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Solving Fuzzy LR Interval Linear Systems using Nonlinear Programming

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ABSTRACT

In this paper, we use the least squares method to solve LR fuzzy interval systems by transforming an interval fuzzy number into two triangular fuzzy numbers. Then, we reduce the distance between the two obtained triangular fuzzy numbers to solve the fuzzy LR interval linear system. Essentially, we convert an LR fuzzy interval linear system into a triangular fuzzy linear system and subsequently solve it using the least squares method introduced in [17, 18].

1. Introduction

Solving a fuzzy linear system has been popular during the past two decades [5, 6]. Fuzzy linear systems have been studied under various assumptions. Buckley et al. [5, 7] studied square fuzzy linear systems, $\tilde{A}\tilde{x} = \tilde{b}$, based on α-cuts, extension principal, and interval arithmetic, where the elements of $\tilde{A}$ and $\tilde{b}$ are fuzzy numbers. Stanimirovi´c and Mici´c [36] described a set of fuzzy relations that solves weakly linear systems to a certain degree and provides ways to compute them. They paid special attention to developing the algorithms for computing fuzzy pre-orders and fuzzy equivalences that are solutions to some extent to weakly linear systems. Stanimirovi´c and Mici´c [36] established additional properties for the set of such approximate solutions over some particular types of complete residuated lattices. They demonstrated the advantage of this approach via many examples that arise from the problem of aggregation of fuzzy networks. Zarei et al. [37] studied first-order linear fuzzy systems under generalized differentiability and presented the general form of their solutions. Then, the fuzzy optimal control problem of these systems was considered to optimize the expected values of the appropriate objective fuzzy functions. The pontryagin maximum principle was used to obtain a necessary

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optimality condition in the form of a fuzzy boundary value problem. Using the necessary optimality condition, the constant formulas for the fuzzy optimal control function and the corresponding fuzzy state function were proposed. Abbasi and Allahviranloo [1] presented a method for solving the fully fuzzy linear systems by using the transmission-average-based fuzzy operations introduced by Abbasi et al. [2]. Also, they presented the necessary conditions for the existence and uniqueness of the fuzzy solution. Park et al. [33] introduced an admissibilization condition for singular interval-valued fuzzy systems with a dynamic output-feedback controller using a linear matrix inequality approach. For the closed-loop system of the singular interval-value fuzzy systems using the dynamic output-feedback controller, the derivation of the admissibility criterion (satisfying regularity, non-impulsiveness, and stability) was concerned. The derived criterion was represented as the parameterized matrix inequalities depending on the membership functions of the system and the controller. Gasilov et al. [12] proposed a new solution method for a nonhomogeneous fuzzy linear system of differential equations. The coefficients of the considered system were crisp while forcing functions and initial values were fuzzy. They considered each forcing function to be in a special form, which they called a triangular fuzzy function and which represents a fuzzy bunch (set) of real functions. They constructed a solution as a fuzzy set of real vector functions, not as a vector of fuzzy-valued functions, as usual. Ghanbari and Mahdavi-Amiri [16] and Ghanbari et al. [17] gave results on the solvability of LR fuzzy interval systems. Also, Ghanbari et al. [16, 17] proposed a new definition of an approximate solution when an exact solution does not exist. In both works [16] and [17] (also [21]), least squares models were used. Indeed, they [16] showed that the approximate solution proposed in [16] is more robust and proper than the solution proposed in [17].

In [16] (also, [15, 17, 21]), the authors studied fuzzy linear systems with fuzzy variables. Here, we intend to develop the recent results given by Ghanbari and Mahdavi-Amiri [16] to the fuzzy interval linear systems when the type of fuzzy interval is LR [38]. We will show that some similar results given by Ghanbari and Mahdavi-Amiri [16] are valid for fuzzy LR interval linear systems. In addition, using a similar definition of an approximate solution for the fuzzy linear systems given by [16], we will give some new results for LR fuzzy interval linear systems. Nafei et al. [26] presented extensions of fuzzy sets such as interval-valued fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, type n-fuzzy sets, and neutrosophic sets, provided powerful and practical tools for dealing with uncertainty in decision-making problems.
They first proposed a modified score function for ranking single-valued neutrosophic numbers. Then, they suggested a TOPSIS method based on the proposed function for decision-making under group recommendation. Also, Nafei et al. [27] presented a new method for group multi-attribute decision-making (GMADM) based on interval neutrosophic sets, where decision-makers determine the weights and the evaluating values of the attributes with respect to the available alternatives by using interval neutrosophic values. In another work,
Nafei et al. [25] developed a new Hamming distance between single-valued neutrosophic numbers and then presented an extension of the TOPSIS method for multi-attribute group decision-making (MAGDM) based on single-valued neutrosophic sets, where the information about attribute values and attribute weights are expressed by decision-makers based on neutrosophic numbers. See the others work in [3, 10, 24, 35].

It is noteworthy that fuzzy interval linear systems are studied by ([9, 13, 20, 31,32, 34]). All of these methods were based on the crisp or fuzzy linear algebraic methods. However, we propose an approximate solution based on a least squares model and the distance function proposed by Ming et al. [22]. To compute an approximate solution, we propose a quadratic programming model with linear constraints.

The concept proposed here has many applications in real case studies. For example, fuzzy linear systems of equations play a major role in various financial and economic applications. We can analyze a particular fuzzy linear system: the derivation of the risk-neutral probabilities in a fuzzy binary tree [14,23].

The rest of the paper is organized as follows. In Section 2, we give some necessary and sufficient conditions for solving fuzzy LR linear systems. We propose a new concept for the approximate solution of a fuzzy linear system and present a quadratic programming model to compute such solutions in Section 3. In Section 4, we use numerical experiments to demonstrate the goodness of fit of our approximate solution compared to other approximate solutions. We conclude in Section 5.

2. Solvability of LR fuzzy interval linear systems

First, we describe some concepts which are used in our paper. Then we discuss a bout solvability of fuzzy LR interval linear systems.

Definition 1. (Transformed interval fuzzy number) For any arbitrary fuzzy LR interval number, there exist $\tilde{a} = (a^l, a^r, a^\alpha, a^\beta)$ two LR triangular fuzzy numbers $\tilde{a}_{left}$ and $\tilde{a}_{right}$ which are defined as follows:

$$\tilde{a}_{left} = (a^l, a^\alpha, a^r - a^l + a^\beta)_{LR}, \quad \tilde{a}_{right} = (a^r, a^r - a^l + a^\alpha, a^\beta)_{LR} \qquad (1)$$

The notations in (1) are shown in Figure 1.

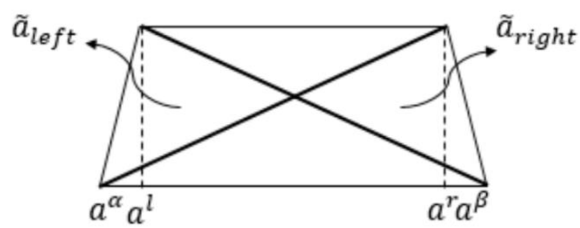

Figure 1. Transformed interval fuzzy number

Example 1. Let $\tilde{a} = (-2,3,4,1)$ be a fuzzy LR interval number. Corresponding to $\tilde{a}$ two LR triangular fuzzy numbers $\tilde{a}_{left}$ and $\tilde{a}_{right}$ are calculated as follows:

$$\tilde{a}_{left} = (-2,4,6)_{LR}, \quad \tilde{a}_{right} = (3,9,1)_{LR}.$$

Definition 2. [30] A fuzzy number is a fuzzy quantity A satisfying the following conditions:
1. $\mu_{\tilde{A}}(x) = 1$, for exactly one x.
2. The support $\{x: \mu_{\tilde{A}}(x) > 0\}$ of A is bounded.
3. The α-cuts of A are closed intervals.

Definition 3. [38] A fuzzy number $\tilde{A}$ is an LR-type if there exist shape function L (for left), R (for right) and scalars $\alpha \geq 0$, $\beta \geq 0$ with,

$$\begin{cases} L\left(\frac{a-x}{\alpha}\right), & x \leq a, \\ R\left(\frac{x-a}{\beta}\right), & x \geq a. \end{cases}$$

The mean value $\tilde{A}$, a, is a real number, α and β are called the left and right spreads, respectively. $\tilde{A}$ is denoted $\tilde{A} = (a, \alpha, \beta)$ (see Figure 2).

Remark 1. Based on Definition (3), another representation of an LR fuzzy number $\tilde{A}$ is $\tilde{A} = (A_L, A_R)$, where A_L is a shape function for the left arm and A_R is a shape function for the right arm.

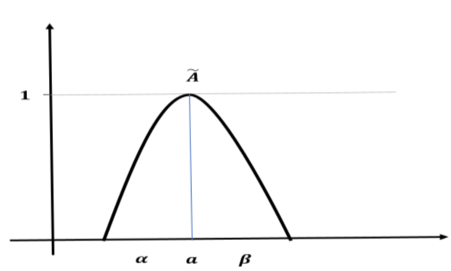

Figure 2. LR fuzzy number

Definition 4. [38] A fuzzy trapezoidal or interval $\tilde{a} = (a_l, a_r, a_\alpha, a_\beta)_{LR}$ is of LR type, if there exist shape functions L and R (for left and right) and scalars $\alpha \geq 0$, $\beta \geq 0$, a_l and a_r with the following membership function:

$$\mu_{\tilde{a}}(x) = \begin{cases} L\left(\frac{a_l - x}{\alpha}\right), & a_l - \alpha \leq x \leq a_l, \\ 1, & a_l \leq x \leq a_r, \\ R\left(\frac{x - a_r}{\beta}\right), & a_r \leq x \leq a_r + \beta, \\ 0, & o.w., \end{cases}$$

which is called a fuzzy LR interval number, where $\alpha, \beta \geq 0$ and $a_r \geq a_l$ and L, R are generating functions nondecreasing and nonincreasing of R^+ into [0, 1], respectively.

Theorem 1. Let $\tilde{a} = (a_l, a_r, a_\alpha, a_\beta)_{LR}$, $\tilde{b} = (b_l, b_r, b_\theta, b_\gamma)_{LR}$, and $\delta \in R^+$. Then [38],

1. $\delta \geq 0 \Longrightarrow \delta\tilde{a} = (\delta a_l, \delta a_r, \delta a_\alpha, \delta a_\beta)_{LR}$.
2. $\delta \leq 0 \Longrightarrow \delta\tilde{a} = (\delta a_r, \delta a_l, -\delta a_\beta, -\delta a_\alpha)_{LR}$.
3. $\tilde{a} \oplus \tilde{b} = (a_l + b_l, a_r + b_r, a_\alpha + b_\theta, a_\beta + b_\gamma)_{LR}$.

Remark 2. Here, the set of LR fuzzy intervals is denoted by $I(\mathbb{R}^1)_{LR}$.

Definition 5. The vector $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T$ is called a fuzzy LR interval vector, if $\tilde{x}_j = (\tilde{x}_l^j, \tilde{x}_r^j, \tilde{x}_\alpha^j, \tilde{x}_\beta^j) \in I(R^1)_{LR}$, $j = 1, \dots, n$. To indicate the fuzzy LR interval vector $\tilde{x}$, we use the notations $\tilde{x} = (x_l, x_r, x_\alpha, x_\beta)_{LR}$, $x_l = (x_l^1, x_l^2, \dots, x_l^n)^T$, $x_r = (x_r^1, x_r^2, \dots, x_r^n)^T$, $x_\alpha = (x_\alpha^1, x_\alpha^2, \dots, x_\alpha^n)^T$, and $x_\beta = (x_\beta^1, x_\beta^2, \dots, x_\beta^n)^T$.
Now, we give the definition of a fuzzy LR interval linear system.

Consider an FLRILS, corresponding to coefficients matrix $A = [a_{ij}]_{m\times n}$.We define the following two matrices

$$[B^+]_{ij} = \begin{cases} a_{ij}, & a_{ij} \geq 0, \\ 0, & a_{ij} < 0, \end{cases} \qquad [B^-]_{ij} = \begin{cases} a_{ij}, & a_{ij} < 0, \\ 0, & a_{ij} \geq 0, \end{cases} \tag{2}$$

for all $i = 1, \dots, m$ and $j = 1, \dots, n$. Also, let $b_l = (b_l^1, \dots, b_l^m)^T$, $b_r = (b_r^1, \dots, b_r^m)^T$, $b_\alpha = (b_\alpha^1, \dots, b_\alpha^m)^T$ and $b_\beta = (b_\beta^1, \dots, b_\beta^m)^T$.

Definition 6. According to (1) the fuzzy numbers in fuzzy LR interval (2) are define as follows:

$$\tilde{x}_{left} = (x^l, x^\alpha, x^r - x^l + x^\beta)_{LR}, \quad \tilde{x}_{right} = (x^r, x^r - x^l + x^\alpha, x^\beta)_{LR},$$
$$\tilde{b}_{left} = (b^l, b^\alpha, b^r - b^l + b^\beta)_{LR}, \quad \tilde{b}_{right} = (b^r, b^r - b^l + b^\alpha, b^\beta)_{LR}.$$

According to Defintion (6), the vectors $\tilde{x}_{new}$ and $\tilde{b}_{new}$ are defined as follows:

$$\tilde{x}_{new} = (x_{new}, x_{new}^\alpha, x_{new}^\beta) = \begin{bmatrix} \tilde{x}_{left} \\ \tilde{x}_{right} \end{bmatrix}, \qquad \tilde{b}_{new} = (b_{new}, b_{new}^\alpha, b_{new}^\beta) = \begin{bmatrix} \tilde{b}_{left} \\ \tilde{b}_{right} \end{bmatrix}$$

$$x_{new} = \begin{bmatrix} x^l \\ x^r \end{bmatrix}, \; x_{new}^\alpha = \begin{bmatrix} x^a \\ x^r - x^l + x^a \end{bmatrix}, \; x_{new}^\beta = \begin{bmatrix} x^r - x^l + x^\beta \\ x^\beta \end{bmatrix}$$

$$b_{new} = \begin{bmatrix} b^l \\ b^r \end{bmatrix}, \; b_{new}^\alpha = \begin{bmatrix} b^\alpha \\ b^r - b^l + b^\alpha \end{bmatrix}, \; b_{new}^\beta = \begin{bmatrix} b^r - b^l + b^\beta \\ b^\beta \end{bmatrix}$$

Definition 7. The following system is called a fuzzy LR converted linear system (FLRCLS):

$$A_{new}\tilde{x}_{new} = \tilde{b}_{new}, \qquad (3)$$

where, $A_{new} \in \mathbb{R}^{2m\times 2n}$ and $\tilde{x}_{new} = (\tilde{x}_{new}^T, \tilde{x}_{new}^T)_{LR}^T \in F(\mathbb{R}^2 n)$ and $\tilde{b}_{new} = (\tilde{b}_{new}^T, \tilde{b}_{new}^T)_{LR}^T \in F(\mathbb{R}^2 m)$.

Theorem 2. (Fundamental Theorem of FLRCLS) Let the system (3), $\tilde{x}_{new} \in F(2\mathbb{R}^n)_{LR}$ is a solution of the system (3), if and only if $(x^{lT}, x^{rT}, x^{\alpha T}, x^{\beta T})$ is the solution of the following crisp system:

$$\begin{cases} B^+x^l + B^-x^r = b^l, \\ B^+x^r + B^-x^l = b^r, \\ B^+x^\alpha - B^-x^\beta = b^\alpha, \\ B^+x^\beta - B^-x^\alpha = b^\beta, \\ x^r \geq x^l, \\ x^\alpha, x^\beta \geq 0. \end{cases} \qquad (4)$$

Proof 1. According to (4), for the equations (1), (2) and (5) we can write:

$$A_{new}x_{new} = b_{new} \qquad (5)$$

and the constraints (3) to (6) are equivalent to:

$$\begin{bmatrix} B_{new}^+ & -B_{new}^- \\ -B_{new}^- & B_{new}^+ \end{bmatrix} \begin{bmatrix} x_{new}^\alpha \\ x_{new}^\beta \end{bmatrix} = \begin{bmatrix} b_{new}^\alpha \\ b_{new}^\beta \end{bmatrix}, \quad \begin{bmatrix} x_{new}^\alpha \\ x_{new}^\beta \end{bmatrix} \geq 0, \qquad (6)$$

Now, it is sufficient to prove $\tilde{x}_{new} \in F(R^2 n)_{LR}$ is the solution of (3), if and only if x_{new} and $(x_{new}^{\alpha^T}, x_{new}^{\beta^T})^T$ are the solution of two systems (5) and (6),we can prove it based on Theorem 3.1 [19].

Theorem 3. The vector $\tilde{x} = (x^l, x^r, x^\alpha, x^\beta)_{LR} \in I(R^n)_{LR}$ is the solution of $A\tilde{x} = \tilde{b}$, if and only if $\tilde{x}_{new}$ is the solution of (4).

Proof 2. Let $\tilde{x} = (x^l, x^r, x^\alpha, x^\beta)$ is a solution of the system $A\tilde{x} = \tilde{b}$, so the following systems have solution

$$\begin{cases} B^+x^l + B^-x^r = b^l, \\ B^-x^l + B^+x^r = b^r, \end{cases} \qquad (7)$$

and

$$\begin{cases} B^+x^\alpha + B^-x^\beta = b^\alpha, \\ -B^-x^\alpha + B^+x^\beta = b^\beta, \\ x^\alpha, x^\beta \geq 0. \end{cases} \qquad (8)$$

Considering to the solvability of the systems (7) and (8), the following system has the solution:

$$\begin{cases} B^+x^\alpha - B^-x^\beta = b^\alpha, \\ B^+x^r - B^-x^l + B^+x^\alpha - B^-x^r + B^-x^l - B^-x^\beta = b^r - b^l + b^\alpha, \\ -B^-x^r + B^-x^l - B^-x^\alpha + B^+x^r - B^+x^l + B^+x^\beta = b^r - b^l + b^\beta, \\ -B^-x^\alpha + B^+x^\beta = b^\beta. \end{cases} \qquad (9)$$

We can rewrite the system (2.9) in the matrix form as follows:

$$\begin{bmatrix} B^+ & 0 & 0 & -B^- \\ 0 & B^+ & -B^- & 0 \\ 0 & -B^- & B^+ & 0 \\ -B^- & 0 & 0 & B^+ \end{bmatrix} \begin{bmatrix} x^\alpha \\ x^r - x^l + x^\alpha \\ x^r - x^l + x^\beta \\ x^\beta \end{bmatrix} = \begin{bmatrix} b^\alpha \\ b^r - b^l + b^\alpha \\ b^r - b^l + b^\beta \\ b^\beta \end{bmatrix} \qquad (10)$$

On the one hand, since $\tilde{x}$ is a fuzzy LR interval number, so $x^\beta \geq 0$, $x^\alpha \geq 0$, $x^r \geq x^l$ ($x^r - x^l \geq 0$), so $x_{new}^\alpha, x_{new}^\beta$. Tuhs, we can write the systems (7) and (10) as follows:

$$A_{new}x_{new} = b_{new},$$

and

$$\begin{bmatrix} B_{new}^{+} & -B_{new}^{-} \\ -B_{new}^{-} & B_{new}^{+} \end{bmatrix} \begin{bmatrix} x_{new}^{\alpha} \\ x_{new}^{\beta} \end{bmatrix} = \begin{bmatrix} b_{new}^{\alpha} \\ b_{new}^{\beta} \end{bmatrix}, \quad \begin{bmatrix} x_{new}^{\alpha} \\ x_{new}^{\beta} \end{bmatrix} \geq 0$$

$\tilde{x}_{new}$ is the solution of system (4).

On the contrary, let $\tilde{x}_{new}$ is the solution of the system (4), so by using the equation (1), (2) and (5) in (4) we conclude the following system:

$$\begin{cases} B^{+}x^{l} + B^{-}x^{r} = b^{l} \\ B^{-}x^{l} + B^{+}x^{r} = b^{r} \\ x^{r} \geq x^{l} \end{cases}$$

by using of the equation (3) to (6), we can conclude:

$$\begin{cases} B^{+}x^{\alpha} - B^{-}x^{\beta} = b^{\alpha}, \\ -B^{-}x^{\alpha} + B^{+}x^{\beta} = b^{\beta} \\ x^{\alpha}, x^{\beta} \geq 0 \end{cases}$$

and this means the system $A\tilde{x} = \tilde{b}$ is solvable.

By reducing the distance between $A\tilde{x}$ and $\tilde{b}$ an approximate or exact solution for the system $A\tilde{x} = \tilde{b}$ can be calculated. On the other hand, considering Theorem 2.7, since the system (5) is equivalent to the system $A\tilde{x} = \tilde{b}$, it can be expected that reducing the distance $A\tilde{x}$ and $\tilde{b}$ will be equivalent to reducing the distance Anew $A_{new}\tilde{x}_{new}$ and $\tilde{b}_{new}$. Therefore, in the next section, we will obtain the exact or approximate solution of the triangular fuzzy LR linear system (5) by solving a quadratic programming problem similar to the method proposed by Ghanbari and Mahdavi Amiri [16], and then using a definition, we will construct the exact or approximate solution of the system $A\tilde{x} = \tilde{b}$.

3. Proposed algorithm for finding an approximate solution

Here, similar to the concept an approximate of the solution proposed in [16], we define an approximate solution for FLRILS. For two fuzzy LR interval vectors $\tilde{x} = (\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n)^T$ and $\tilde{y} = (\tilde{y}^1, \tilde{y}^2, \dots, \tilde{y}^n)^T$ with $\tilde{x}^j = \left(x_l^j, x_r^j, x_\alpha^j, x_\beta^j\right)_{LR}$ and $\tilde{y}^j = \left(y_l^j, y_r^j, y_\alpha^j, y_\beta^j\right)_{LR}$, $j = 1, \dots, n$, Ming et al. [22] defined the following distance function:

$$\begin{aligned} 2D_n^2(\tilde{x}, \tilde{y}) &= 2(x_l - y_l)^T(x_l - y_l) \\ &+2(x_r - y_r)^T(x_r - y_r) - 2(x_l - y_l)^T(x_\alpha - y_\alpha) \\ &+2(x_r - y_r)^T(x_\beta - y_\beta) + (x_\alpha - y_\alpha)^T(x_\alpha - y_\alpha) \\ &+(x_\beta - y_\beta)^T(x_\beta - y_\beta). \end{aligned} \tag{11}$$

Now, for each $\tilde{x} \in I(\mathbb{R}^n)_{LR}$, we define the residual at $\tilde{x}$ as follows:

$$r(\tilde{x}) = 2D_n^2(A\tilde{x}, \tilde{b}). \tag{12}$$

Therefore, for each $\tilde{x}_{new} \in F\left(\mathbb{R}^2 n\right)_{LR}$ he residual at $\tilde{x}_{new}$ is defined as follows:

$$r(\tilde{x}_{new}) = 2D_n^2(A_{new}\tilde{x}_{new}, \tilde{b}_{new})$$

Thus, to compute an approximate or exact solution of (5), we must solve the following optimization problem:

$$\begin{cases} \min r\,(\tilde{x}_n ew) = 2D_n^2(A_{new}\tilde{x}_{new}, \tilde{b}_{new}), \\ s.t., \\ \tilde{x}_{new} \in F\left(\mathbb{R}^2 n\right)_{LR}. \end{cases} \tag{13}$$

where, for each vector $\tilde{x}_{new} = \left(x_{new}, x_{new}^{\alpha}, x_{new}^{\beta}\right)$ *and* $\tilde{b}_{new} = \left(b_{new}, b_{new}^{\alpha}, b_{new}^{\beta}\right)$.

According to definition B_{new}^{+} and B_{new}^{-} in 2.2, we can write:

$$A\tilde{x}_{new} = \left(A_{new}x_{new}, B_{new}^{+}x_{new}^{\alpha} - B_{new}^{-}x^{\beta}, -B_{new}^{-}x_{new}^{\alpha} + B_{new}^{+}x_{new}^{\beta}\right)_{LR},$$

We have,

$$r\left(\tilde{x}_{\text{new}}\right) = 2D_n^2\left(A_{\text{new}}\tilde{x}_{\text{new}},\tilde{b}_{\text{new}}\right) = 4\left(A_{\text{new}}x_{\text{new}} - b_{\text{new}}\right)^T\left(A_{\text{new}}x_{\text{new}} - b_{\text{new}}\right)$$
$$+\left(B^+{}_{\text{new}}x^{\alpha}_{\text{new}} - B^-{}_{\text{new}}x^{\beta}_{\text{new}} - b^{\alpha}_{\text{new}}\right)^T\left(B^+{}_{\text{new}}x^{\alpha}_{\text{new}} - B^-{}_{\text{new}}x^{\beta}_{\text{new}} - b^{\alpha}_{\text{new}}\right)$$
$$+\left(-B^-{}_{\text{new}}x^{\alpha}_{\text{new}} + B^+{}_{\text{new}}x^{\beta}_{\text{new}} - b^{\beta}_{\text{new}}\right)^T\left(-B^-{}_{\text{new}}x^{\alpha}_{\text{new}} + B^+{}_{\text{new}}x^{\beta}_{\text{new}} - b^{\beta}_{\text{new}}\right) \quad (14)$$
$$+2\left(A_{\text{new}}x_{\text{new}} - b_{\text{new}}\right)^T\left(-B^-{}_{\text{new}}x^{\alpha}_{\text{new}} + B^+{}_{\text{new}}x^{\beta}_{\text{new}} - B^+{}_{\text{new}}x^{\alpha}_{\text{new}}\right.$$
$$\left.+B^-{}_{\text{new}}x^{\beta}_{\text{new}} + b^{\alpha}_{\text{new}} - b^{\beta}_{\text{new}}\right).$$

The matrix for of (14) by calculating $Q_{6n\times 6n}$, $f_{6n\times 1}$ and $c_{1\times 1}$ is as follows:

$$r(\tilde{x}_{new}) = \frac{1}{2}\left[x^T_{new},\quad x^{\alpha^T}_{new},\quad x^{\beta^T}_{new}\right] Q \begin{bmatrix} x^T_{new} \\ x^{\alpha^T}_{new} \\ x^{\beta^T}_{new}\end{bmatrix} + f^T \begin{bmatrix} x^T_{new} \\ x^{\alpha^T}_{new} \\ x^{\beta^T}_{new}\end{bmatrix} + c,$$

where c is constant and Q is a symmetric matrix. Therefore, to compute an approximate or exact solution of (5), we can solve the following quadratic programming problem:

$$\begin{cases} \frac{1}{2}\left[x^T_{new},\quad x^{\alpha^T}_{new},\quad x^{\beta^T}_{new}\right] Q \begin{bmatrix} x^T_{new} \\ x^{\alpha^T}_{new} \\ x^{\beta^T}_{new}\end{bmatrix} + f^T \begin{bmatrix} x^T_{new} \\ x^{\alpha^T}_{new} \\ x^{\beta^T}_{new}\end{bmatrix} + c, \\ s.t., \\ x^{\alpha}_{new}, x^{\beta}_{new} \geq 0. \end{cases} \quad (15)$$

After solving (15), we can find x_{new}, x^{α}_{new} and x^{β}_{new}. We can find an approximate or exact solution by using following definition:

Definition 8. Let $\tilde{x}_{new}$ is the solution of (15), so we define:

$$\alpha^1_j = x^{\alpha}_j, \quad \alpha^2_j = x^r_j - x^l_j + x^{\alpha}_j, \quad \alpha^3_j = x^r_j - x^l_j + x^{\beta}_j, \quad \alpha^4_j = x^{\beta}_j,$$

So,

$$\tilde{x}_{left} = \left(x^l_j, \min \alpha^1_j, \alpha^2_j, \max \alpha^3_j, \alpha^4_j\right)_{LR},$$
$$\tilde{x}_{right} = \left(x^r_j, \max \alpha^1_j, \alpha^2_j, \min \alpha^3_j, \alpha^4_j\right)_{LR}, \text{ for all } j = 1,\cdots,n.$$

Now, considering the obtained values for $\tilde{x}_{left}$ and $\tilde{x}_{right}$ we have the following cases:

1. If we have $x^l_j \leq x^r_j, for\ all\ j = 1,\cdots,n$ in the solution then, the interval LR fuzzy vector $\tilde{x}_j = \left(x^l_j, x^r_j, x^{\alpha}_j, x^{\beta}_j\right)_{LR}$ equal to the j-th component of the solution $\tilde{x}$ for the system $A\tilde{x} = \tilde{b}$.
2. If we have $x^l_j \geq x^r_j$, for all $j = 1,\cdots,n$ in the solution then, the interval LR fuzzy vector $\tilde{x}_j = \left(x^r_j, x^l_j, x^r_j - x^l_j + x^{\alpha}_j, x^r_j - x^l_j + x^{\beta}_j\right)_{LR}$ equal to the j-th component of the solution $\tilde{x}$ for the system $A\tilde{x} = \tilde{b}$.

Algorithm 1 shows the proposed algorithm of this method.

Algorithm 1 Algorithm to compute a solution to the system FLRCLS.

1. Get A and $\tilde{b}$.
2. Compute Q_{new} and f_{new} [18].
3. Solve quadratic programming problem (3.15) by the proposed algorithm in [16], and introduce fuzzy LR vector $\tilde{x}_{new} = \left(x_{new}, x^{\alpha}_{new}, x^{\beta}_{new}\right)$ as a solution to the system (3.15).
4. Using Definition 3.1 and calculate the solution of the system $A\tilde{x} = \tilde{b}$.
5. Get $\tilde{x}^*$. If $r(\tilde{x}^*) = 0$, then $\tilde{x}^*$ is an exact solution for the system $A\tilde{x} = \tilde{b}$ else $\tilde{x}^*$ is an approximate solution for the system $A\tilde{x} = \tilde{b}$.

4. Examples and numerical results

In the following examples, we show the superiority of our method compared to the proposed method in [28].

Example 4.1 Consider

$$\begin{cases} 3\tilde{x}_1 + 2\tilde{x}_2 = (4,13,5,8)_{LR} \\ -5\tilde{x}_1 + 3\tilde{x}_2 = (-18,-4,13,8)_{LR} \end{cases}$$

We have:

$$\begin{bmatrix} 3 & 2 \\ -5 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 2 & 0 & 0 \\ 0 & 3 & -5 & 0 \\ 0 & 0 & 3 & 2 \\ -5 & 0 & 0 & 2 \end{bmatrix}$$

According to $\tilde{b}$ we can write:

$$\tilde{b}_{left} = \begin{bmatrix} (4,5,17)_{LR} \\ (-18,13,22)_{LR} \end{bmatrix}, \qquad \begin{bmatrix} (13,14,8)_{LR} \\ (-4,27,8)_{LR} \end{bmatrix}$$

So, we have:

$$\tilde{b}_{new} = \begin{bmatrix} (4,5,17)_{LR} \\ (-18,13,22)_{LR} \\ (13,14,8)_{LR} \\ (-4,27,8)_{LR} \end{bmatrix}.$$

We find the following system:

$$\begin{cases} 3\tilde{y}_1 + 2\tilde{y}_2 = (4,5,17)_{LR}, \\ 3\tilde{y}_2 - 5\tilde{y}_3 = (-18,13,22)_{LR}, \\ 3\tilde{y}_3 + 2\tilde{y}_4 = (13,14,8)_{LR}, \\ -5\tilde{y}_1 + 3\tilde{y}_4 = (-4,27,8)_{LR}. \end{cases} \tag{16}$$

Then, by using [16] Q, f, and c are computed, we solve the quadratic programming problem (3.15) by using the interioir point algorithm with the initial point $x^0 = ((-1,1,3)_{LR}(-1,1,3)_{LR}(1,3,1)_{LR}(1,3,1)_{LR})^T$. The solution of the system (16) as form $\tilde{x}_{new} = (\tilde{x}_{left}^T, \tilde{x}_{right}^T)^T$ can be calculated as follows:

$$\tilde{x}_{left} = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} \left(x^l, x^\alpha, x^r - x^l + x^\beta\right)_{LR} \\ \left(x^l, x^\alpha, x^r - x^l + x^\beta\right)_{LR} \end{bmatrix} = \begin{bmatrix} (2,1,3)_{LR} \\ (-1,1,4)_{LR} \end{bmatrix}, \tag{17}$$

$$\tilde{x}_{right} = \begin{bmatrix} \tilde{y}_3 \\ \tilde{y}_4 \end{bmatrix} = \begin{bmatrix} \left(x^r, x^r - x^l + x^\alpha, x^\beta\right)_{LR} \\ \left(x^r, x^r - x^l + x^\alpha, x^\beta\right)_{LR} \end{bmatrix} = \begin{bmatrix} (3,2,2)_{LR} \\ (2,4,1)_{LR} \end{bmatrix}, \tag{18}$$

According to the values of the $\tilde{x}_{left}$ and $\tilde{x}_{right}$ and by using the Definition 3.1 we can write:

$$\tilde{x}_1 = \left(x^l, x^r, x^\alpha, x^\beta\right)_{LR} = (2,3,1,2)_{LR}, \tilde{x}_2 = \left(x^l, x^r, x^\alpha, x^\beta\right)_{LR} = (-1,2,1,1)_{LR}. \tag{19}$$

So, $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ with $r(\tilde{x}) = 0$. Then $\tilde{x}$ is an exact solution to the system.

We report some numerical results inspired by [16, 29]. For our test problems, the coefficient matrix A in (2.3) must be generated in such a way that it is singular, nonsingular, full rank, and rank deficient and m and n are selected from the three following sets:

$$small = \{10,20,\dots,90\}, medium = \{100,200,\dots,500\}, large = \{600,\dots,1000\}.$$

We investigated the gave results into two categories:

1. **Category 1** is generated completely randomly.
2. **Category 2** is generated such that corresponding crisp system of the $Ax = b$ (this system is derived from the fuzzy environment) is solvable.

Now, inspired by [16, 29], we compute the relative error for each solution (3). Next, we compare mean relative errors for an approximate solution in Tables 1 and 2.

Table 1: The mean relative error for Category 1

Sizes	Rank	Scale	Friedman	Nasseri and Gholami	SIP	KKTIP	LSIP
		small	1.50E+01	3.25E+01	4.72E-06	4.12E-05	1.95E-08
m = n	Full rank	medium	1.78E+02	4.32E+01	1.31E-07	5.20E-07	6.27E-08
		large	2.45E+02	3.68E+03	4.36E-08	1.77E-08	1.17E-08
		small	3.25E+02	2.78E+01	6.83E-08	1.82E-08	2.38E-08
m = n	Deficient rank	medium	3.01E+02	1.50E+01	5.35E-07	3.88E-08	3.87E-08
		large	2.56E+02	3.12E+01	9.88E-09	3.74E-07	2.89E-06
		small	4.37E+02	5.30E+01	1.48E-07	8.20E-07	2.14E-06
m < n	Full rank	medium	4.31E+02	5.63E+01	7.91E-08	7.57E-08	2.83E-07
		large	2.06E+02	2.58E+01	7.75E-08	1.31E-07	3.14E-07
		small	3.43E+02	1.96E+01	1.01E-07	1.80E-07	6.70E-08
m < n	Deficient rank	medium	1.89E+02	2.39E+01	5.86E-08	6.55E-08	2.16E-07
		large	3.98E+02	4.63E+01	3.21E-07	3.52E-07	2.35E-07
		small	2.83E+02	3.85E+01	4.02E-09	2.29E-09	4.68E-09
m > n	Full rank	medium	1.95E+02	2.65E+01	5.33E-09	9.70E-09	8.45E-09
		large	2.54E+02	3.62E+01	6.39E-09	8.75E-09	2.50E-08
		small	3.91E+02	2.98E+01	2.94E-09	2.59E-09	7.11E-10
m > n	Deficient rank	medium	4.08E+02	4.78E+01	4.78E-10	1.28E-09	8.03E-09
		large	1.02E+02	2.13E+01	7.50E-09	2.34E+08	2.01E-08

In Table 1, we compare the mean relative error to compute the approximate solution obtained by Algorithm 1 in category 1 with three initial points, simple initial point (SIP) [16, 29], Karush–Kuhn–Tucker initial point (KKTIP) [16, 29], and local search initial point (LSIP) [16, 29]. Results show that SIP and LSIP methods have better performance in all different cases. Also, in Table 2, we compare the mean relative error to compute the approximate solution obtained by Algorithm 1 in category 2 with three initial points SIP, KKTIP, and LSIP. Results show that LSIP, SIP, and KKTIP methods have similar performance.

We give all results in MATLAB environment version 7.14.0 and run the algorithm on a notebook Intel Core i5-4200M 2.50GHZ with 6 GB of RAM.

In addition, all problems are solved by Friedman [11] and Nassri and Gholami's methods [28]. Based on the obtained results, it is found that the average relative error in the calculation of the approximate solution obtained by the Algorithm 1 for three different starting points is lower in all test tasks than that caused by other methods.

Table 2: The mean relative error for Category 2

Sizes	Rank	Scale	Friedman	Nasseri and Gholami	SIP	KKTIP	LSIP
		small	2.23E+03	2.45E+04	1.47E-03	**2.44E-09**	1.29E-01
m = n	Full rank	medium	2.36E+03	2.58E+04	1.11E-07	1.57E-06	**6.90E-09**
		large	3.26E+03	4.15E+04	3.29E-08	**2.98E-08**	3.63E-07
		small	3.95E+03	4.02E+04	**1.99E-08**	6.32E-08	3.10E-07
m = n	Deficient rank	medium	3.62E+03	4.32E+04	3.02E-07	1.05E-06	**1.59E-07**
		large	3.68E+03	4.98E+04	7.32E-08	1.57E-07	**1.72E-08**
		small	4.15E+03	4.80E+04	**4.03E-08**	5.75E-08	2.23E-06
m < n	Full rank	medium	4.75E+03	4.98E+04	1.26E-07	**1.25E-07**	1.26E-07
		large	3.47E+03	3.87E+04	3.50E-07	3.35E-03	**1.85E-07**
		small	5.39E+03	6.36E+04	5.88E-08	**4.21E-08**	6.62E-08
m < n	Deficient rank	medium	2.95E+03	3.02E+04	**1.56E-08**	3.25E-08	3.90E-07
		large	4.60E+03	4.25E+04	5.79E-07	5.78E-07	**1.67E-07**
		small	2.60E+03	3.01E+04	**4.87E-09**	6.81E-09	1.00E-08
m > n	Full rank	medium	2.31E+03	3.16E+04	**5.46E-09**	5.57E-09	1.28E-08
		large	1.71E+03	2.13E+04	1.13E-08	**9.23E-09**	1.77E-08
		small	1.69E+03	2.31E+04	2.72E-09	3.72E-09	**1.60E-09**
m > n	Deficient rank	medium	5.16E+03	2.79E+04	2.05E-08	3.16E-09	**8.50E-10**
		large	2.93E+03	3.18E+04	**4.13E-09**	2.27E-08	1.33E-08

5. Conclusions

In this paper, we applied the least squares method to address LR fuzzy interval systems by converting an interval fuzzy number into two triangular fuzzy numbers. We minimized the distance between these two triangular fuzzy numbers to solve the fuzzy LR interval linear system. Essentially, we transformed the LR fuzzy interval linear system into a triangular fuzzy linear system, which we then solved using the least squares method as outlined in [18, 19]. Our analysis categorized the results into two groups: in category 1, where results were generated randomly, both the SIP and LSIP methods demonstrated superior performance across various scenarios. Conversely, in category 2, where results were derived from a corresponding crisp system, the LSIP, SIP, and KKTIP methods exhibited comparable performance.

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