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Estimation of inputs and outputs in the general production possibility set with negative data based on the inverse DEA

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Abstract
The primary models in data envelopment analysis (DEA), consider the inputs and outputs of the decision-making units (DMUs) as non-negative. However, in the real world, we face many cases where the data is negative. In this paper, we investigate the inverse DEA models to estimate the optimal level of inputs and outputs of DMUs based on target efficiency scores. We also assume that some input and output components are negative. In this way, we propose three different models in variable returns to scale (VRS) to determine optimal levels. In order to solve each model, we determine the counterpart DMU corresponding to the DMUs under evaluation. This DMU is obtain based on the additive model, and then we get the level of the target and the observed outputs corresponding to the DMU under evaluation to determine which of these three models to use to measure the efficiency of the DMU under evaluation. We apply the proposed approach with a numerical example and consider it to measure the optimal levels of inputs and outputs of bank branches. Also, we propose the results of paper.

Keywords: Data envelopment analysis; Inverse DEA; Negative data; Target efficiency.

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1. Introduction

DEA is an effective technique to measure the relative efficiency of a set of production units. These units use multiple inputs to produce multiple outputs. In the primary models, the inputs and outputs from the decision-making units are positive. In many applications, it is possible that some of the inputs and outputs of the decision-making units are negative. Many ideas for dealing with negative data have been proposed under the DEA framework, which can be classified into four categories. The first category is data transformation, whereby constants are added to the observations of factors with negative values to make them positive. Pastor [1] showed that input efficiency measured from the BCC model (Banker et al. [2]) under variable returns to scale will be the same when observations of output factors are transferred. Similarly, the efficiency of output will not be affected by the transfer of observations of input factors. However, input and output factors cannot be transferred simultaneously and efficiency is measured. When the inputs and outputs are negative data and the production technology has constant returns to scale, Seiford and Zhu [3] proposed to assign the smallest constant to any factor that can make them positive, and from conventional DEA models to measure add efficiency. While data transmission does not render efficient DMUs inefficient, and vice versa, the efficiency score increases as the constant is added to the output coefficient (Scheel, [4]).

The second category is factor separation, which separates a factor with negative data into two factors. There are two types of separation, one is used to express the coefficient as the difference between two factors, and the other is used to separate the factor into two factors, one containing data with positive values and the other containing data with positive values. Data with negative values in most cases, negative observations are the difference between two values. For example, profit is the difference between sales and costs. Based on this idea, Halme et al. [5] used two factors to express a factor with negative data, where the minuend maintains the original input or output state and the subtrahend changes to the other state. A drawback noted in the paper was that as the number of agents increases after agent separation, efficiency may increase and some inefficient DMUs may become efficient. Emrouznejad et al. [6,7] proposed a semi-oriented radial measure (SORM), in which the factor with negative data is divided into two factors, one for positive data and the other for negative data.

The third category is slack-based measurement (SBM). The limitations of this type of model are being able to handle negative data. The model presented in Mehdiloozad et al. [8] for density detection is an example. These methods are difficult to calculate the efficiency because negative values are involved. To solve this problem, Sharp et al. [9] proposed a modified slack-based measurement (MSBM) that uses the minimum values of individual inputs and the maximum values of individual outputs to adjust the evaluated DMU values to make them positive. In the next section, it will be shown that the transformed values do not meet the definition of efficiency and will produce biased results. Efficiency in Sharp et al. [9] is the product of the arithmetic mean of input factor efficiency and the harmonic mean of output factor efficiency. Instead of using the product, Pastor and Ruiz [1] used the average of two returns as the final efficiency and used model of Aparicio et al. [10] to find the target closest to the evaluated DMU. Lin et al. [11] used ranges, rather than original observations, in an SBM model to calculate superlative efficiency. Lo and Lu [12] defined the efficiency of SBM in a complex way for output factors with negative data. This form is also biased as shown in the next section. Lin and Liu [13] proposed a super-efficiency model based on the directional distance function in the presence of negative data. They defined a similar bound to adjust the direction vector to make it positive in the extreme efficiency measure.

The fourth category is the directional distance function (DDF) measurement. Conventional DDF models can handle negative data as long as the direction vector is positive, so inputs decrease and outputs expand as we move toward the efficient frontier. Portela et al. [14] proposed a range directional model (RDM) to measure performance in the presence of negative data. To ensure that the direction vector is positive, the largest value of each output factor and the smallest value of each input factor are used to adjust the corresponding observations of the evaluated DMU. The complement of the distance parameter is defined as efficiency. As noted in their paper, since the efficiency score is unable to incorporate all sources of inefficiency, it does not reflect the actual performance of the DMU. In the next section, it will be proven that this model will produce biased returns. The idea of Portela et al. [14] was used by Tavana et al. [15] to measure the efficiency of a two-stage system. Cheng et al. [16] proposed a type of radial measurement (VRM), which is a specific DDF model.

The DDF measure has the limitation that it only shows the degree of inefficiency, without being able to provide measures of efficiency. While all existing methods for handling negative data have merits, they also have disadvantages and limitations, notably that the economic foundations are weak, although mathematically sound. For example, most methods allow all outputs to have negative data. From an economic point of view, this production process consumes resources to absorb inventory outputs instead of producing outputs. This type of production process generally does not exist. If there is, it will not be done by sane people. Conversely, most methods also allow all inputs to have negative data. In this case, the production process produces inputs at the same time as output. This kind of production process cannot exist in reality. In this paper, we propose an inverse DEA model that considers basic production theory to merge when some agents have negative observations.

The objective of the inverse DEA approach is to estimate practical output levels under increasing inputs and maintain the efficiency score in the DEA framework. This inverse optimization problem can be used as a useful planning tool for managerial decisions by providing information such as the number of resources that should be invested to achieve an optimal level of competitiveness. In contrast, conventional DEA mainly focused on evaluating organizational performance. The idea of reverse DEA was first introduced by Zhang and Cui [17] and Wei et al. [18] then formalized it using multi-objective linear programming (MOLP) techniques. After that, this problem has been investigated in many theoretical and applied publications.

Hadi Venche et al. [19] was introduced a new inverse DEA to estimate inputs under increasing outputs and maintaining the efficiency score. Apart from these, a stream of literature has contributed to methodological developments such as input estimation for resource allocation (Zhang, Wang, and Cui [20]), intertemporal application. (Jahanshahloo, Soleimani-damaneh, and Ghobadi [21]), fuzzy inverse DEA (Ghobadi and Jahangiri [22]), inverse DEA with boundary changes (Lim [23]), undesirable outputs or inputs (Eyni, Tohidi, and

Mehrabeian [24]), Cost and income efficiency (Ghiyasi [25]). In addition to demonstrating the method presented in the above texts, inverse DEA has been widely used to deal with various real-world situations. For example: application on sustainability accession (Hassanzadeh, Yousefi, Saen, and Hosseininia [26]), Enterprise merger (Amin et al. [27]; Gattoufi et al. [28]). Gattoufi et al. [28] applied the inverse DEA approach in merger analysis for determining the level of inputs and outputs from the merging entities. The inverse DEA model of Gattoufi et al. [28] determines the quantities of inputs that can be saved by a merger when the output ordination model obtains additional outputs that can be produced for a given efficiency target. Amin et al. [27] proposed a general model for firms' restructuring. The restructuring scenarios, namely consolidation and split. Emrouznejad et al. [29] proposed a new application of inverse DEA in environmental efficiency to determine the optimal allocation of CO_2 emissions reduction in Chinese manufacturing industries. Wegener and Amin [30] proposed an inverse DEA model for minimizing greenhouse gas emissions in the gas and oil industries. Gerami et al. [31] proposed a generalized inverse DEA model for firm restructuring based on value efficiency.

It can be said that the main contribution of this paper is as follows. In this paper, we present three inverse DEA models to obtain the optimal level of inputs and outputs based on the target efficiency in conditions where some of the input and output components are negative. In this regard, a simple algorithm to determine the optimal levels of inputs and outputs from the merging entities. We obtain models based on the process of merging inputs and outputs from two DMUs to create a new unit.

It can be said that the continuation of this paper is organized as follows. The second section presents the inverse DEA model in merging process in presence of positive data. The third section present new three models in inverse DEA in presence of negative data. The fourth section illustrate models with a numerical example. The fifth section proposed an application in banking and at the end we present the results of the research.

2. An inverse DEA approach for dealing with negative data

Let n DMUs as $DMU_j = (X_j, Y_j)$, where the input vector $X_j = (x_{1j}, \dots, x_{mj}) \in R^m$ are used to produce the output vector $Y_j = (y_{1j}, \dots, y_{sj}) \in R^s$. In the traditional case where all observations are nonnegative, the output orientation model for calculating efficiency of $DMU_o = (X_o, Y_o)$ is as follows (Banker et al. [1]).

$$\begin{aligned} &\varphi^{P^*} = \max\ \varphi^P && (1)\\ &S.t.\ \sum_{j=1}^{n} \mu_j\, x_{ij} \le x_{io}\ ,\ i = 1, \dots, m,\\ &\sum_{j=1}^{n} \mu_j\, y_{rj} \ge \varphi^P y_{ro}, \quad r = 1, \dots, s,\\ &\sum_{j=1}^{n} \mu_j = 1,\ \ \mu_j \ge 0,\ j = 1, .., n. \end{aligned}$$

Definition 1. $DMU_o = (X_o, Y_o)$ is efficient DMU if and only if $\varphi^{P^*} = 1$, else it is inefficient.

The efficiency score of $DMU_o = (X_o, Y_o)$ based on the model (1) define as $\frac{1}{\varphi^{P^*}}$.

Suppose $\left(\mu_j^*, \varphi^{P^*}: j = 1, \dots, n\right)$ be an optimal solution of model (1). The target points on the production frontier that $DMU_o = (X_o, Y_o)$ uses to calculate efficiency obtain as $\left(\sum_{j=1}^{n} \mu_j^* x_{ij}, \sum_{j=1}^{n} \mu_j^* y_{rj}\right)$.

In this section, we propose three inverse DEA for measuring efficiency in DEA. These models obtain the optimal level of outputs and based on the target efficiency score of the new unit created in the merger process. Let $\bar{\varphi}^p$ is the target efficiency score for the new unit created (the merged entity T) in the merger process by selecting DMU_k and DMU_h as units to merge in the inverse DEA process. μ_j is the intensity variable. DMUs k and h are consolidating their activities. Let's T show the merged entity generated by the consolidation and also F is the set of indices of all DMUs except k and h. Let β_{rk} and β_{rh} be the levels of the r-th input from the merging DMU k and DMU h, that is kept by the merged entity T respectively. In the conventional case where all observations are nonnegative, in this consolidation, we proposed the following output oriented inverse DEA model.

$$\max \sum_{r=1}^{s} \beta_r \tag{2}$$
$$S.t. \quad \sum_{j\in F} \mu_j x_{ij} + \mu_T(x_{ik} + x_{ih}) \leq (x_{ik} + x_{ih}),\ i = 1, \dots, m,$$
$$\sum_{j\in F} \mu_j y_{rj} + \mu_T\ (y_{rk} + y_{rh} + \beta_r) \geq \bar{\varphi}^p(y_{rk} + y_{rh} + \beta_r),\ \ r = 1, \dots, s,$$
$$\sum_{j=1}^{n} \mu_j = 1, \quad \mu_j \geq 0,\ j = 1, .., n,$$
$$\beta_r, \text{ free in sign},\ \ r = 1, \dots, s.$$

To calculate efficiency in the presence of negative data, we consider three different cases.

First case: the output values of the observed DMU under evaluation and the corresponding target output values are on the positive efficiency frontier.

The second case: the observed unit output values under negative evaluation and the corresponding target output values are on the positive efficiency frontier.

The third case: the output values of the observed unit under evaluation and the corresponding target are both negative on the efficiency frontier.

Now we present three different models to calculate efficiency and perform the merger process. These models are presented based on which of the above states the unit under evaluation has.

In the following, we present inverse DEA models based on efficiency evaluation models.

First, we consider, first case:

If the DMU to be evaluated belongs to first case, in that both the target and observed outputs are positive, then we use model (1) to measure efficiency. We apply model (2) to perform merge process.

Now, we consider second case, in this case, the target output is positive, and the observed output is negative, the efficiency is negative. We know that a DMU with negative observed output has a counterpart DMU with positive observed output in the first. This DMU has the same efficiency, but the opposite sign, as the DMU being assessed. The model for calculating the efficiency of $DMU_o = (X_o, Y_o)$ of second case is thus to calculate the efficiency of its counterpart DMU, $-y_{ro}$, and the efficiency of DMU_o is the negation of that of its counterpart DMU. In this case, we propose model (3) for evaluation efficiency of DMU_o as follows.

$$\varphi^{N^*} = \max\ \varphi^N \tag{3}$$
$$S.t.\ \sum_{j=1}^{n} \mu_j\, x_{ij} \leq x_{io}\ ,\ i = 1, \dots, m,$$
$$\sum_{j=1}^{n} \mu_j\, y_{rj} \geq \varphi^N(-y_{ro}), r = 1, \dots, s,$$
$$\sum_{j=1}^{n} \mu_j = 1, \ \ \mu_j \geq 0,\ j = 1, .., n.$$

Definition 2. $DMU_o = (X_o, Y_o)$ is efficient DMU if and only if $\varphi^{N^*} = 1$, else it is inefficient.

The efficiency score of $DMU_o = (X_o, Y_o)$ based on the model (3) define as $\frac{1}{\varphi^{P^*}}$.

Suppose $\left(\mu_j^*, \varphi^{N^*}: j = 1, \dots, n\right)$ be an optimal solution of model (3). The target points on the production frontier that $DMU_o = (X_o, Y_o)$ uses to calculate efficiency obtain as $\left(\sum_{j=1}^{n} \mu_j^*\, x_{ij}, \sum_{j=1}^{n} \mu_j^*\, y_{rj}\right)$.

Note that the counterpart of a DMU may lie outside of the production possibility set. Then, in this case, the counterpart DMU is super-efficient, and the efficiency score corresponding to this DMU is greater than one, and the original DMU has an efficiency less than minus one. See [32-33].

Similarly, first case, let $\bar{\varphi}^N$ is the target efficiency score for the new unit created (the merged entity T) in the merger process by selecting DMU_k and DMU_h as units to merge in the inverse DEA process in the second case. μ_j is the intensity variable. DMUs k and h are consolidating their activities. Let's T show the merged entity generated by the consolidation and also F is the set of indices of all DMUs except k and h. Let β_{rk} and β_{rh} be the levels of the r-th input from the merging DMU k and DMU h, that is kept by the merged entity T respectively. In this consolidation for second case, we proposed the following output oriented inverse DEA model.

$$\max \sum_{r=1}^{s} \beta_r \quad (4)$$
$$S.t. \quad \sum_{j\in F} \mu_j x_{ij} + \mu_T (x_{ik} + x_{ih}) \leq (x_{ik} + x_{ih}),\ i = 1, \dots, m,$$
$$\sum_{j\in F} \mu_j y_{rj} + \mu_T\ (y_{rk} + y_{rh} + \beta_r) \geq \bar{\varphi}^N (y_{rk} + y_{rh} + \beta_r), \quad r = 1, \dots, s,$$
$$\sum_{j=1}^{n} \mu_j = 1, \quad \mu_j \geq 0,\ j = 1, .., n,$$
$$\beta_r, \text{ free in sign}, \ r = 1, \dots, s.$$

Finally, we consider third case, in this case both the target and observed outputs of the DMU to be assessed are negative, the efficiency score is the ratio of the target output to the observed output. The target point is the contraction point of the assessed DMU on the production frontier. That is, efficiency is the minimum contraction of parameter φ^{ND} that keeps the point $(X_o, \varphi^{ND} Y_o)$ in the production possibility set. The model for calculating efficiency in this case is as follows.

$$\varphi^{ND^*} = \min\ \varphi^{ND} \quad (5)$$
$$S.t.\ \sum_{j=1}^{n} \mu_j\, x_{ij} \leq x_{io}\ ,\ i = 1, \dots, m,$$
$$\sum_{j=1}^{n} \mu_j\, y_{rj} \geq \varphi^{ND} y_{ro}, \quad r = 1, \dots, s,$$
$$\sum_{j=1}^{n} \mu_j = 1, \ \ \mu_j \geq 0,\ j = 1, .., n.$$

In this case, the target output namely $\sum_{j=1}^{n} \mu_j\, y_{rj}$ and the observed output namely y_{ro} are negative, the constraint of $\sum_{j=1}^{n} \mu_j\, y_{rj} \geq \varphi^{ND} y_{ro}, \quad r = 1, \dots, s,$ implies that φ^{ND} positive. Also, the constraint of $\sum_{j=1}^{n} \mu_j = 1, \ \ \mu_j \geq 0,\ j = 1, .., n,$ implies that the target output $\sum_{j=1}^{n} \mu_j\, y_{rj}$ is finite. Therefore, the minimum value of $\frac{\sum_{j=1}^{n} \mu_j y_{rj}}{y_{ro}} = \varphi^{ND}$, $r = 1, \dots, s,$ occurs for an output with the slack output equal to zero, which is also finite. Then, we conclude that efficiency score is a positive finite value in the third case. See [34-35].

Similarly, first, and second case, let $\bar{\varphi}^{ND}$ is the target efficiency score for the new unit created (the merged entity T) in the merger process by selecting DMU_k and DMU_h as units to merge in the inverse DEA process in the third case. DMUs k and h are consolidating their activities.

Let's T show the merged entity generated by the consolidation and also F is the set of indices of all DMUs except k and h. Let β_{rk} and β_{rh} be the levels of the r-th input from the merging DMU k and DMU h, that is kept by the merged entity T respectively. In this consolidation for third case, we proposed the following output oriented inverse DEA model.

$$\max \sum_{r=1}^{s} \beta_r \quad (6)$$
$$S.t. \quad \sum_{j\in F} \mu_j x_{ij} + \mu_T (x_{ik} + x_{ih}) \leq (x_{ik} + x_{ih}),\ i = 1, \dots, m,$$
$$\sum_{j\in F} \mu_j y_{rj} + \mu_T\ (y_{rk} + y_{rh} + \beta_r) \geq \bar{\varphi}^{ND} (y_{rk} + y_{rh} + \beta_r), \quad r = 1, \dots, s,$$
$$\sum_{j=1}^{n} \mu_j = 1, \quad \mu_j \geq 0,\ j = 1, .., n,$$
$$\beta_r, \text{ free in sign}, \ r = 1, \dots, s.$$

To select the appropriate model to calculate efficiency and perform merger process, one must know the signs of the target and observed outputs of the DMU to be assessed. When there are multiple outputs, the outputs of a DMU may not have the same sign, which is also the case for the target outputs. In this case, all the observed outputs and target outputs are aggregated, respectively, to result in one value with one sign for classification. This can be conducted in two steps. First, we can use of additive model of Charnes et al. [32] to find a tentative target point as follows.

$$\max \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \quad (7)$$
$$S.t. \sum_{j=1}^{n} \mu_j x_{ij} + s_i^- = x_{io}, i = 1, \dots, m,$$
$$\sum_{j=1}^{n} \mu_j y_{rj} - s_r^+ = \varphi^{ND} y_{ro}, r = 1, \dots, s,$$
$$\sum_{j=1}^{n} \mu_j = 1, \ \ \mu_j \geq 0, \ j = 1, .., n,$$
$$s_i^- \geq 0, s_r^+ \geq 0, i = 1, \dots, m, r = 1, \dots, s.$$

where $\left(\sum_{j=1}^{n} \mu_j x_{ij}, \sum_{j=1}^{n} \mu_j y_{rj}\right)$ is the target point. We use the dual of this model to aggregate the target outputs and the observed outputs, respectively as follows.

$$\min \sum_{i=1}^{m} v_i x_{io} + u_o - \sum_{r=1}^{s} u_r y_{ro}$$
$$s.t. \ \ \sum_{i=1}^{m} v_i x_{ij} + u_o - \sum_{r=1}^{s} u_r y_{rj} \geq 0, \ \ j = 1, \dots, n, \quad (8)$$
$$v_i \geq 1, \ u_r \geq 1, \ i = 1, \dots, m, \ \ r = 1, \dots, s.$$

Suppose, $(u_r^*, v_i^*, u_o^*: i = 1, \dots, m, r = 1, \dots, s)$ is an optimal solution of model (8). In this case, $(\sum_{i=1}^{m} v_i^* x_{io} + u_o^*)$ and $(\sum_{r=1}^{s} u_r^* y_{ro})$ show the aggregate target and aggregate observed outputs, respectively. See [36].

Based on their signs, appropriate models are applied to calculate efficiency. Models (1), (3), and (5) are solved for cases first, second, and third, respectively, to measure the efficiency for DMU_o . To perform the merger process, we can solve models (2), (4), and (6) respectively.

In the event that, there may exist multiple solutions. The DMUs in first case should be investigated to check if they can also be placed into third case to obtain a higher efficiency score if possible. We can look into this by solving model (8) again, by adding the constraints $\sum_{i=1}^{m} v_i x_{ij} + u_o \leq 0, \sum_{r=1}^{s} u_r y_{ro} \leq 0$.

It is also possible model (5) have a higher efficiency. To review this issue, the DMUs in third case should be investigated to check if they can also be placed into first case, we must solve model (8) again, by adding the constraints $\sum_{i=1}^{m} v_i x_{ij} + u_o \geq 0, \sum_{r=1}^{s} u_r y_{ro} \geq 0$.

The DMUs for second case should be investigated to check if they can also be placed into first, and third case to obtain a higher efficiency score if possible. For all three cases, the highest efficiency score measured from different cases is the final efficiency score of the evaluated DMU. In the following, we will solve the corresponding model for the merger process, based on which model obtains the maximum efficiency score of the unit under evaluation.

3. Case study

In this section, we use of a data in Sharp et al. [9] for illustrating the proposed inverse DEA model in this paper in merger process. Each DMUs consume two inputs for producing three outputs that have been brought in Table 1. The data set are of the notional effluent processing system. DMUs have one positive input including cost, one non-positive input including effluent, one positive output including saleable output and two non-positive outputs including methane and CO2.

At first, we solve model (8) to tentatively determine the model to be applied to measure the efficiency score of the DMUs. The objective value of model (8) is shown in the second column of Table (2). In the event that the objective value is zero, then the corresponding DMU is efficient; otherwise, it is inefficient. The third and fourth columns show the values of the aggregate target and aggregate observed outputs, respectively.

We have three cases for the aggregate target and aggregate observed outputs values: Both the aggregate target and aggregate observed outputs are positive, in this case model (1) is used to measure efficiency; the aggregate target is positive, and aggregate observed outputs is negative, in which case model (3) is used to measure efficiency score, and both the aggregate target and aggregate observed outputs are negative, in this case model (5) is used to measure efficiency score. Of the thirteen DMUs, DMUs 3, 4, 6, 7, 9, 10, 11 and 13 belong to first case; DMUs 1, 2, 5, and 12 belongs to second case, and DMU 5 is only DMU belong to third case. Column five shows the model to be apply for each DMU to measure efficiency score. Similarly, we have three cases for the aggregate target and aggregate observed outputs values: Both the aggregate target and aggregate observed outputs are positive, in this case model (2) is used to performing the merger process in the inverse DEA; the aggregate target is positive, and aggregate observed outputs is negative, in which case model (4) is used to performing the merger process in the inverse DEA, and both the aggregate target and aggregate observed outputs are negative, in this case model (6) is used to performing the merger process in inverse DEA. The six column of Table (2) show the model to be use for each DMU to performing the merger process.

Table 1: Input–output data. (Sharp et al. [9])

DMU	Input1	Input2	Output1	Output2	Output3
1	1.03	-0.05	0.56	-0.09	-0.44
2	1.75	-0.17	0.74	-0.24	-0.31
3	1.44	-0.56	1.37	-0.35	-0.21
4	10.8	-0.22	5.61	-0.98	-3.79
5	1.3	-0.07	0.49	-1.08	-0.34
6	1.98	-0.1	1.61	-0.44	-0.34
7	0.97	-0.17	0.82	-0.08	-0.43
8	9.82	-2.32	5.61	-1.42	-1.94
9	1.59	0	0.52	0	-0.37
10	5.96	-0.15	2.14	-0.52	-0.18
11	1.29	-0.11	0.57	0	-0.24
12	2.38	-0.25	0.57	-0.67	-0.43
13	10.3	-0.16	9.56	-0.58	0

Table 2: The results of models for DMUs

DMU	Objective	Aggregate target	Aggregate observed outputs	DEA model	Efficiency	Inverse DEA model
1	0.4756	0.3656	-0.11	Model (3)	0.8963	Model (4)
2	1.1489	0.9656	-0.1833	Model (3)	0.7005	Model (4)
3	0	0.2656	0.2656	Model (1)	1	Model (2)
4	9.1057	10.6923	1.5866	Model (1)	0.5936	Model (2)
5	1.9649	1.0349	-0.93	Model (3)	0.7	Model (4)
6	0.9481	1.9923	1.0443	Model (1)	0.8613	Model (2)
7	0	0.1856	0.1856	Model (1)	1	Model (2)
8	0	-36.8533	-36.8533	Model (5)	1	Model (6)
9	0.59	0.74	0.15	Model (1)	0.9123	Model (2)
10	4.1443	4.65	0.5057	Model (1)	0.3857	Model (2)
11	0	0.33	0.33	Model (1)	1	Model (2)
12	2.59	2.06	-0.53	Model (3)	0.4496	Model (4)
13	0	8.98	8.98	Model (1)	1	Model (2)

Table 3: The results of inverse DEA process.

Merge DMU	Target Efficiency	β_1	β_2	β_3	$\beta_1 + y_{1k} + y_{1h}$	$\beta_2 + y_{2k} + y_{2h}$	$\beta_3 + y_{3k} + y_{3h}$
$DMU_k = DMU_3$, $DMU_h = DMU_4$	0.8	1.7764	0.1518	1.6520	10.5328	-1.0264	-0.6961
$DMU_k = DMU_3$, $DMU_h = DMU_4$	0.9	1.1912	0.2088	1.6906	9.3624	-0.9123	-0.6187
$DMU_k = DMU_3$, $DMU_h = DMU_4$	1	0.7231	0.2544	1.7216	8.4262	-0.8211	-0.5569
$DMU_k = DMU_9$, $DMU_h = DMU_{10}$	0.8	3.0562	-0.0579	0.2343	8.7724	-0.6358	-0.0815
$DMU_k = DMU_9$, $DMU_h = DMU_{10}$	0.9	2.5689	-0.0226	0.2388	7.7977	-0.5651	-0.0724
$DMU_k = DMU_9$, $DMU_h = DMU_{10}$	1	2.1790	0.0057	0.2424	7.0180	-0.5086	-0.0652
$DMU_k = DMU_1$, $DMU_h = DMU_2$	0.8	0.9804	-0.0755	0.2636	3.2608	-0.4810	-0.2228
$DMU_k = DMU_1$, $DMU_h = DMU_2$	0.9	0.7993	-0.0488	0.2760	2.8985	-0.4275	-0.1980
$DMU_k = DMU_1$, $DMU_h = DMU_2$	1	0.6543	-0.0274	0.2859	2.6087	-0.3848	-0.1782
$DMU_k = DMU_5$, $DMU_h = DMU_8$	0.8	2.0333	-0.8333	-0.7600	10.1667	-4.1667	-3.8000
$DMU_k = DMU_5$, $DMU_h = DMU_8$	0.9	0.7625	-0.3125	-0.2850	7.6250	-3.1250	-2.8500
$DMU_k = DMU_5$, $DMU_h = DMU_8$	1	0	0	0	6.1	-2.5	-2.28

We show the merged DMU by T. In the output-orientation, model keeps the amount of output of both DMUs, that is $x_T = x_k + x_h$, and the models (2), (4), (6) find the maximum amount of the outputs of DMU_k and DMU_h in order to reach the desired given efficiency target. Table (3) shows the levels of the outputs from the merging DMU k and DMU h, for predetermined target efficiency score of DMU T. We present the results for different selection of DMU_k and DMU_h as units to merge in the inverse DEA process and different target efficiency score in Table (3).

4. Conclusion

In traditional DEA models to measure efficiency, we assume that the data are non-negative. But, in the real world, we come across many cases where the data are negative. Existing methods for dealing with negative data have advantages. They also have disadvantages and limitations. In this paper, we first examined the performance evaluation models when the inputs and outputs have negative values. We showed that we can evaluate efficiency models based on whether the total target output and the total observed output have positive or negative values, we presented three different models to evaluate the efficiency of DMUs. These three cases are used to measure efficiency as long as total target output and total observed output are both positive, the first is positive and the second is negative and both are negative. Also, we explained how to use each of these models if these models have alternative optimal solutions. In the following, we presented inverse DEA models to perform the merger process of DMUs corresponding to each of the three modes. The presented models obtain the new level of the new unit created in the merger process to reach the target efficiency level. As future work, we can develop the presented models in other technologies, such as semi additive technology and FDH. We can also develop the proposed approach in this paper for other data structures such as two-stage network.

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