sum_{j \in E'-\{k\}} \lambda_j^{k*} x_{1j}, \ \ldots, \\ \sum_{j \in E'-\{k\}} \lambda_j^{k*} x_{lj}, \ \ldots, \\ \sum_{j \in E'-\{k\}} \lambda_j^{k*} x_{mj}, \\ \sum_{j \in E'-\{k\}} \lambda_j^{k*} y_{1j}, \ \ldots, \\ \sum_{j \in E'-\{k\}} \lambda_j^{k*} y_{sj} \end{pmatrix}$$

Dominates the CCR-efficient DMU_k, a contradiction (see Theorem 1). Now, if $\theta_l^{k*} = 1$, we have:

$$\begin{aligned} &\sum_{j \in E'-\{k\}} \lambda_j^{k*} x_{lj} = x_{lk} \\ &\sum_{j \in E'-\{k\}} \lambda_j^{k*} x_{ij} \le x_{ik}, \quad i = 1, \ldots, m, \ i \ne l \\ &\sum_{j \in E'-\{k\}} \lambda_j^{k*} y_{rj} \ge y_{rk}, \quad r = 1, \ldots, s \end{aligned} \tag{12}$$

At least one of the inequality constraints of (12) is a strict inequality, because, otherwise, the CCR-efficient DMU_k, is not extreme DMU and so $\theta_l^{k*} > 1$. Therefore, there exist $\beta > 0$ so that $\theta_{lk}^{*} x_{lk} = x_{lk} + \beta$. This means that, the virtual DMU $DMU_k' = \begin{pmatrix} x_{1k}, \ \ldots, x_{(l-1)k}, \ x_{lk} + \beta, \\ x_{(l+1)k}, \ldots, x_{mk}, \ \mathrm{y}_{1k}, \ \ldots, \ \mathrm{y}_{sk} \end{pmatrix}$ is, in fact, an observed DMU belongs to the PPS of the CCR model. This is a contradiction because; the CCR-inefficient DMUs had been eliminated from the PPS of the CCR model. The proof is completed.

Theorem 6: *In a single output case, for each* $DMU_k = (x_{1k}, \ldots, x_{mk}, \ \mathrm{y}_{1k})$ *the* $DOV_k^1 \ DMU_k = (x_{1k}, \ldots, x_{mk}, \ \mathrm{y}_{1k} - \gamma)$ *in which* $\gamma > 0$ *is an interior point of the PPS of the CCR model.*

Proof. The proof is similar to the theorem 3 and so, the details are omitted.
In Figure 3, DOV_1^1 DMU $D' = (x_{11}, \ \mathrm{x}_{12}, \ \mathrm{y}_{11} - \gamma)$, corresponding to DMU $D_1 = (x_{11}, \ \mathrm{x}_{12}, \ \mathrm{y}_{11})$ is an interior point of the PPS.

Theorem 7: *In multiple outputs case, if for some q, model (4) is infeasible, then, CCR-efficient* DMU_k *is an anchor DMU.*

Proof. The proof is similar to theorem 4 except that it can be shown that in the performance evaluation of DMU_k' using model (1); $\theta^* = 1$.
In Figure 2, model (4) corresponding to DMU $D_1 = (x_{11}, \ \mathrm{y}_{11}, \ \mathrm{y}_{21})$, with *q* = 2, is infeasible, so, the DOV_1^2 DMU is a $D_1' = (x_{11}, \ \mathrm{y}_{11}, \ \mathrm{y}_{21} - \gamma)$ WEV DMU
The following theorem is, in fact, the converse of theorem 7.

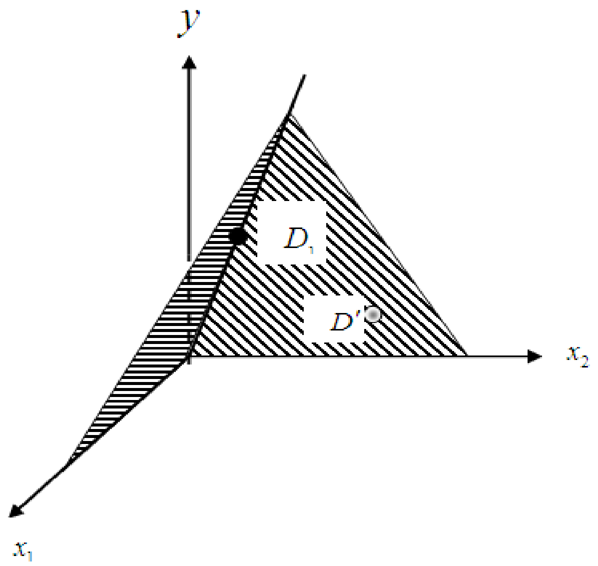

Figure 3: Theorem 6. DOV_1^1 D' **is interior point of the PPS.**

Theorem 8: *In multiple outputs case, let extreme CCR-efficiency DMU* $DMU_k = (x_{1k}, \ldots, x_{mk}, y_{1k}, \ldots, y_{qk}, \ldots, y_{sk})$ *is an anchor DMU and the* DOV_k^q *DMU is a WEV DMU; then model (4) is infeasible.*

Proof. The proof is similar to the theorem 5 except that by contradiction it must be assumed that the model (4) is feasible. So, we omit it.

Now, by theorem 2; all extreme DMUs of the PPS of the CCR model can be found. Also, by theorems 4 and 5; all anchor DMUs for which the DIV_k^l DMUs

$$DMU'_k = \begin{pmatrix} x_{1k}, \ldots, x_{(l-1)k}, x_{lk} + \alpha, \\ x_{(l+1)k} \ldots, x_{mk}, y_{1k}, \ldots, y_{sk} \end{pmatrix}$$

are WEV DMUs can be found and by theorems 7 and 8; all anchor DMUs for which the DOV_k^q DMUs $DMU'_k = (x_{1k}, \ldots, x_{mk}, y_{1k}, \ldots, y_{qk} - \beta, \ldots, y_{sk})$ are WEV DMUs can be found and therefore, all anchor points of the PPS of the CCR model can be found.

Now we are in the position to put all together the ingredients of the method.

Summary of finding all anchor DMUs algorithm

- **Step 1.** Evaluate n DMUs with a suitable form of models (1) and, (2). Hold all CCR-efficient DMUs and remove other DMUs. Put indices of these CCR-efficient DMUs in E'.
- **Step 2.** Evaluate each DMU in E' with models (3) and (4). (Note that in the single input case we don't use model (3) and in the single output case we don't use model (4)).
- **Step 3.** If for some *l* (or *q*) the model (3) (or (4)) is infeasible, then, DMU_k is an anchor DMU and DIV_k^l (or DOV_k^q) DMU'_k is WEV DMU.
- **Step 4.** If each DMU in E' are evaluated by models (3) and (4), stop. Otherwise, go to step 1.

Table 1: Data of Numerical Example 1.

DMU	D1	D2	D3	D4
x_1	1	2	5	6
x_2	4	2	1	1
y	3	4	5	2

Table 2: Example 2. Multiple input and output.

DMU	D_1	D_2	D_3	D_4	D_5
x_1	2	1	2	4	3
x_2	3	2	2	2	5
y_1	7	3	4	6	5
y_2	4	5	3	1	2

4. Numerical Examples

Example 1 *(Single output case)*
Table 1 shows data for 4 DMUs with two inputs and one output. Using the CCR model (1), CCR-efficient DMUs are determined to be D_1, D_2 and D_3. So, $E' = \{1, 2, 3\}$. Remove CCR-inefficient DMU D_3 from PPS and solve model (3) corresponding to CCR-efficient DMUs D_1, D_2 and D_3.
The following results are yielded:
By theorem 2, DMUs D_1, D_2 and D_3 lie on the extreme rays of the PPS. Model (3) corresponding to DMU D_1 with l = 2 and DMU D_3 with l = 1 is infeasible. So, by theorem 4 DMUs D_1 and D_3 are anchor DMUs and DIV_1^2 and DIV_3^l DMUs are WEV DMUs. Note that model (3) corresponding to DMU D_2 with l = 1, 2 is feasible. So, by theorem 5, DMU D_2 is not anchor DMU.

Example 2 *(Multiple outputs and inputs case)*
Table 2 shows data for 5 DMUs with two inputs and two outputs. Running model (1) (or (2)) shows that D_1, D_2 and D_4 are CCR-efficient and other DMUs are CCR-inefficient. So, $E' = \{1, 2, 4\}$. By applying models (3) and (4) to each DMU_k, $k \in E'$ the results reported in table 3 are obtained. In table 3, "INFES" and "FES" denotes "*infeasible*" and "*feasible*", respectively. For instance, "INFES" in the first row and the second column means that model (3), corresponding to DMU D_1 with l=2, is infeasible. So, by theorem 4, D_1 is an anchor DMU and DIV_1^2 DMU $D_1' = (2, 3+\alpha, 7, 4)$ is a WEV DMU. Using theorems 4 and 7 and the information of table 3, all DMU_k, $k \in E'$ are anchor DMUs.

Example 3 *(Real word data)*
We evaluated the data of 20 branches of a bank in Iran using the proposed method. The data was previously analyzed by Amirteimoori et al. (2005), (see table (4)). Running the DEA model (1) (or (2)) resulted in $E' = \{1, 4, 7, 12, 15, 17, 20\}$. Using the proposed method, all DMUs in E' are found to be anchor DMUs. Also $DIV_1^{1,2}$, $DIV_4^{1,2,3}$, $DIV_7^{1,2,3}$, DIV_{12}^{2}, $DIV_{15}^{1,2,3}$, $DIV_{17}^{1,2,3}$, $DIV_{20}^{2,3}$ and DOV_1^2,

$DOV_4^{1,2}$, $DOV_7^{1,3}$, $DOV_{12}^{1,3}$, $DOV_{15}^{2,3}$, $DOV_{17}^{1,3}$, $DOV_{20}^{1,2}$ DMUs are WEV DMUs. For instance, $DIV_{15}^{1,2,3}$ means that, DMU_{15} is an anchor point and the first, second and the third inputs of DMU_{15} can be increased without penetrating the interior of the PPS. Also, $DOV_7^{1,3}$ means that, DMU_7 is an anchor point and the first and third outputs of DMU_7 can decrease without penetrating the interior of the PPS.

Table 3: Example 2. The results of evaluation CCR-efficient DMUs by models (3) and (4).

DMU	l		q	
	1	2	1	2
D_1	FES	INFES	FES	INFES
D_2	INFES	INFES	INFES	FES
D_4	INFES	FES	FES	INFES

Table 4: Example 3. DMUs' data (extracted from [Amirteimoori et al. (2005), p. 689]).

	input			output		
Branch	Staff	Computer terminals	Space m2	Deposits	Loans	Charge
1	0.9503	0.70	0.1550	0.1900	0.5214	0.2926
2	0.7962	0.60	1.0000	0.2266	0.6274	0.4624
3	0.7982	0.75	0.5125	0.2283	0.9703	0.2606
4	0.8651	0.55	0.2100	0.1927	0.6324	1.0000
5	0.8151	0.85	0.2675	0.2333	0.7221	0.2463
6	0.8416	0.65	0.5000	0.2069	0.6025	0.5689
7	0.7189	0.60	0.3500	0.1824	0.9000	0.7158
8	0.7853	0.75	0.1200	0.1250	0.2340	0.2977
9	0.4756	0.60	0.1350	0.0801	0.3643	0.2439
10	0.6782	0.55	0.5100	0.0818	0.1835	0.0486
11	0.7112	1.00	0.3050	0.2117	0.3179	0.4031
12	0.8113	0.65	0.2550	0.1227	0.9225	0.6279
13	0.6586	0.85	0.3400	0.1755	0.6452	0.2605
14	0.9763	0.80	0.5400	0.1443	0.5143	0.2433
15	0.6845	0.95	0.4500	1.0000	0.2617	0.0982
16	0.6127	0.90	0.5250	0.1151	0.4021	0.4641
17	1.0000	0.60	0.2050	0.0900	1.0000	0.1614
18	0.6337	0.65	0.2350	0.0591	0.3492	0.0678
19	0.3715	0.70	0.2375	0.0385	0.1898	0.1112
20	0.5827	0.55	0.5000	0.1101	0.6145	0.7643

5. Conclusions

Anchor points play an important role in DEA theory and application. They delineate the efficient frontier from the free-disposability portion of the PPS frontier. Their identification has several notable DEA applications such as the construction of \unobserved" DMUs in order to reduce appropriately the DEA-inefficient boundary of the PPS. This paper proposed a method for finding all anchor DMUs of the PPS of the CCR model using two super-efficiency models (see models (3) and (4)). The necessary and sufficient conditions for a DMU to be an anchor DMU were stated and proved. The advantage of our approach is in determining inputs (outputs) of anchor

DMUs that can increase (decrease) without penetrating into the interior of the production possibility set. Another advantage of our approach is in clarifying the edges of the PPS on which anchor DMUs lie; whereas the aforesaid methods are unable to do these two advantages. Initial studies had shown that our approach can also be applied to BCC model. We suggest a deeper analysis in this subject as future works. Finally, the GAMs software has been utilized to run the models (3) and (4).

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