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A general Approach to find Non-Zero Multiplier Weights in DEA

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Abstract
Data Envelopment Analysis (DEA) models can be stated as two mutually dual linear programs referred to as the envelopment and multiplier models. The multiplier models are stated in terms of variable input and output weights (multipliers). Zero multiplier weight for an input or output causes efficient problems in multiplier model. This paper concentrates on a previously proposed DEA model developed by Wang and Chin (2010) and later improved by Wang et al. (2011) to find non-zero multi-plier weights. We will show that these models reveal shortcoming for certain classes of DMUs. In addition, we propose a general developed model to find a maximal element for a multiplier DEA model.

Keywords: Data envelopment analysis, Maximal element, Cross-efficiency.

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1. Introduction

Data Envelopment Analysis (DEA) is a well-known nonparametric method-ology to assess the relative efficiency of a sample of homogeneous Decision Making Units (DMUs) with multiple inputs and multiple outputs. DEA models can be stated as two mutually dual linear programs referred to as the envelopment and multiplier models. The multiplier models are stated in terms of variable input and output weights (multipliers). We call the optimal solution of a multiplier DEA model as multiplier weights, in short (Banker et al. (1984); Charnes et al. (1978)). The analyst who employs DEA or the information provided by the multiplier weights face some well-known difficulties; (i) alternate optima (ii) zero weights.

A zero multiplier weight for a input or output usually creates two important ambiguities. First, the need for parsimony in the selection of model inputs and outputs conveys a message that these are regarded as especially important to the appraisal process. Zero multiplier weight for an input or output sends a contradictory message to a manager as it implies that these are not regarded as important for the evaluation. Second, a zero multiplier weight for a input implies that the output can be produced without it. Alter-natively, a zero multiplier weight for a output implies that this output does not need to be produced (Bougnol and Dula (2012)). Multiplier weights with the heights number of positive component, a maximal element, for a multiplier DEA model is important from two different outlook; addressing the aforementioned ambiguities, other areas of DEA such as return to scale (Sueyoshi and Goto (2011); Sueyoshi and Sekitani (2007)).

Recently, Wang and Chin (2010) and Wu et al. (2012) in order to deal with the alternate optimal solution in cross-efficiency method (Sexton et al. (1986); Doyle and Green (1994); Liang et al. (2008); Oukil and Amin (2015); Wang and Chin (2010); Wu et al. (2016)), proposed to reduce the number of zero multiplier weights as a secondary goal. The models presented in this studies are almost identical, and they called the neutral DEA models. The first one attempts to reduce only the number of output zero weights and the second one attempts to reduce the both number of output and input zero multiplier weights simultaneously.

The main idea behind the neutral DEA models is to find a maximal element in optimal solution set for a multiplier DEA model. Although this idea is innovative and addresses both mentioned problems, but we will see how these models are incapable of achieving their goal for those DMUs that themselves or their projection are located on the weak efficient portion of the efficient frontier.

The aim of this paper is to provide a more comprehensive model to find a maximal element in optimal weight set of a multiplier DEA model, for brevity, optimal maximal element. For this purpose, first we see that the support set of all maximal element of a convex set for example K is identical. Next, an algorithm to identify the support set of optimal solution set will be proposed. Finally, as a result, a linear programming (LP) model which is a development of the neutral DEA models will be proposed, so their previous defects are eliminated.

The paper is structured as follows: in Section 2 a survey of the theoretical background of the LP and the neutral DEA model are provided. In Section 3 the neutral DEA models are explored. In Section 4 the main model is presented and in Section 5 a numerical example is provided. Finally, Section 7 concludes the paper, and describes future works.

2. Theoretical background

First of all, we declare the notations used in this study. The n-dimensional

Euclidean space is denoted by R^n and non-negative section denoted by R^n_+. We symbolize the sets by capital letters, set members by lower-case letters, vectors and matrices in bold letters: vectors in lower case and matrices in upper case, all vectors are column vectors and the transpose of vectors and matrices by a superscript T. We also use 0_n and 1_n to show n-dimensional vectors with zero and one components, respectively. Furthermore the superscript $*$ for a variable displays the optimal value of variable.

1.1. The Karush-Kuhn-Tucker (KKT) optimality conditions

Consider the primal and dual LP models in canonical form:

$$
\begin{aligned}
&\min \quad cx \\
(P)\ \text{s.t} \quad & A^t x \le b \\
& x \ge 0_n
\end{aligned}
$$

$$
\begin{aligned}
&\max\ b^T y \\
(D)\ \ \text{s.t} \quad & A^t y \ge c \\
& y \ge 0_m
\end{aligned}
$$

Where $c \in R^n$ and $b \in R^m$. Let $s \in R^m_+$ and $v \in R^n_+$ are slack variables corresponding to the constraints $A^t x \le b$ and $A^t y \ge c$ from (P) and (D), respectively. Also assume that S_P and S_D denote the feasible solution of primal and dual problems, respectively.

Definition 2.1. (Bazaraa and et al. (2011)) Let $\begin{pmatrix} x \\ s \end{pmatrix} \in S_P$ and $\begin{pmatrix} y \\ v \end{pmatrix} \in S_D$. They Satisfy the complementary slackness condition (CSC) if:

$x^T v=0$ and $y^T s= 0$.

If in addition $x + v > 0_n$ and $y + s > 0_m$, we say these solutions satisfy the strictly complementary slackness condition (SCSC) and, for brevity, is called a strict complementary solution.

Theorem 2.1. (Bazaraa and et al. (2011)) Let $\begin{pmatrix} x \\ s \end{pmatrix} \in S_P$ and $\begin{pmatrix} y \\ v \end{pmatrix} \in S_D$. Then they are respectively optimal if and only if satisfy the (CSC).

Theorem 2.2. (Goldman and Tucker (1965)) If (P) and (D) are feasible then there exists a strictly complementary pair of optimal solutions.

2.2. DEA

Assume that there are n DMUs with m inputs and s output to be evaluated. A DEA data domain D is characterized by a data matrix

$$P = \begin{bmatrix} Y \\ -X \end{bmatrix} = [p_1, p_2, \dots, p_n] \qquad (1)$$

With $s + m$ rows and n columns. **Y** and **X** are output and input matrixes, respectively. We denote by J the set of all n DMUs. The jth column $p_j = \begin{bmatrix} y_j \\ -x_j \end{bmatrix}$ where $y_j \in R^m_+$ and $y_j \in R^s_+$ are the output and input vectors, respectively, corresponding to the DMU_j. We assume that (i) for no **j** is either $x_j = 0$ or $y_j = 0$ (ii) no two columns of **P** are proportional. We also assume that $u \in R^m_+$ and $v \in R^s_+$ are output and input weight vectors corresponding to the y_j and x_j.

Definition 2.2. (Efficiency)

The efficiency of DMU_k with respect to the output and input weight vectors, **u** and **v**, is defined as follows:

$$E(k,u,v) = \frac{u^T y_k}{v^T x_k}$$

Definition 2.3. (CCR Relative efficiency)
The relative efficiency of DMU_k relative to the other DMU_s and with respect to the output and input weight vectors, **u** and **v**, is defined as follows:

$$RE(k,u,v) = \frac{E(k,\ u,\ v)}{\max_{j\in J} E(j,u,v)}$$

Charnes et al. (1978) measured the maximum relative efficiency of DMU_k using a (LP) model which is known as multiplier form. This model together its dual model which is also known as envelopment form are as follows:

$$\begin{aligned} &\max\ u^T\ y_k \\ &\text{CCR multiplier model} \\ &\text{s.t}\quad v^T x_k\ = 1 \\ &\qquad u^T Y - v^T X + zI\ = 0 \\ &\qquad u \geq\ 0,\ v \geq\ 0 \end{aligned} \tag{2}$$

$$\begin{aligned} &\theta_o^* = \ \min\ \theta \\ &\text{CCR envelopment model} \\ &\text{s.t}\quad X\lambda +\ s^- = \theta x_k \\ &\qquad Y\lambda - s^+ = \ y_k \\ &\qquad \lambda \geq 0 \end{aligned} \tag{3}$$

Where $\lambda \in R^n$.

Definition 2.4. Let (θ_0^*, λ^*) is an optimal solution of the CCR envelopment model. If θ_0^*= 1, then the DMU_o is called CCR scale-efficient, otherwise it is called CCR scale-inefficient.
We follow Charnes et al. (1986), and partition the set of all DMUs into the following six classes E', EF, NE', NE and NF which called, respectively, extreme efficient, non-extreme efficient, weak efficient, extreme inefficient, non-extreme inefficient, weak inefficient as follows.

Definition 2.5. Let W_o display the optimal solution set of CCR multiplier model for DMU_o. If DMU_o is CCR scale-efficient, then it will be classified into classes E', E and F, according to that all members of W_o are positive, one member of W_o is positive and no member of W_o is positive, respectively. Also, if DMU_o is CCR scale-inefficient, then it will be classified into classes NE', NE and NF, according to that the its CCR projection, $(\theta^* x_o, y_o)$, belongs to which class.

Definition 2.6. (Cross- efficiency)
Let (u_k^*, v_k^*, θ^*) is an optimal solution of (2) for DMU_k. Then $\theta_{kk} = (k, u_k^*, v_k^*)$ is referred to as the CCR-efficiency and reflects the self-evaluated efficiency. Furthermore, $\theta_{jk} = E(j, u_k^*, v_k^*)$ is referred to as a cross-efficiency value of DMU_j and reflects the peer evaluation of DMU_k to DMU_j $(j \in J \text{ and } j \neq k)$.
Due to the problem of alternate optima, the amount of θ_{jk} varies from one optimal solution to the other. The following neutral DEA models were proposed to avoid the non-uniqueness in solutions by Wang and Chin (2010) proposed a neutral DEA model, aimed at reducing the number of zero output weights as a secondary goal, for DMU_k, as follows:

$$\max\ \delta$$

$$
\begin{aligned}
\text{s.t}\quad & \sum_{i\in I}\upsilon_i x_{ik}=1\\
& \sum_{r\in O}u_r y_{rk}=\theta_{kk}\\
& \sum_{r\in O}u_r y_{rj}-\sum_{i\in I}\upsilon_i x_{ij}\le 0,\\
& j\in J, j\ne k\\
& u_r y_{rk}\ge\delta,\ r\in O\\
& u_r\ge 0,\ \upsilon_i\ge 0,\ i\in I, r\in O
\end{aligned}
\qquad (4)
$$

Where u_r and υ_i are components of **u** and **v**, respectively.
Wang et al. (2011) proposed another more general neutral DEA model that reduce the zero weights of inputs and outputs, simultaneously, as follows:

$$
\begin{aligned}
& \max\ \alpha\delta+\beta\gamma\\
\text{s.t}\quad & \sum_{i\in I}\upsilon_i x_{ik}=1\\
& \sum_{r\in O}u_r y_{rk}=\theta_{kk}\\
& \sum_{r\in O}u_r y_{rj}-\sum_{i\in I}\upsilon_i x_{ij}\le 0,\\
& j\in J, j\ne k\\
& u_r y_{rk}\ge\delta,\ r\in O\\
& \upsilon_i x_{ik}\ge\delta,\ i\in I\\
& u_r\ge 0,\ \upsilon_i\ge 0,\ i\in I, r\in O
\end{aligned}
\qquad (5)
$$

Where $\alpha\ge 0$ and $\beta\ge 0$ are the weighting coefficients for the two objectives δ and γ, satisfying the condition $\alpha+\beta=1$.

3. Analysis of neutral DEA models

Here, we examine the ability of the neutral DEA models to reduce zero weights individually. You can simply see from Models (4) that if the variable δ in Model (4) is positive at optimality, we have a solution with positive output weights. Also, if the variables δ,γ or both of Model (5) are positive at optimality, we have a solution with positive output weights, positive input weights or positive output and input weights, respectively. Hence, the main question is, under what conditions these variables are positive at optimality?

Proposition 3.1. Consider Model (4).We have δ^* is positive for each DMU_j in E, E', NE and NE'.
Proof. It is a direct result of Definition (2.5).

Proposition 3.2. Consider Model (4).We have δ^* is zero for each DMU_j in F and NF that have $S^{+*}\ne 0$ in at least one optimal solution of the CCR envelopment model.
Proof. It is followed from Definition (2.5) and Theorem (2.1).

Proposition 3.3. Consider Model (5).We have δ^* and γ^* are positive for each DMU_j in E, E', NE and NE'.
Proof. It is a direct result of Definition (2.5).

Proposition 3.4. Consider Model (5). We have δ^* is zero for each DMU_j in F and NF that have $S^{+*}\ne 0$ in at least one optimal solution of the CCR envelopment model. Also, we have γ^* is zero for each DMU_j in F and NF that have $S^{+*}\ne 0$ in at least one optimal solution of the CCR envelopment model.
Proof. It is followed from Definition (2.5) and Theorem (2.1).
According to the above propositions, the neutral DEA models, models (4) and (5), are unable to reduce the number of zero weights of inputs, outputs or both for each $DMU_j\in F\cup NF$. To give an illustration of what e.g. Proposition (3.4) mean, lets look at the following example.

Example 3.1. A illustrative example
Consider six DMUs each with two inputs

and one output. Data are shown in Table (1). The CCR production frontier is depicted in Figure (1). It is easy to see that $E=\{DMU_2, DMU_3, DMU_4\}$, $F=\{DMU_1, DMU_5\}$ and $NF=\{DMU_6\}$ The Table (2) displays an optimal solution of the CCR envelopment model for each DMUs. We have $s_2^{-*}=1$ for DMU_1. Regarding to Theorem (2.1), $\upsilon_1^*=0$ in each optimal solution of CCR multiplier model for DMU_1. In addition, $\upsilon_2^*=0$ in each optimal solution of CCR multiplier model for DMU_5 by the same reason.

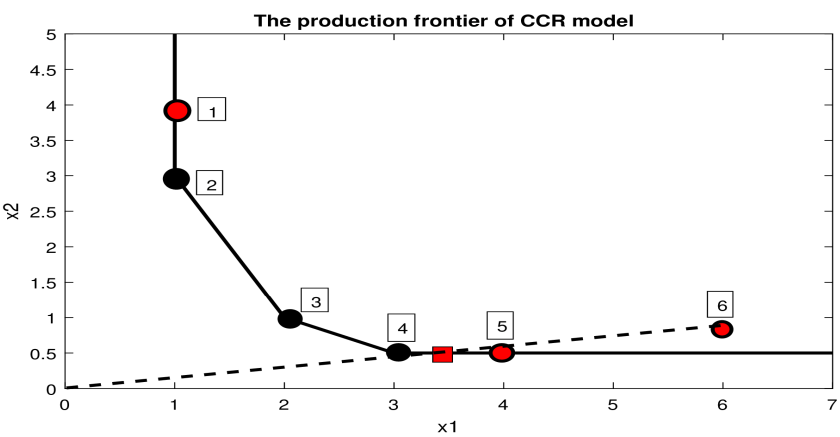
Therefore, we have $\gamma^*=0$ for DMU_1 and DMU_5 so the Model (5) will no longer be able to reduce the number of zero-input weights for these DMUs. As mentioned in the Introduction, this simple example reveals the weakness of Model (4) to reduce the number of zero weights for 4.

Table 1: Data for the illustrative example

DMUs	1	2	3	4	5	6
i_1	1	1	2	3	4	6
i_2	4	3	1	0.5	0.5	1
o	1	1	1	1	1	1

Table 2: CCR envelopment model optimal solutions

	λ_1^*	λ_2^*	λ_3^*	λ_4^*	λ_5^*	λ_6^*	s_1^{-*}	s_2^{-*}	s^{+*}	θ
DMU_1	0	1	0	0	0	0	0	1	0	1
DMU_2	0	1	0	0	0	0	0	0	0	1
DMU_3	0	0	1	0	0	0	0	0	0	1
DMU_4	0	0	0	1	0	0	0	0	0	1
DMU_5	0	0	0	1	0	0	1	0	0	1
DMU_6	0	0	0	0	1	0	0	0	0	0.667



4. A more general model for reducing zero weights

In this section, we are going to present a developed model to find an optimal maximal element for the CCR multiplier model. First of all, we need to identify the support sets of the variables of the CCR envelopment and multiplier models. Below we mention some characterizations and properties of maximal element.

4.1. Maximal elements and complementary solutions

Definition 4.1. The support of a vector x denote by $\sigma(x)$ and is defined as the index set of positive coordinates of x, that is, $\sigma(x)=\{i\ :\ x_i>0\}$

Definition4.2 (Guler and Ye (1993)). We define

1. a weak partial order, $\preceq$, over Rn by declaring $x^1 \preceq x^2$ if $\sigma(x^1)\subseteq\sigma(x^2)$

2. $x^1 \sim x^2$, x^1 and x^2 are equivalent, if $x^1 \preceq x^2$ and $x^1 \preceq x^2$.

3. maximal element x of $K\subseteq R^n$ under $\preceq$, whenever $x'\in K$ and $x\preceq x'$ imply $x \sim x'$, i.e., $x\in \operatorname{argmax}_{x\in K}\sigma(x)$

Lemma 4.1 (Guler and Ye (1993)). If K is a convex set, then all maximal elements of K are equivalent.

Definition 4.3. we define the support of K, $\sigma(K)$, as follows:

$$\sigma(K)=\sigma(x),\quad \text{for } x\in \operatorname{argmax}_{x\in K}\sigma(x)$$

A set of two nonempty subset of P, $\{P_1,\ P_2\}$, is a partition of P if

$$P=\ P_1\cup P_2,\ P_1\cap P_2=\varnothing$$

Let $\begin{pmatrix}\lambda^o\\ \theta^o\end{pmatrix}\in S_p^0$ and $\begin{pmatrix}u^o\\ v^o\end{pmatrix}\in S_d^0$.

We also assume that

$s^+=\left(\theta^o x_o-X\lambda^o\right)\in R_+^s$,

$s^-=\left(Y\lambda^o-y_o\right)\in R_+^m$ and μ^+

$=-\left(u^o y-v^0 X\right)\in R^n$

are slack vector corresponding to the output, input and DMU constraint sets in Model (3) and (2).

Lemma 4.2. The following statements are equivalent for each pair $\left(\begin{pmatrix}\lambda^o\\ \theta^o\end{pmatrix},\begin{pmatrix}u^o\\ v^o\end{pmatrix}\right)\in S_p^o\times S_d^o$.

1. The pair $\left(\begin{pmatrix}\lambda^o\\ \theta^o\end{pmatrix},\begin{pmatrix}u^o\\ v^o\end{pmatrix}\right)$ is strictly complementary solution.

2. The sets $\{\sigma(\lambda^o),\sigma(\mu^o)\},\{\sigma\left(s^+\right),$ $\sigma\left(u^o\right)\}$ and $\{\sigma(s^-),\sigma\left(v^o\right)\}$

are partitions of J, O, I, respectively.

Proof. 1. → 2. is the direct result of Definition (1.1). For 2. → 1., we have

$\lambda_j^o\mu_j^o=0$ and $\lambda_j^o+\mu_j^o>0$

for each $j\in J$

From

$\sigma(\lambda^o)\cap\sigma(\mu^o)=\varnothing$ and

$\sigma(\lambda^o)\ \cup\ \sigma(\mu^o)=J$.

With the same argument, we also have

$s_r^+u_r^o=0$ and $s_r^+ + u_r^o>0$

for each $r\in O$

and

$s_i^-\upsilon_i^o=0$ and $s_i^{-+}+\upsilon_i^o>0$

for each $i\in I$.

Assume that $\in S_p^{o*}$ and $\in S_d^{o*}$ display the optimal solution sets corresponding to the CCR envelopment and multiplier models, respectively.

Lemma 4.3. The following statements are equivalent for each pair $\left(\begin{pmatrix}\lambda^o\\ \theta^o\end{pmatrix},\begin{pmatrix}u^o\\ v^o\end{pmatrix}\right)\in S_p^o\times S_d^o$.

1. The pair $\left(\begin{pmatrix}\lambda^o\\ \theta^o\end{pmatrix},\begin{pmatrix}u^o\\ v^o\end{pmatrix}\right)$ is strictly complementary solution.

2. The solutions $\begin{pmatrix}\lambda^o\\ \theta^o\end{pmatrix}$ and $\begin{pmatrix}u^o\\ v^o\end{pmatrix}$ are maximal elements of S_p^{o*} and S_d^{o*}, respectively.

Proof. (1. → 2.). The optimality of $\begin{pmatrix}\lambda^o\\ \theta^o\end{pmatrix}$ and $\begin{pmatrix}u^o\\ v^o\end{pmatrix}$ is deduced by Theorem (1.1). Now, assume that $\begin{pmatrix}\lambda^o\\ \theta^o\end{pmatrix}$ is not a maximal element. So, there is $\begin{pmatrix}\lambda_1^o\\ \theta^o\end{pmatrix}\in S_p^{o*}$ such that $\begin{pmatrix}\lambda^o\\ \theta^o\end{pmatrix}\leq\begin{pmatrix}\lambda_1^o\\ \theta^o\end{pmatrix}$ and $\begin{pmatrix}\lambda^o\\ \theta^o\end{pmatrix}\not\geq\begin{pmatrix}\lambda_1^o\\ \theta^o\end{pmatrix}$. Hence, there is $j_0\in\sigma\left(\lambda_1^o\right)\setminus\sigma\left(\lambda^o\right)$. By Theorem (1.1), $\lambda_{1jo}^o>0$ then $\mu_{jo}^o=0$. On the other hand, $\lambda_{1jo}^o>0$ then we must have $\mu_{jo}^o>0$ by Lemma (3.2). This is a contradiction and therefore we have $\begin{pmatrix}\lambda^o\\ \theta^o\end{pmatrix}$ is a maximal element and this complete the proof. (2. → 1.). By the Lemma (4.2), it is sufficient to show that the sets $\{\sigma(\lambda^o),\sigma(\mu^o)\},\{\sigma\left(s^+\right),\sigma\left(u^o\right)\}$ and $\{\sigma(s^-),\sigma\left(v^o\right)\}$ are partitions of J, O, I, respectively. Consider $\{\sigma(\lambda^o),\sigma(\mu^o)\}$. Assume that it is not a partition of J. By assumption and Theorem (2.1), we have $\sigma\left(\lambda^o\right)\cap\sigma\left(\mu^o\right)=\varnothing$.

So there is $j_0\in J$ such that $j_0\notin\left(\sigma\left(\lambda^o\right)\cup\sigma\left(\mu^o\right)\right)$. Regarding Theorem (2.2), there is a strictly complementary solution pair, $\left(\begin{pmatrix}\bar{\lambda}^o\\ \theta^o\end{pmatrix},\begin{pmatrix}\bar{u}^o\\ \bar{v}^o\end{pmatrix}\right)\in S_p^{o*}\times S_d^{o*}$. By using Lemma (4.2), we conclude $j_0\in\left(\sigma\left(\bar{\lambda}^o\right)\cup\sigma\left(\bar{\mu}^o\right)\right)$. Without loss of generality, we assume that $j_0\in\sigma\left(\bar{\lambda}^o\right)$ and define

$$\begin{pmatrix}\hat{\lambda}^o\\ \theta^o\end{pmatrix}:=\alpha\begin{pmatrix}\lambda^o\\ \theta^o\end{pmatrix}+(1-\alpha)\begin{pmatrix}\bar{\lambda}^o\\ \theta^o\end{pmatrix}.$$

for $0<\alpha<1$

It is easy to verify that $\begin{pmatrix}\hat{\lambda}^o\\ \theta^o\end{pmatrix}\in S_p^*\begin{pmatrix}\lambda^o\\ \theta^o\end{pmatrix}\underset{\not\approx}{\prec}\begin{pmatrix}\hat{\lambda}^o\\ \theta^o\end{pmatrix}$ which is a contradiction. We have therefore shown that $\{\sigma(\lambda^o),\sigma(\mu^o)\}$ is a partition of J. Likewise, we also have $\left\{\sigma\left(s^+\right),\sigma\left(u^o\right)\right\}$ and $\left\{\sigma(s^-),\sigma\left(v^o\right)\right\}$ are partitions of O and I, respectively.

Our key tool is the next algorithm. It clarify the support sets of $\sigma\left(S_p^{o*}\right)$ and

$\sigma\left(S_d^{o*}\right)$ to determine which one of the weights can be positive for $DMU_o, o \in J$.

4.2. An algorithm to find support sets

In outline, the algorithm is found on CSC. Regarding Theorem, we always have; (I) one and only one variable has a positive value from the two complementary variables in the optimality, (II) If the value of a variable is positive in an optimal solution, then the value of that variable is positive in each optimal maximal solution. We first obtain optimal solutions $\left(\lambda_0, s_0^-, s_0^+\right)$ and $\left(u_0, v_0, z_0\right)$ of problems (3) and (2), respectively. Next we define $IP := \sigma\left(s_0^-\right), \cdot$

$$ID := \sigma\left(v_0\right),\ I^{\#} = I \setminus I_P \cup I_D,$$

$$O_P := \sigma\left(s_0^+\right),\ O_D := \sigma\left(u_0\right),$$

$$O^{\#} = I \setminus O_P \cup O_D,\ \ J_P := \sigma(\lambda_0),$$

$$J_D := \sigma\left(z_0\right) \text{and } J^{\#} = J \setminus J_P \cup J_D$$

According to the Lemma (3.2), the current solutions are strictly complementary if $I^{\#} = \varnothing$, $O^{\#} = \varnothing$ and $J^{\#} = \varnothing$. We take this as stop criterion.

Algorithm. Finding support sets for variables of the CCR models

Step 1. Find arbitrary optimal solutions for models (3) and (2); $\left(\lambda_0, s_0^-, s_0^+\right)$ and $\left(u_0, v_0, z_0\right)$

Step 2. Provide sets $I_P, I_D, O_P, O_D, J_P, J_D, I^{\#}, O^{\#}$ and $J^{\#}$.

Step 3. If $I^{\#} = \varnothing$, $O^{\#} = \varnothing$ and $J^{\#} = \varnothing$, then go to the Step 4. Otherwise, go to the

Step 4. Stop and I_P, I_D, O_P, O_D, J_P and J_D are support set for s^-, v, s^+, v, λ and z, respectively.

Step 5. If $I^{\#} = \varnothing$ go to the next step, else solve

$$\max \sum_{i \in I^{\#}} S^- \qquad \max \sum_{i \in I^{\#}} v_i ,$$

$$\text{s.t } (\lambda, s^-, s^+) \in S_p^* \qquad \text{s.t}(u,\ v,\ z) \in S_d^*$$

update $I_P, I_D, O_P, O_D, J_P, J_D, I^{\#}, O^{\#}$ and $J^{\#}$ and repeat this step.

Step 6. If $O^{\#} = \varnothing$ go to the next step, else solve

$$\max \sum_{r \in O^{\#}} S^+ \qquad \max \sum_{r \in O^{\#}} u_r ,$$

$$\text{s.t } (\lambda, s^-, s^+) \in S_p^* \qquad \text{s.t}(u,\ v,\ z) \in S_d^*$$

update $O_P, O_D, J_P, J_D, O^{\#}$ and $J^{\#}$ and repeat this step.

Step 7. If $J^{\#} = \varnothing$ go to the Step 4, else solve

$$\max \sum_{j \in J^{\#}} \lambda_j \qquad \max \sum_{j \in J^{\#}} z_j ,$$

$$\text{s.t } (\lambda, s^-, s^+) \in S_p^* \qquad \text{s.t}(u,\ v,\ z) \in S_d^*$$

update J_P, J_D and $J^{\#}$ and repeat this step.

Step 4 attempts to unfold that the members of $I^{o\#}=\varnothing$ belong to I_P^o or I_D^o. In addition, it is stop in a finite number of iterations with a partition of I du to Theorem (2.2). A similar argument is valid for Step 5 and Step 6.
Suppose that $\{I_P^o, I_D^o\}$ and $\{O_P^o, O_D^o\}$ are partitions of I and O, respectively, obtained by the above algorithm. In the following, we present a developed model of the neutral DEA models to find a maximal element of S_d^{o*}.

$$
\begin{aligned}
&\max\ \delta \\
&\text{s.t}\quad \sum_{i\in I}\upsilon_{ik}^{T}x_{ik}=1 \\
&\qquad \sum_{r\in O}u_{rk}y_{rk}=\theta_{kk} \qquad\qquad (6)\\
&\qquad \sum_{r\in O}u_{rk}y_{rj}-\sum_{i\in I}\upsilon_{ik}x_{ij}\le 0, \\
&\qquad j\in J, j\neq k \\
&\qquad u_{rk}y_{rj}\ge\delta,\ r\in O_D \qquad\qquad (7)\\
&\qquad \upsilon_{ik}x_{ij}\ge\delta,\ i\in I_D \\
&\qquad u_{rk}\ge 0\ ,\ \upsilon_{ik}\ge 0, \qquad\qquad (8)\\
&\qquad i\in I\ , r\in O
\end{aligned}
$$

I It is clear that, the optimal value of the above problem always is positive if the data are positive. However, if constraint sets (7) and (8) are replaced by constraint sets $u_{rk}\ge\delta,\ r\in O_D$ and $\upsilon_{rk}\ge\delta,\ i\in I_D$, respectively, data no longer need to be positive.

Lemma4.4. Let $\left(u^{o*}, v^{o*}, \delta_o^*\right)$ is an optimal solution for Model (6).

1. If the data domain, D, contains only positive data, we have $\delta_o^*>0$.

2. If $\delta_o^*>0$, then the weight vector $\left(u^{o*}, v^{o*}\right)$ has the lowest number of zero multiplier weights among all optimal multiplier weights.

Proof. (1.) CCR multiplier model has at least one optimal maximal element by Theorem (2.2) and Lemma (4.3). We denote it by $\left(\bar{u}^{o*}, \bar{v}^{o*}\right)$ and define

$$\bar{\delta}:=\ \mathrm{Min}\left\{y_{ro}\bar{u}_r^{o}, x_{io}\bar{\upsilon}_i^{o}\,\middle|\, r\in O_d, i\in I_d\right\}$$

Since all maximal element are equivalent, we deduce that $\bar{\delta}>0$. So, Model (6) has a feasible solution with positive objective value and therefor $\delta_o^*>0$. (2.) In this case, we have a solution with the same support set with $\sigma\left(S_d^{o*}\right)$ and this together with Lemma (4.1) result this part.

5. Numerical examples

To compare the neutral DEA models and the proposed model, we use a small numerical example involving six DMUs. Each DMU has three inputs and two outputs, which are shown in Table (3).

Table 3: Data of DMUs

DMUs	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6
i_1	2	2	4	4	5	5
i_2	3	3	5	3	5	3
i_3	2	4	3	3	4	2
$^{o}1$	5	5	2	3	3	2

We evaluate the DMUs by CCR envelopment model and CCR multiplier model. The results of the evaluations are displayed in Table (4). Table (5) displays the support sets of variables **u**, and **v** and also the class of each DMU. Since **u** has only one component and its value always is positive at optimality, hence we exhibit only the support set of **v**.

Table 4: Optimal weights obtained from CCR multiplier model (2)

DMUs	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6
$^{v}1$	0	0.500	0	0	0	0
v_2	0	0	0	0.333	0.200	0
v_3	0.500	0	0.333	0	0	0.500
$^{u}1$	0.200	0.200	0.133	0.200	0.120	0.200
CCR-Efficiency	1.000	1.000	.267	0.600	0.360	0.400

Table 5: The support set of v and the class of DMUs

DMUs	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6
$\sigma(v)$	$\{1,2,3\}$	$\{1,2\}$	$\{3\}$	$\{2\}$	$\{2\}$	$\{2,3\}$
Class	E	F	NF	NF	NF	NF

One of the main goals of this study was to proposed a model to resolve the zero weights issue. Especially, it was to develop the previously presented Models (4) and (5), which were difficult to deal with $DMU_j \in F \cup NF$. Because of the same nature of these models, we compare our model (6) with Model (5) only here. For $DMU_1 \in E$, both models succeed to find a solution with three nonzero input weights as claimed. For $DMU_2 \in F$, the CCR multiplier model gives $\upsilon_1^* = 0.5$, $\upsilon_2^* = 0$, and, $\upsilon_3^* = 0$. Model (6) give $\upsilon_1^* = 0.250$, $\upsilon_2^* = 0.167$, and, $\upsilon_3^* = 0$., and, v∗ 3 = 0 while Model (5) give $\upsilon_1^* = 0.5$, $\upsilon_2^* = 0.0$, and, $\upsilon_3^* = 0$. With regard to $\sigma(v) = \{1, 2\}$ for DMU_2, we can say that the proposed model give us an maximal element, however the solution of Model (5) is identical with the solution of CCR multiplier model. The CCR multiplier solutions for DMU_3 and DMU_4 are maximal elements according to the support set of v for these DMUs. Hence, none of the Models (6) and (5) can reduce the number of zero weights. The last DMU, DMU_6, belongs to NF. It has two input zero weights, $\upsilon_1^* = 0$, $\upsilon_2^* = 0$, and, $\upsilon_3^* = 0.5$, nevertheless, it can have one zero weight. Again it can be seen that Model (6) reduce the number of zero weights, $\upsilon_1^* = 0$, $\upsilon_2^* = 0.2$, and, $\upsilon_3^* = 0.2$, although (5) has no success in this work. In short, it is observed that both models are able to find an optimal maximal element solution for all $DMU_j \in E \cup E' \cup NE \cup NE'$.

Whereas this is only the proposed model which can also do this for $DMU_j \in EF \cup NF$.

DMUj ∈ EF ∪ NF. Our Study only focuses on finding a maximal element in a DEA model, whereas it might be important to include a LP model as well. In fact, the inclusion of LP models would enable us to find a complementary solution for LP models in general.

6. Conclusion

In this paper, it has been shown that the neutral DEA models has weak-ness in reducing the number of zero weighs for weak efficient and weak in-efficient DMUs. A novel approach has been introduced to resolve this issue. With a simple numerical example, it was shown how the proposed method can always find an optimal solution with the lowest number of zero components. Although the focus of the research was on the CCR multiplier model, however, it can be extended to other DEA models. Further studies are needed to develop the proposed model to LP models.

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