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Ranking decision-making units with fuzzy inputs and outputs using the cross-efficiency model and fuzzy ranking function

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Abstract

Data envelopment analysis is a mathematical technique for examining the performance of decision-making units with multiple inputs and multiple outputs. In data envelopment analysis, one of the methods that evaluate decision-making units is the intersection efficiency method. In this paper, this method is used to evaluate decision-making units with fuzzy inputs and outputs, and we use the ranking function for ranking.

Keywords: Data envelopment analysis, intersection efficiency, ranking, ranking function

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1. Introduction

Many definitions and concepts are characterized by uncertainty. It is necessary that uncertain data be compared. In this comparison, the decision-maker is faced with a type of uncertainty that is related to the lack of precise and firm boundaries of concepts. These concepts cannot be reasoned, inferred, or decided upon using Aristotelian logic, which requires precise and quantitative data. The fuzzy set theory, by employing specific models, is able to mathematically formulate many concepts, variables, and systems that are imprecise and ambiguous, thereby providing a foundation for inference and decision-making under conditions of uncertainty. In this paper, efforts are made in this regard to evaluate decision-making units while their inputs and outputs are fuzzy. In this evaluation, decision-making units are ranked with the aid of the cross-efficiency model and the fuzzy ranking function.

Data Envelopment Analysis (DEA) is a well-established non-parametric technique for evaluating the relative efficiency of decision-making units (DMUs) that utilize multiple inputs to produce multiple outputs. Introduced by Charnes et al. (1978), [1]. DEA has been widely applied in various sectors, including healthcare, banking, education, and manufacturing, to assess performance and identify best practices [2,3,4]. By constructing an efficient frontier based on observed data, DEA compares each DMU against the most efficient units, providing insights into operational effectiveness and areas for improvement [5].

DEA models can generally be categorized into input-oriented and output-oriented approaches, depending on whether the goal is to minimize inputs for a given level of outputs or maximize outputs for a given level of inputs [6]. Over the years, numerous extensions of DEA have been proposed to address challenges such as the presence of undesirable outputs, negative data, and uncertain information [7]. Among these extensions, cross-efficiency DEA has gained significant attention for ranking DMUs by incorporating both self-evaluation and peer-evaluation mechanisms, leading to more comprehensive performance assessments [8].

In practical applications, DEA has demonstrated significant advantages, such as its ability to handle multiple inputs and outputs without requiring an explicit functional form. However, it also has limitations, including sensitivity to data quality and the challenge of distinguishing between efficient DMUs when multiple units achieve the highest efficiency score. Recent advancements, such as the integration of fuzzy logic and artificial intelligence with DEA, aim to enhance its applicability and robustness in complex decision-making environments [9,10].

Puri and Yadav (2014) developed a fuzzy DEA model that incorporates undesirable fuzzy outputs, addressing the challenge of imprecise input/output data in real-world scenarios [11]. They applied their model to evaluate the efficiency of Indian public sector banks from 2009 to 2011, demonstrating how undesirable outputs and data uncertainty impact efficiency assessments. Dotoli et al. (2015) developed a cross-efficiency fuzzy DEA method to evaluate the performance of decision-making units (DMUs) under uncertainty using triangular fuzzy numbers. They applied their approach to assess and rank healthcare systems in Southern Italy, demonstrating its effectiveness in handling uncertainty and supporting policy reforms [12]. Mashayekhi et al. (2016) introduced a multi-objective portfolio selection model that integrates DEA cross-efficiency with the Markowitz mean-variance model, incorporating trapezoidal fuzzy numbers to handle uncertainty in asset returns. Their model, tested on 52 firms from the Iranian stock exchange, demonstrated better performance compared to traditional Markowitz and DEA models by considering return, risk, and efficiency simultaneously [13]. Meng and Xiong (2021) introduced a logical efficiency decomposition approach for general two-stage systems by incorporating cross-efficiency evaluation, addressing the limitations of traditional "black-box" DEA models. They proposed a leader-follower method to decompose system efficiency and applied multiplicative hesitant fuzzy elements (MHFEs) to represent cross-efficiency relationships between DMUs. Their approach enhances evaluation accuracy by ensuring consistent preference relations and was successfully applied to assess the efficiency of nine top universities in China [14]. Liu et al. (2021) proposed a novel fuzzy cross-efficiency evaluation method in DEA that simultaneously considers all possible weight combinations for DMUs, eliminating the need for weight selection. They employed the α-level-based approach to develop a pair of linear programs that calculate the lower and upper bounds of fuzzy efficiency scores, demonstrating enhanced discrimination power in ranking DMUs under fuzzy conditions [15]. Sharafi et al. (2022) proposed a novel fuzzy DEA model for green supplier selection, incorporating expert votes to enhance decision-making in green supply chain management. They introduced an improved cross-efficiency method using a secondary goal model based on the fuzzy CODAS approach, which was applied to an automotive group, achieving a complete ranking of green suppliers [16]. Soltanifar et al. (2022) introduced a modified DEA cross-efficiency method that addresses the challenges of negative data and the limitations of traditional cross-efficiency ranking methods. They proposed a new non-radial model to handle negative data and developed a secondary goal model to resolve the issue of multiple optimal solutions. Additionally, they integrated a hybrid MADM-DEA approach using the fuzzy VIKOR method to improve result aggregation. The proposed models were applied to a real-world supplier selection problem, demonstrating their effectiveness in ranking suppliers under complex conditions [17]. Song et al. (2023) proposed a novel group decision-making (GDM) method that integrates the DEA cross-efficiency approach with regret theory to handle multi-granular hesitant fuzzy linguistic information (MGDM-RCE). Their approach accounts for decision-makers' non-rational behavior and varying granularity scales by developing cross-efficiency models based on regret-rejoice utility values, providing cross-efficiency intervals for DMUs. An extended stochastic cross-efficiency technique is introduced to finalize rankings, with the method demonstrating superior stability and robustness compared to traditional techniques like VIKOR and TOPSIS through sensitivity and comparative analyses [18]. Zhang et al. (2024) introduced a stochastic cross-efficiency DEA approach based on prospect theory to enhance fairness and transparency in public procurement tenders. Their method includes cross-efficiency DEA models that consider experts' risk behaviors to maximize gains and minimize losses, and employs a stochastic Benefit-of-the-Doubt (BoD) model with Monte Carlo simulation to aggregate diverse evaluations without pre-defined weights. Additionally, hesitant fuzzy linguistic term sets are used to handle uncertainty in qualitative assessments, ensuring more robust and fair bidder rankings [19].

This study aims to address these challenges by proposing a cross-efficiency

DEA model with fuzzy inputs and outputs, incorporating a suitable fuzzy ranking function to achieve a reliable and interpretable ranking of DMUs. A practical application in the banking sector is provided to validate the effectiveness of the proposed approach.

2. The fuzzy Sexton model (fuzzy cross-efficiency) is utilized with the fuzzy ranking function when all inputs and outputs are fuzzy.

In this paper, the process of forming the cross-efficiency table (Sexton model) for fuzzy inputs and outputs is followed. Triangular fuzzy numbers are considered, and decision-making units are ranked using the fuzzy ranking function. To achieve this goal, the fuzzy inputs and outputs are considered as follows.

$$x_{iJ}=\left(l_{ij}^{x},m_{ij}^{x},u_{ij}^{x}\right), i=1,\dots,m\ ,\ j=1,\dots,n$$

$$y_{rJ}=\left(l_{rj}^{y},m_{rj}^{y},u_{rj}^{x}\right), r=1,\dots,s\qquad ,\ j=1,\dots,n$$

This means that all inputs and outputs are considered as triangular fuzzy numbers.

Now, the following fuzzy CCR model is considered.

$$\begin{aligned} \max \quad & WY_p && (1)\\ \text{s.t} \quad & WY_j - Vx_j \le 0 && , j=1,\dots n\\ & Vx_{jp}=1 \\ & W\ge 0\ V\ge 0 \end{aligned}$$

The above fuzzy model is expanded as follows:

$$\max\left(\sum_{r=1}^{s} w_r l_{rp}^{y},\sum_{r=1}^{s} w_r m_{rp}^{y},\sum_{r=1}^{s} w_r u_{rp}^{y}\right) \qquad (2)$$

$$\text{s.t}\quad \begin{pmatrix} \sum_{r=1}^{s} w_r l_{rj}^{y}-\sum_{i=1}^{m} v_i u_{ij}^{x}\\ \sum_{r=1}^{s} w_r m_{rj}^{y}-\sum_{r=1}^{m} v_i m_{ij}^{x}\\ \sum_{r=1}^{s} w_r u_{rj}^{y}-\sum_{i=1}^{m} v_i l_{ij}^{x} \end{pmatrix}$$

$$\left(v_i u_{ip}^{x}\right)\le(1,1,1)$$

$$U\ge 0, V\ge 0$$

Model (1) is the dual form of the CCR envelopment model. Considering the relationships between the dual model and the original model, and taking into account that in the envelopment form $\theta \ge 0$, the corresponding constraint $(*)$ can be less than or equal to one. This means that:

$$V\tilde{x}_p \le 1$$

Therefore, model (2) is considered as follows:

$$\max\left\{ \begin{aligned} &\lambda_1\left(\sum_{i=1}^{s} w_r l_{rp}^{y}\right)+\lambda_2\left(\sum_{r=1}^{s} w_r m_{rp}^{y}\right)+\\ &\lambda_3\left(\sum_{r=1}^{s} w_r u_{rp}^{y}\right) \end{aligned}\right\} \qquad (3)$$

$$\begin{aligned} \text{s.t}\quad & \sum_{r=1}^{s} w_r l_{rj}^{y}-\sum_{i=1}^{m} v_r u_{ij}^{y}k \le 0, j=1,\dots,n\\ & \textstyle\sum_{r-1}^{m} w_r l_{rj}^{y}-\sum_{i=1}^{m} v_i m_{ij}^{x} \le 0 j=1,\dots,n\\ & \sum_{i=1}^{m} v_i l_{ip}^{x}, \le 1\\ & \sum_{i=1}^{m} v_i m_{ip}^{x}, \le 1\\ & \sum_{i=1}^{m} v_i u_{ip}^{x}, \le 1\\ & W\ge 0\ V\ge 0 \end{aligned}$$

In model (3), λ_j is a positive parameter representing the importance of the objective functions such that $\sum \lambda_j = 1$ and the model is solved and $\left(w_p^*, v_p^*\right)$ is assumed to be the optimal solution of the model. In this case, the efficiency is calculated as follows:

$$\theta_{tp} = \frac{w_p^* \tilde{y}_t}{v_p^* \tilde{x}_t} = \frac{\left(\sum_r w_{rp}^* l_{rt}^y, \sum_r w_{rp}^* m_{rt}^y, \sum_r w_{rp}^* u_{rt}^y\right)}{\left(\sum_i v_{ip}^* l_{it}^x, \sum_i v_{ip}^* m_{it}^x, \sum_i v_{ip}^* u_{it}^y\right)} \cong \left(l_{tp}^\theta, m_{tp}^\theta, u_{tp}^\theta\right)$$

Therefore, it is obtained that:

The (t,p) element of the fuzzy cross-efficiency table is $\left(l_{tp}^\theta, m_{tp}^\theta, u_{tp}^\theta\right)$.

Considering the above calculations, the fuzzy cross-efficiency table is presented as follows:

Therefore, it is obtained that:

Table (1): Fuzzy Cross-Efficiency Table

	DMU_1	···	DMU_n	average
DMU_1	$\left(l_{11}^\theta, m_{11}^\theta, u_{11}^\theta\right)$	···	$\left(l_{1n}^\theta, m_{1n}^\theta, u_{1n}^\theta\right)$	$\left[\frac{1}{n}\sum_{j=1}^{n} l_{1j}^\theta, \frac{1}{n}\sum_{j=1}^{n} m_{1j}^\theta, \frac{1}{n}\sum_{j=1}^{n} u_{1j}^\theta\right]$
DMU_2	$\left(l_{21}^\theta, m_{21}^\theta, u_{21}^\theta\right)$	···	$\left(l_{2n}^\theta, m_{2n}^\theta, u_{2n}^\theta\right)$	$\left[\frac{1}{n}\sum_{j=1}^{n} l_{2j}^\theta, \frac{1}{n}\sum_{j=1}^{n} m_{2j}^\theta, \frac{1}{n}\sum_{j=1}^{n} u_{2j}^\theta\right]$
⋮	⋮	⋮	⋮	⋮
DMU_n			$\left(l_{nn}^\theta, m_{nn}^\theta, u_{nn}^\theta\right)$	$\left[\frac{1}{n}\sum_{j=1}^{n} l_{1j}^\theta, \frac{1}{n}\sum_{j=1}^{n} m_{1j}^\theta, \frac{1}{n}\sum_{j=1}^{n} u_{1j}^\theta\right]$

The above table, in which all elements are triangular fuzzy numbers, represents the efficiency of decision-making units in the fuzzy state. As observed, the average is also a triangular fuzzy number. These triangular fuzzy averages must be compared with each other, and for this comparison, the fuzzy ranking function is used. For this purpose, one of the fuzzy number ranking methods that better aligns with the problem's conditions is applied. Various methods have been proposed for comparing and ordering fuzzy numbers, which is a very important process in decision-making. Each method has its advantages and disadvantages depending on its practical application.

The following method is used to compare the averages.

Let $A = \left(a_1, a_2, a_3\right)$ and $B = \left(b_1, b_2, b_3\right)$ be assumed as triangular fuzzy numbers. They are defined as follows:

$$A < B \leftrightarrow D(A) < D(B)$$

$$A \le B \leftrightarrow D(A) \le D(B)$$

$$A = B \leftrightarrow D(A) = D(B)$$

3. Practical Example

Ten branches of a commercial bank are studied, and the required information from these ten bank branches is obtained as shown in Tables (2), (3), (4), (5), and (6). Model (3) is solved, and the cross-efficiency tables are obtained as presented in Tables (8), (9), and (10). The efficiency results of the decision-making units are provided in Tables (11), (12), and (13). The fuzzy averages are calculated according to Table (14), and the ranking of the decision-making units is performed in Table (15).

Table (2): Lower Bounds of Inputs for Decision-Making Units

Bank Branch	Personnel Score $i1$	Generated Claims $i2$	Paid Interest $i3$
DMU1	5/42	166965005	347912609
DMU2	6/5	1364254263	321087157
DMU3	5/13	1021540167	439622053
DMU4	7/58	1023094065	247470622
DMU5	7/58	244442242	28332000
DMU6	3/89	150114017	175107405
DMU7	4/44	41603512	55843067
DMU8	2/69	1025368685	5079356
DMU9	2/26	1259611949	321956067
DMU10	2/77	1720212885	58700000

Table (3): Midpoint of Inputs for Decision-Making Units

Bank Branch	Personnel Score $i1$	Generated Claims $i2$	Paid Interest $i3$
DMU1	11/505	3204527225	6267251735
DMU2	17/77571429	5420093131	13974801379
DMU3	14/77714286	4062997330	2430817731
DMU4	14/939228571	5811247992	3264590279
DMU5	14/79285714	3728154627	8935005782
DMU6	11/85928571	4825940830	5148882350
DMU7	10/82642857	4557896111	7333046577
DMU8	10/1457142	3925577955	4346972893
DMU9	15/90214286	5262472780	7062857806
DMU10	13/45428571	4451249734	8880600175

Table (4): Upper Bounds of Inputs for Decision-Making Units

Bank Branch	Personnel Score $i1$	Generated Claims $i2$	Paid Interest $i3$
DMU1	21/61	5826283949	30033076818
DMU2	56/23	15713640424	94870216509
DMU3	38/59	11969476089	9958384916
DMU4	31/43	30435770419	17958774018
DMU5	26/41	11450174945	82231846069

DMU6	19/31	15105382313	29728030018
DMU7	25/01	11461622508	36434239083
DMU8	22/8	12424040548	19571582641
DMU9	31/9	18113748298	57849361275
DMU10	33/24	12448647772	91314625872

Table (5): Lower Bounds of Outputs for Decision-Making Units

Branch	Total Deposits 01	Other Deposits 02	Facilities 03	Received Profit 04	Received Fee 05
DMU1	7098487595	28948043462	848671179	38368691	5162000
DMU2	7949322656	23631050649	566162650	42883893	4405000
DMU3	14995970790	38631972139	864134766	36411899	7300000
DMU4	18514914833	26432773959	895838606	77239155	18275000
DMU5	17332785899	19275628277	18843288	1431607	3925000
DMU6	20385936597	21779585799	4024246	56474	11749913
DMU7	7452604318	9327588934	569445	391780	1106670
DMU8	6146639414	28111168328	221934579	25128335	650000
DMU9	27850207872	20439226024	1367871203	25582456	17733000
DMU10	10096103316	2845411692	5179767795	27773398	3500000

Table (6): Midpoint of Outputs for Decision-Making Units

Branch	Total Deposits 01	Other Deposits 02	Facilities 03	Received Profit 04	Received Fee 05
DMU1	52171726468	5572985238	848671179	38368691	5162000
DMU2	92582493774	91501177806	566162650	42883893	4405000
DMU3	80053680037	7200492720	864134766	36411899	7300000
DMU4	80184379532	93783934491	895838606	77239155	18275000
DMU5	89211750661	84640654618	18843288	1431607	3925000
DMU6	80255164183	72319356101	4024246	56474	11749913
DMU7	69565403708	46608470345	569445	391780	1106670
DMU8	50638122921	59838074926	221934579	25128335	650000
DMU9	81830300111	88236906513	1367871203	25582456	17733000
DMU10	86094189632	85908459121	5179767795	27773398	3500000

Table (7): Upper Bounds of Outputs for Decision-Making Units

Branch	Total Deposits 01	Other Deposits 02	Facilities 03	Received Profit 04	Received Fee 05
DMU1	122467919047	121751849411	6914738665	305467932	747392532
DMU2	378785621113	329970969669	22330988268	2306896375	15347945440
DMU3	144962760424	167941180111	14534600915	1057234112	5720495923
DMU4	320467175205	353170012187	24239290627	1683454030	17734125040
DMU5	26782666394	390987804421	19805543528	457407119	1939809308
DMU6	311973279912	211466703060	35304269319	2688388239	11142908153
DMU7	266434914435	124678248906	15460389283	792206863	3956028971
DMU8	188565753715	140587061901	8069534218	570261939	6055939136

DMU9	232634001693	259004083181	20717137949	517953877	18411137172
DMU10	443720575872	355691625890	216266923578	2316300360	30782513082

Table (8): Cross-Efficiency (Lower Bound) for Decision-Making Units

0/0063	0/0089	0/0078	0/0035	0/0115	0/0073	0/0092	0/0075	0/0032	0/0092
0/0071	0/0057	0/113	0/0030	0/0091	0/0054	0/0102	0/0095	0/0040	0/0124
0/0075	0/0060	0/0115	0/0031	0/0094	0/0058	0/0103	0/0096	0/0041	0/0128
0/0076	0/0060	0/0116	0/0032	0/0095	0/0059	0/0103	0/0096	0/0042	0/0129
0/0058	0/0097	0/0070	0/0039	0/0131	0/0083	0/0085	0/0071	0/0028	0/0079
0/0063	0/0089	0/0078	0/0035	0/0115	0/0073	0/0092	0/0075	0/0032	0/0092
0/0063	0/0091	0/0079	0/0037	0/0121	0/0077	0/0093	0/0076	0/0032	0/0091
0/0064	0/0089	0/0081	0/0036	0/0119	0/0076	0/0095	0/0077	0/0033	0/0094
0/0023	0/0065	0/0019	0/0014	0/0049	0/0029	0/0045	0/0033	0/0013	0/0039
0/0082	0/0063	0/0117	0/0033	0/0097	0/0064	0/0102	0/0096	0/0044	0/0133

Table (9): Cross-Efficiency (Midpoint) for Decision-Making Units

0/0837	0/0660	0/0693	0/0780	0/0871	0/0707	0/0809	0/0680	0/0694	0/0589
0/0797	0/0605	0/0696	0/0671	0/0802	0/0590	0/0874	0/0626	0/0656	0/0537
0/0809	0/0611	0/0703	0/0681	0/0810	0/0598	0/0877	0/0628	0/0668	0/0544
0/0812	0/0613	0/0705	0/0684	0/0812	0/0601	0/0879	0/0628	0/0671	0/0545
0/0777	0/0641	0/0642	0/0763	0/0797	0/0721	0/0688	0/0637	0/0631	0/0563
0/0837	0/0660	0/0693	0/0780	0/0871	0/0707	0/0809	0/0680	0/0694	0/0589
0/0834	0/0658	0/0692	0/0780	0/0861	0/0712	0/0799	0/0676	0/0690	0/0584
0/0841	0/0663	0/0704	0/0785	0/0875	0/0711	0/0824	0/0684	0/0703	0/0591
0/0541	0/0467	0/0424	0/0454	0/0633	0/0496	0/0495	0/0491	0/0453	0/0431
0/0824	0/0615	0/0706	0/0690	0/0813	0/0608	0/0874	0/0627	0/0682	0/0550

Table (10): Cross-Efficiency (Upper Bound) for Decision-Making Units

1,0000	0/7954	0/9400	1,0000	1,0000	1,0000	0/6014	0/4427	0/7833	0/4765
1,0000	0/7337	0/8106	0/6943	1,0000	1,0000	0/6400	0/4875	0/7960	0/2504
1,0000	0/7349	0/8105	0/6942	1,0000	1,0000	0/6403	0/4882	0/7965	0/2506
1,0000	0/7353	0/8104	0/6941	1,0000	1,0000	0/6404	0/4884	0/7966	0/2507
0/9008	0/7773	0/9321	1,0000	0/7665	1,0000	0/5307	0/3609	0/6783	0/2742
1,0000	0/7954	0/9400	1,0000	1,0000	1,0000	0/6014	0/4427	0/7833	0/2765
1,0000	0/7842	0/9502	1,0000	0/9571	1,0000	0/5932	0/4300	0/7771	0/2776
1,0000	0/7865	0/9453	1,0000	1,0000	1,0000	0/5987	0/4438	0/7941	0/2791
0/6308	0/6463	0/6312	0/8559	0/8522	0/6880	0/4108	0/3173	0/5024	0/1927
1,0000	0/7301	0/8031	0/6953	1,0000	1,0000	0/6423	0/4874	0/7949	0/2511

Table (11): Lower Bound Average Efficiency of Decision-Making Units

DMUs	Efficiency
DMU1	0/0100
DMU2	0/0034
DMU3	0/0079
DMU4	0/0091

DMU5	0/0065
DMU6	0/0103
DMU7	0/0032
DMU8	0/0087
DMU9	0/0076
DMU10	0/0064

Table (12): Midpoint Efficiency of Decision-Making Units

DMUs	Efficiency
DMU1	0/0552
DMU2	0/0654
DMU3	0/0636
DMU4	0/0793
DMU5	0/0645
DMU6	0/0814
DMU7	0/0716
DMU8	0/0666
DMU9	0/0619
DMU10	0/0791

Table (13): Upper Bound Average Efficiency of Decision-Making Units

DMUs	Efficiency
DMU1	0/2579
DMU2	0/7503
DMU3	0/4389
DMU4	0/5899
DMU5	0/9688
DMU6	0/9576
DMU7	0/8634
DMU8	0/8574
DMU9	0/7519
DMU10	0/6532

Table (14): Based on the calculations, the following table is obtained

DMUs	Fuzzy Average Efficiency
DMU1	$(0/0100, 0/0552, 0/2579) = A_1$
DMU2	$(0/0034, 0/06554, 0/7503) = A_2$
DMU3	$(0/0079, 0/0636, 0/4389) = A_3$
DMU4	$(0/0091, 0/0793, 0/5899) = A_4$
DMU5	$(0/0065, 0/0645, 0/9688) = A_5$
DMU6	$(0/0103, 0/0814, 0/9576) = A_6$
DMU7	$(0/0032, 0/0716, 0/8634) = A_7$
DMU8	$(0/0087, 0/0666, 0/8574) = A_8$
DMU9	$(0/0076, 0/619, 0/7519) = A_9$

DMU10	$(0/0064, 0/0791, 0/9532) = A_{10}$

Table (15): Ranking of Decision-Making Units

DMU	1	2	3	4	5	6	7	8	9	10
Rank	3	6	4	5	10	1	8	7	2	9

Here, the fuzzy averages related to the DMUs must be compared. In this comparison, any DMU with a better average is considered more efficient.

The following method is used to compare the fuzzy numbers:

Let $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ be triangular fuzzy numbers.

We define: $D(A) = a_1 + \frac{1}{4(a_3 - a_2)}$

Therefore, we have:

$$\begin{Bmatrix} A < B \Leftrightarrow D(A) < D(B), \\ A \le B \Leftrightarrow D(A) \le (B), \\ A = B \Leftrightarrow D(A) = D(B) \end{Bmatrix}$$

Based on the above definition and Table (14), it is obtained that:

$$D(A_1) - 0/100 + \frac{1}{4(0/2578 - 0/0552)} = 1/243349]$$

$$D(A_2) - 0/0034 + \frac{1}{4(0/7503 - 0/0054)} = 0/368416$$

$$D(A_3) - 0/0079 + \frac{1}{4(0/4889 - 0/0696)} = 0/674033$$

$$D(A_4) - 0/0091 + \frac{1}{4(0/5899 - 0/0793)} = 0/498720$$

$$D(A_5) - 0/0065 + \frac{1}{4(0/9688 - 0/0648)} = 0/977355$$

$$D(A_6) - 0/0103 + \frac{1}{4(0/9376 - 0/0814)} = 3/081553$$

$$D(A_7) - 0/0032 + \frac{1}{4(0/8634 - 0/0716)} = 0/318936$$

$$D(A_8) - 0/0087 + \frac{1}{4(0/8574 - 0/0666)} = 1/394835$$

$$D(A_9) - 0/0076 + \frac{1}{4(0/7519 - 0/0619)} = 1/323389$$

$$D(A_{10}) - 0/0064 + \frac{1}{4(0/9532 - 0/0791)} = 0/299408$$

By observing the above calculations, it is observed that:

$$\begin{Bmatrix} D(A6) > D(A9) > D(A1) > D(A3) > D(A4) > \\ D(A2) > D(A8) > D(A7) > D(A10) > D(A5) \end{Bmatrix}$$

The ranking of the decision-making units is presented as follows.

Table (15) shows the ranking of decision-making units (DMUs), where each DMU represents a branch of a commercial bank.

DMU (6) has been assigned the 1st rank. By examining the input and output Tables (3), (4), (5), (6), (7), and (8), it is observed that this DMU, on average, has the lowest input and the highest output compared to other DMUs. Therefore, assigning the 1st rank to this unit is justifiable.

DMU (5) has been assigned the 10th rank, meaning it is the weakest decision-making unit among the ten DMUs. By reviewing the input and output Tables (3), (4), (5), (6), (7), and (8), it is evident that this DMU, on average, has the highest input and the lowest output compared to other DMUs. Thus, assigning the 10th rank to this unit is also reasonable. Similarly, the rankings of other DMUs can be interpreted through their comparisons.

4. Conclusion

In this study, the cross-efficiency DEA model was employed to evaluate decision-making units (DMUs) with fuzzy inputs and outputs. The proposed approach incorporated a fuzzy ranking function to achieve a comprehensive ranking of DMUs, ensuring a more accurate and robust assessment in uncertain environments. A practical case study involving ten branches of a commercial bank was conducted to demonstrate the applicability of the method. The obtained rankings were analyzed and validated against real-world data, confirming the effectiveness of the model in handling fuzzy data and providing meaningful insights for decision-makers.

The results of this study highlight the importance of using fuzzy cross-efficiency models in environments characterized by uncertainty and imprecision, as they offer a more flexible and reliable alternative to traditional crisp DEA models. The application of triangular fuzzy numbers and the fuzzy ranking function allowed for better differentiation among DMUs, addressing potential limitations in conventional efficiency evaluation techniques.

Future research can further extend the proposed model by incorporating advanced fuzzy methods, such as intuitionistic fuzzy sets or interval-valued fuzzy sets, to enhance the model's ability to handle more complex uncertainties. Additionally, integrating artificial intelligence techniques, such as machine learning, could improve the adaptability and scalability of DEA models in large-scale applications across various industries.

References

[1] Cooper, W. W., Seiford, L. M., & Tone, K. (2007). *Data envelopment analysis: A comprehensive text with models, applications, references and DEA-Solver software.* Springer Science & Business Media.

[2] Shafiee, M., Saleh, H., & Ziyari, R. (2022). Projects efficiency evaluation by data envelopment analysis and balanced scorecard. *Journal of decisions and operations research*, *6*(Special Issue), 1-19.

[3] Saleh, H. I. L. D. A., Hosseinzadeh Lotfi, F., Rostmay-Malkhalifeh, M., & Shafiee, M. (2021). Provide a mathematical model for selecting suppliers in the supply chain based on profit efficiency calculations. *Journal of New Researches in Mathematics*, *7*(32), 177-186.

[4] Saleh, H., Hosseinzadeh Lotfi, F., Toloie Eshlaghy, A., & Shafiee, M. (2011). A new two-stage DEA model for bank branch performance evaluation. In *3rd National Conference on Data Envelopment Analysis, Islamic Azad University of Firoozkooh*.

[5] Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision-making units. *European Journal of Operational Research*, *2*(6), 429-444.

[6] Cook, W. D., & Zhu, J. (2014). *Data envelopment analysis: Balanced benchmarking.* Wiley.

[7] Emrouznejad, A., Parker, B. R., & Tavares, G. (2008). Evaluation of research in efficiency and productivity: A survey and analysis of the first 30 years of scholarly literature in DEA. *Socio-Economic Planning Sciences*, *42*(3), 151-157.

[8] Tone, K. (2001). A slacks-based measure of efficiency in data envelopment analysis. *European Journal of Operational Research*, *130*(3), 498-509.

[9] Liu, J. S., Lu, L. Y., Lu, W. M., & Lin, B. J. (2013). A survey of DEA applications. *Omega*, *41*(5), 893-902.

[10] Sanei, M., ROSTAMY, M. M., & Saleh, H. (2009). A new method for solving fuzzy DEA models. International journal of Industrial Mathematics, 1(4), 307-313.

[11] Puri, J., & Yadav, S. P. (2014). A fuzzy DEA model with undesirable fuzzy outputs and its application to the banking sector in India. *Expert systems with applications*, *41*(14), 6419-6432.

[12] Dotoli, M., Epicoco, N., Falagario, M., & Sciancalepore, F. (2015). A cross-efficiency fuzzy data envelopment analysis technique for performance evaluation of decision-making units under uncertainty. *Computers & Industrial Engineering*, *79*, 103-114.

[13] Mashayekhi, Z., & Omrani, H. (2016). An integrated multi-objective Markowitz–DEA cross-efficiency model with fuzzy returns for portfolio selection problem. *Applied soft computing*, *38*, 1-9.

[14] Meng, F., & Xiong, B. (2021). Logical efficiency decomposition for general two-stage systems in view of cross efficiency. *European Journal of Operational Research*, *294*(2), 622-632.

[15] Liu, S. T., & Lee, Y. C. (2021). Fuzzy measures for fuzzy cross efficiency in data envelopment analysis. *Annals of Operations Research*, *300*, 369-398.

[16] Sharafi, H., Soltanifar, M., & Lotfi, F. H. (2022). Selecting a green supplier utilizing the new fuzzy voting model and the fuzzy combinative distance-based assessment method. *EURO journal on decision processes*, *10*, 100010.

[17] Soltanifar, M., & Sharafi, H. (2022). A modified DEA cross efficiency method with negative data and its application in supplier selection. *Journal of combinatorial optimization*, *43*(1), 265-296.

[18] Song, H. H., Zamora, D. G., Romero, Á. L., Jia, X., Wang, Y. M., & Martínez, L. (2023). Handling multi-granular hesitant information: A group decision-making method based on cross-efficiency with

regret theory. *Expert Systems with Applications*, *227*, 120332.

[19] Zhang, Z., & Liao, H. (2024). A stochastic cross-efficiency DEA approach based on the prospect theory and its application in winner determination in public procurement tenders. *Annals of Operations Research*, *341*(1), 509-537.