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A Compromise Solution Approach for Fuzzy Data Envelopment Analysis: A Case of the Efficiency Prediction

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A B S T R A C T

The data envelopment analysis (DEA) a data-oriented approach for evaluating the relative performance of decision-making units (DMUs). The traditional DEA applies only to crisp data, whereas the data collected in the real world may be ambiguous and imprecise. The fuzzy DEA is an extension of the DEA using the fuzzy variable to deal with uncertain or imprecise data. This paper proposes two new fuzzy arithmetic-based DEA models with dynamic weights and common weights, formulated as multiple objective decision-making (MODM), and proposed models are represented as the linear programs providing the compromise solutions. The numerical experiment is illustrated to examine the validity of the proposed models, and the experiment shows that the proposed models give better results than other models. The proposed fuzzy DEA models are applied to predict the energy efficiency of 40 iron and steel enterprises in China.

1. Introduction

Data Envelopment Analysis (DEA) is a data-oriented approach for evaluating the relative efficiency of a set of homogeneous decision-making units (DMUs) which usually consume multiple inputs to produce multiple outputs. Traditional data envelopment analysis does not require any prior information, but it applies only to crisp data of all inputs and outputs [42]. However, the input and output values observed in real-world applications sometimes contain missing data, judgment data, or predictive data, and in general, those are imprecise or vague data [1]. The existence of any missing value or outlier in the data might cause the efficiency measurement of most DMUs to change drastically [21].

The uncertain or imprecise data can be expressed by fuzzy numbers [38]. The fuzzy DEA was first suggested by Sengupta [36] as an extension of the DEA using the fuzzy variable to deal with uncertain or imprecise data. The fuzzy DEA has some advantages [16, 29]. First, uncertainty in measurement can be incorporated to different degrees. Second, it can incorporate qualitative variables such as expert judgment trade-offs and environmental variables. Third, fuzzy DEA can handle missing observations, and finally, decision-

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makers can analyze how efficiency scores change with different levels of uncertainty measured.

Many researchers have formulated different fuzzy DEA models in recent years. The approach to fuzzy DEA in the literature can mainly be classified into four categories, namely, the tolerance approach, the α -level-based approach, the possibility approach, and the fuzzy ranking approach [29]. The tolerance approaches, such as Sengupta [36] and Kahraman and Tolga [18] , use the concept of fuzziness in DEA modelling by defining tolerance levels on constraint violations. The limitation is related to the design of a DEA model with a fuzzy objective function and fuzzy constraints which may or may not be satisfied. The α -level-based approaches, such as Kao and Liu [21], Angiz et al. [2], Ren et al. [32], and Cinaroglu [8], are based on the basic principle that first uses the concept of the α -cuts method to the fuzzy inputs and outputs for a given α -level, then mathematically reform the model by α variable. The drawback is that considerable computational effort is required to obtain fuzzy efficiencies of DMUs. In the possibility approach, such as Lertworasirikul et al. [24], Ruiz and Sirvent [34], Darijani et al. [10], and Omrani et al. [30], there exist two kinds of the approach called the "possibility approach" and the "credibility approach", which modelled the uncertainty in the fuzzy objective function and fuzzy constraints with possibility measures from both optimistic and pessimistic viewpoints. In the possibility approach, the decision-maker has to determine parameters such as possibility level. In the fuzzy ranking approach, such as Lotfi et al. [26], Hatami-Marbini et al. [13], Liu et al. [25], Tabatabaei et al. [39], Mahmoudabadi et al. [27], Majdi et al. [28], and Gerami et al. [12], the main idea is to find the fuzzy efficiency scores of the DMUs using fuzzy linear programs which require ranking the fuzzy set.

As a kind of fuzzy ranking approach, there is the fuzzy arithmetic approach on which the paper focuses. In some literature, it is even considered an equivalent classification with the fuzzy ranking approach. The fuzzy arithmetic approach focuses on the fact that decision-makers are not allowed to convert a fuzzy fractional program to a linear problem model using conventional methods. That is to say, $\sum_{r=1}^{s} u_r \tilde{y}_{rj} / \sum_{i=1}^{m} v_i \tilde{x}_{ij}$ cannot be transformed into $\sum_{r=1}^{s} u_r \tilde{y}_{rj}$ by setting $\sum_{i=1}^{m} v_i \tilde{x}_{ij} = \tilde{1}$ [11]. Accordingly, Wang et al. [41] represented the fuzzy data as the triangular fuzzy number and introduced a fuzzy arithmetic approach to evaluate the fuzzy efficiency of DMUs by using three linear program problems without the need for making any assumptions. Bhardwaj et al. [5] mentioned the flaws of Wang et al. [41] and proposed an improved model to address those problems. As pointed out by Bhardwaj et al. [5], the method of Wang et al. [41] provides only the best possible solution for each item in fuzzy DEA, which is not a real one. To obtain the optimal fuzzy efficiency of each DMU, Bhardwaj et al. [5] proposed a three-step optimization process to maximize the three-mark values of fuzzy efficiency in order. In detail, the method first calculates an optimal lower value of the fuzzy efficiency, then calculates an optimal modal value of fuzzy efficiency under the constraint of getting the lower value, and finally, calculates an optimal upper value of fuzzy efficiency under the constraint of getting both of the lower value and the modal value. This method gives the optimal solution for the lower value of fuzzy efficiency, and it is not discussed in the whole sense. That is, the optimization process was performed independently three times in Wang et al. [41], whereas three-step of serial processing in Bhardwaj et al. [5].

Thus, this paper designs the fuzzy DEA based on fuzzy arithmetic by multiple objective decision-making (MODM), which simultaneously optimizes the three-mark values of fuzzy efficiency represented as a triangular fuzzy number. The best possible solutions obtained from Wang et al. [41] are regarded as the ideal solutions of the three markers of fuzzy efficiency. We derive a new model of a compromise solution from producing the minimum distance between real mark values and the ideal mark values of fuzzy efficiency and get a linear program model.

Then, we extend the model to one on common weights. Standard DEA allocates the most favourable weights to maximize the efficiency of individual DMUs. If DMUs are experiencing similar circumstances, then the pricing of inputs and outputs should apply uniformly across all DMUs, and the use of different weights for any DMUs makes their efficiencies unable to be compared and ranked on the same basis [19]. For this, the common set of weights (CSW) method in DEA was first proposed by Cook and Kress [9], which calculates the efficiency only by common weights for each input and output data and eliminates inter-unit weight flexibility [33]. Kao and Hung [20] stated that the frontier produced by a dynamic set of weights (DSW) is ideal and the efficiency generated by CSW should be closest to the ideal efficiency generated by DSW. Similar to the need for CSW in standard DEA, in the fuzzy DEA, we might need to evaluate DMUs with imprecise data in the same circumstance on the same basis. Hence, several approaches also suggested the fuzzy DEA models on CSW, such as Saati and Memariani [35], Payan [31], Hu et al. [15], Shabani et al. [37], Kachouei et al. [17] and

Kazemi [22]. Some of them are also based on fuzzy arithmetic. Azar et al. [3] extended the fuzzy arithmetic approach of Wang et al. [41] to the approach of fuzzy DEA on CSW. Bagheri et al. [4] mentioned that the model of Azar et al. [3] has no guarantee to obtain a unique optimal solution and suggested a method to solve the multiple objective linear program using a lexicographic approach for it. Although it had advantages such as the models being linear and simple to solve, since those models are considered only at the lower value of the triangular fuzzy number, the persuasiveness of the solutions is insufficient. With our study, extending the fuzzy DEA with dynamic weights mentioned above, a fuzzy DEA with common weights is also developed in the direction of multiple objective decision-making.

After all, this paper proposes two new fuzzy DEA models with dynamic weights and common weights, which can be applicable regardless of the circumstances of DMUs. Two models are formulated as linear programs for obtaining the compromise solutions. The two proposed models are applied to the experimental data of Wang et al. [41], and the results are compared to previous studies. Then, we introduce the two models to an actual application, the prediction of energy efficiency of the iron and steel industry in China. Although there exist several studies on energy efficiency of the iron and steel industry, such as Chen et al. [7], He et al. [14], and Wang et al. [40], those are all limited to the evaluation of existing data. The up-to-dateness of data plays an important role in the managerial establishment, but in most cases, we can only use past data due to data access rights and data incompleteness. We predict the energy efficiency of 40 iron and steel enterprises in China from past periods' data, through a new prediction framework provided in this paper. The indicator values for next year are firstly predicted as a triangle fuzzy number from the past year's data, then two fuzzy DEA models proposed are introduced to evaluate the efficiency of virtual DMUs with predicted indicator values. Then, the energy efficiencies of enterprises in 2018 are estimated and analyzed.

Some main advantages of this paper are 1) depending on the nature of the application, we can choose to use one of the two fuzzy models with dynamic weights and common weights 2) the models are linear and easy to solve, 3) two models give reliable results and make all DMUs are fully ranked, and 4) this is the first approach for prediction of energy efficiency about iron and steel industry.

The paper is organized as follows: Section 2 reviews the background of the fuzzy operator, CCR model, and fuzzy DEA. Section 3 introduces two fuzzy DEA models with dynamic weights and common weights. Section 4 and Section 5 illustrate a numerical experiment and an empirical situation, respectively, and the last section concludes the paper.

2. Background

2.1. Fuzzy Operator

A fuzzy number is a convex fuzzy set, characterized by a given interval of real numbers, each with a grade of membership between 0 and 1. The most commonly used fuzzy numbers are triangular, defined as follows.

Definition 1: A fuzzy number $\tilde{A}=(a_L,a_M,a_U)$ is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x)=\begin{cases}(x-a_L)/(a_M-a_L), & a_L\le x<a_M\\ 1, & x=a_M\\ (a_U-x)/(a_U-a_M), & a_M<x\le a_U\\ 0, & otherwise\end{cases} \tag{1}$$

Definition 2: A triangular fuzzy number $\tilde{A}=(a_L,a_M,a_U)$ is a positive triangular fuzzy number if and only if $a_L>0$.

Let $\tilde{A}=(a_L,\, a_M,\, a_U)$ and $\tilde{B}=(b_L,\, b_M,\, b_U)$ be two positive triangular fuzzy numbers. Then basic fuzzy arithmetic operations on these fuzzy numbers are defined as

Addition : $\tilde{A}+\tilde{B}=(a_L+b_L,\, a_M+b_M,\, a_U+b_U)$

Subtraction : $\tilde{A}-\tilde{B}=(a_L-b_U,\, a_M-b_M,\, a_U-b_L)$

Multiplication : $\tilde{A}\times\tilde{B}\approx(a_Lb_L,\, a_Mb_M,\, a_Ub_U)$

Division : $\tilde{A}/\tilde{B} \approx (\frac{a_L}{b_U}, \frac{a_M}{b_M}, \frac{a_U}{b_L})$

The operations on a fuzzy number $\tilde{A}$ and a crisp number $c(c>0)$ can proceed by considering c as a triangular fuzzy number in the form of (c, c, c).

2.2. CCR model

The CCR model, proposed by Charnes, Cooper, and Rhodes [6], is one of the most basic DEA models. The factional program of the CCR model with m inputs and s outputs is as follows.

$$\begin{aligned} \max\ \theta_0 &= \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}} \\ s.t.\ & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1,\ \ j=1,\cdots,n \\ & u_r,\ v_i \ge 0,\ i=1,\cdots,m,\ \ r=1,\cdots,s \end{aligned} \tag{2}$$

where $u_r(r=1,\cdots,s)$ and $v_i(i=1,\cdots,m)$ are the output and input weights assigned to the r th output and the i th input; respectively, and DMU_0 refers to the DMU for assessing.

The following linear program can replace the above fractional program,

$$\begin{aligned} \max\ \theta_0 &= \sum_{r=1}^{s} u_r y_{r0} \\ s.t.\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0,\ \ j=1,\cdots,n \\ & \sum_{i=1}^{m} v_i x_{i0} = 1 \\ & u_r \ge 0,\ v_i \ge 0,\ i=1,\cdots,m,\ \ r=1,\cdots,s \end{aligned} \tag{3}$$

and it is called the input-oriented CCR multiplier model. The following two facts are satisfied: the fractional program model (2) is equivalent to the multiplier model (3); the optimal values of max $\theta_0 = \theta^*$ in model (2) and model (3) are independent of the units in which the inputs and outputs are measured provided these units are the same for every DMU.

2.3. Fuzzy DEA

CCR model (2) can only be used for cases where the data are crisp. Model (4) is formulated by introducing fuzzy input-output variables to model (2).

$$\begin{aligned} \max\ \ \tilde{\theta}_0 &= \frac{\sum_{r=1}^{s} u_r \tilde{y}_{r0}}{\sum_{i=1}^{m} v_i \tilde{x}_{i0}} \\ s.t.\ \ & \frac{\sum_{r=1}^{s} u_r \tilde{y}_{rj}}{\sum_{i=1}^{m} v_i \tilde{x}_{ij}} \le 1,\ \ j=1,\cdots,n \\ & u_r, v_i \ge 0,\ \ i=1,\cdots,m\ \ \ r=1,\cdots,s \end{aligned} \tag{4}$$

The fuzzy arithmetic approach considers that the objective $\sum_{r=1}^{s} u_r \tilde{y}_{r0} / \sum_{i=1}^{m} v_i \tilde{x}_{i0}$ cannot be transformed into $\sum_{r=1}^{s} u_r \tilde{y}_{r0}$ by setting $\sum_{i=1}^{m} v_i \tilde{x}_{i0} = \tilde{1}$ unless $\tilde{1}$ is assumed to be a crisp number.

Without loss of generality, all input and output data x_{ij} and y_{rj} are assumed to be uncertain and characterized by triangular fuzzy numbers $\tilde{x}_{ij} = \left(x_{ij}^L, x_{ij}^M, x_{ij}^U\right)$ and $\tilde{y}_{ij} = \left(y_{rj}^L, y_{rj}^M, y_{rj}^U\right)$. Crisp input and output data can be considered a special case of triangular fuzzy input and output data x_{ij} and y_{rj} with $x_{ij}^L = x_{ij}^M = x_{ij}^U$

and $y_{rj}^L = y_{rj}^M = y_{rj}^U$. Then the efficiency of DMU_j is defined as

$$\tilde{\theta}_j = \frac{\sum_{r=1}^{s} u_r \left(y_{rj}^L, y_{rj}^M, y_{rj}^U\right)}{\sum_{i=1}^{m} v_i \left(x_{ij}^L, x_{ij}^M, x_{ij}^U\right)} = \frac{\left(\sum_{r=1}^{s} u_r y_{rj}^L, \sum_{r=1}^{s} u_r y_{rj}^M, \sum_{r=1}^{s} u_r y_{rj}^U\right)}{\left(\sum_{i=1}^{m} v_i x_{ij}^L, \sum_{i=1}^{m} v_i x_{ij}^M, \sum_{i=1}^{m} v_i x_{ij}^U\right)} \approx \left(\frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i x_{ij}^U}, \frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i x_{ij}^M}, \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i x_{ij}^L}\right) \tag{5}$$

The following fuzzy DEA model is constructed to measure the performance of each DMU:

$$\begin{aligned} \max \quad & \tilde{\theta}_0 \approx \left(\theta_0^L, \theta_0^M, \theta_0^U\right) = \left(\frac{\sum_{r=1}^{s} u_r y_{r0}^L}{\sum_{i=1}^{m} v_i x_{i0}^U}, \frac{\sum_{r=1}^{s} u_r y_{r0}^M}{\sum_{i=1}^{m} v_i x_{i0}^M}, \frac{\sum_{r=1}^{s} u_r y_{r0}^U}{\sum_{i=1}^{m} v_i x_{i0}^L}\right) \\ s.t. \quad & \tilde{\theta}_j \approx \left(\theta_j^L, \theta_j^M, \theta_j^U\right) = \left(\frac{\sum_{r=1}^{s} u_r y_{rj}^L}{\sum_{i=1}^{m} v_i x_{ij}^U}, \frac{\sum_{r=1}^{s} u_r y_{rj}^M}{\sum_{i=1}^{m} v_i x_{ij}^M}, \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i x_{ij}^L}\right) \le 1, \quad j = 1, \cdots, n \\ & u_r, v_i \ge 0, \quad i = 1, \cdots, m \quad r = 1, \cdots, s \end{aligned} \tag{6}$$

For model (6), as long as θ_j^U is kept being less than or equal to one, then $\theta_j^L \le 1$ and $\theta_j^M \le 1$ will be automatically satisfied. The model can therefore be simplified as

$$\begin{aligned} \max \quad & \tilde{\theta}_0 \approx \left(\theta_0^L, \theta_0^M, \theta_0^U\right) = \left(\frac{\sum_{r=1}^{s} u_r y_{r0}^L}{\sum_{i=1}^{m} v_i x_{i0}^U}, \frac{\sum_{r=1}^{s} u_r y_{r0}^M}{\sum_{i=1}^{m} v_i x_{i0}^M}, \frac{\sum_{r=1}^{s} u_r y_{r0}^U}{\sum_{i=1}^{m} v_i x_{i0}^L}\right) \\ s.t. \quad & \theta_j^U = \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i x_{ij}^L} \le 1, \quad j = 1, \cdots, n \\ & u_r, v_i \ge 0, \quad i = 1, \cdots, m \quad r = 1, \cdots, s \end{aligned} \tag{7}$$

For model (7), Wang et al. [41] proposed a method to obtain the best possible values of $\left\{\theta_0^L, \theta_0^M, \theta_0^U\right\}$ captured by the following models:

$$\begin{aligned} \max \; & \theta_0^L = \sum_{r=1}^{s} u_r y_{r0}^L \\ s.t. \; & \sum_{r=1}^{s} u_r y_{rj}^U - \sum_{i=1}^{m} v_i x_{ij}^L \le 0, \quad j = 1, \cdots, n \\ & \sum_{i=1}^{m} v_i x_{i0}^U = 1 \\ & u_r, v_i \ge 0, \quad i = 1, \cdots, m \quad r = 1, \cdots, s \end{aligned} \tag{8}$$

$$\begin{aligned} \max \; & \theta_0^M = \sum_{r=1}^{s} u_r y_{r0}^M \\ s.t. \; & \sum_{r=1}^{s} u_r y_{rj}^U - \sum_{i=1}^{m} v_i x_{ij}^L \le 0, \quad j = 1, \cdots, n \\ & \sum_{i=1}^{m} v_i x_{i0}^M = 1 \\ & u_r, v_i \ge 0, \quad i = 1, \cdots, m \quad r = 1, \cdots, s \end{aligned} \tag{9}$$

$$\begin{aligned} &\max \theta_0^U = \sum\nolimits_{r=1}^{s} u_r y_{r0}^U \\ &s.t.\ \sum\nolimits_{r=1}^{s} u_r y_{rj}^U - \sum\nolimits_{i=1}^{m} v_i x_{ij}^L \le 0,\ \ j=1,\cdots,n \\ &\qquad \sum\nolimits_{i=1}^{m} v_i x_{i0}^L = 1 \\ &\qquad u_r, v_i \ge 0,\ \ i=1,\cdots,m\ \ \ r=1,\cdots,s \end{aligned} \tag{10}$$

The first n constraints in the model (8)-(10) are the same; objective functions are different from each other. In the Wang et al.'s methods, the values are obtained by solving the crisp DEA models (8)–(10), separately. In other words, Wang et al. decomposed the simultaneous optimization problem (7) and converted it into three individual optimization problems (8), (9), and (10). As a result, the solutions obtained from models (8), (9), and (10) are possible solutions, but are not real solutions for the model (7). Hence, the solutions obtained above model (8)–(10) are not the optimal solutions of fuzzy DEA model (4), i.e., it is not possible to find an optimal solution of fuzzy DEA model (4) using the Wang et al.'s methods.

3. Proposed Method

3.1. Fuzzy DEA based on DSW (DSW-FDEA)

To obtain the maximum value of $\tilde{\theta}_0$, above model (7) can be considered as MODM which simultaneously maximizes all mark values θ_0^L, θ_0^M, and θ_0^U, as follows:

$$\begin{aligned} &\max \left\{\theta_0^L, \theta_0^M, \theta_0^U\right\} = \left\{ \frac{\sum\nolimits_{r=1}^{s} u_r y_{r0}^L}{\sum\nolimits_{i=1}^{m} v_i x_{i0}^U}, \frac{\sum\nolimits_{r=1}^{s} u_r y_{r0}^M}{\sum\nolimits_{i=1}^{m} v_i x_{i0}^M}, \frac{\sum\nolimits_{r=1}^{s} u_r y_{r0}^U}{\sum\nolimits_{i=1}^{m} v_i x_{i0}^L} \right\} \\ &s.t.\ \sum\nolimits_{r=1}^{s} u_r y_{rj}^U - \sum\nolimits_{i=1}^{m} v_i x_{ij}^L \le 0,\ \ j=1,\cdots,n \\ &\qquad u_r, v_i \ge 0,\ \ i=1,\cdots,m\ \ \ r=1,\cdots,s \end{aligned} \tag{11}$$

The model (11) is a fractional program with multiple objectives and the issue is how to determine the final solution from the set of Pareto optimal solutions.

Let $\left\{\theta_0^{L*}, \theta_0^{M*}, \theta_0^{U*}\right\}$ denote the ideal efficiency level that $\left\{\theta_0^L, \theta_0^M, \theta_0^U\right\}$ can be attainable in the model (11). Then, $\left\{\theta_0^{L*}, \theta_0^{M*}, \theta_0^{U*}\right\}$ is determined by solving each of three models aiming at each objective under the constraint of the model (11), as result, these individual models are equal to (8), (9), and (10). The solution of $\left\{\theta_0^L, \theta_0^M, \theta_0^U\right\}$ that we want to obtain can be regarded as finding the solution closest to $\left\{\theta_0^{L*}, \theta_0^{M*}, \theta_0^{U*}\right\}$. So, we can set the values $\left\{\theta_0^{L*}, \theta_0^{M*}, \theta_0^{U*}\right\}$ as the ideal solution of $\left\{\theta_0^L, \theta_0^M, \theta_0^U\right\}$ in the model (11) and obtain a compromise solution nearest to the ideal. To determine the degree of closeness between the ideal solution $\left\{\theta_0^{L*}, \theta_0^{M*}, \theta_0^{U*}\right\}$ and a real solution $\left\{\theta_0^L, \theta_0^M, \theta_0^U\right\}$, a generalized family of distance measures is applied.

$$d_p^3\left(\left(x_1, x_2, x_3\right), \left(y_1, y_2, y_3\right)\right) = \left(\left(x_1 - y_1\right)^p + \left(x_2 - y_2\right)^p + \left(x_3 - y_3\right)^p\right)^{\frac{1}{p}}, \quad p \ge 1 \tag{12}$$

where superscript 3 denotes the dimension of space discussed and p represents the distance parameter.

The following model is derived from model (11).

$$\min\ d_p^3\left(\left(\theta_0^{L*},\theta_0^{M*},\theta_0^{U*}\right),\left(\theta_0^{L},\theta_0^{M},\theta_0^{U}\right)\right)=$$

$$=\left(\left(\theta_0^{L*}-\frac{\sum_{r=1}^{s}u_r y_{r0}^{L}}{\sum_{i=1}^{m}v_i x_{i0}^{U}}\right)^p+\left(\theta_0^{M*}-\frac{\sum_{r=1}^{s}u_r y_{r0}^{M}}{\sum_{i=1}^{m}v_i x_{i0}^{M}}\right)^p+\left(\theta_0^{U*}-\frac{\sum_{r=1}^{s}u_r y_{r0}^{U}}{\sum_{i=1}^{m}v_i x_{i0}^{L}}\right)^p\right)^{\frac{1}{p}} \tag{13}$$

$$s.t.\quad \sum_{r=1}^{s}u_r y_{r0}^{U}-\sum_{i=1}^{m}v_i x_{i0}^{L}\le 0,\quad j=1,\cdots,n$$

$$u_r,v_i\ge 0,\ i=1,\cdots,m\quad r=1,\cdots,s$$

Model (13) is a non-linear program with one objective, so we try to obtain a linear model.

Before discussing the above, let us first discuss the following transformation of the standard CCR multiplier model (3) mentioned in Zohrehbandian's approach[43].

Remark 1. In model (3), denoting the efficiency of DMU_j as θ_j^*, virtual DMU $(\theta_j^* x_j, y_j)$ becomes an efficient DMU placed on the frontier. The following model (14) gives the same result as in model (3).

$$\begin{aligned}&\max\ \sum_{r=1}^{s}u_r y_{r0}-\sum_{i=1}^{m}v_i\left(\theta_0^* x_{i0}\right)\\&s.t.\quad \sum_{r=1}^{s}u_r y_{rj}-\sum_{i=1}^{m}v_i\left(\theta_j^* x_{ij}\right)\le 0,\quad j=1,\cdots,n\\&\qquad \sum_{r=1}^{s}u_r+\sum_{i=1}^{m}v_i=1\\&\qquad u_r,\ v_i\ge 0,\ i=1,\cdots,m,\ r=1,\cdots,s\end{aligned} \tag{14}$$

Proof. The efficiency in model (3) is calculated from the actual DMUs, whereas the one in model (14) is calculated from the virtual efficient DMUs corresponding to the actual DMUs. The second constraint in the model (14) is only for weight normalization. In detail, as can be seen in the model (14), the optimal value of the objective function is 0. Let be the weight vector for DMU_0 obtained from model (3), then $\sum_{i=1}^{m}v_i^* x_{i0}=1$, and $\theta_0^*=\sum_{r=1}^{s}u_r^* y_{r0}$ becomes a maximum value. Let being $\sum_{r=1}^{s}u_r^*+\sum_{i=1}^{m}v_i^*=h$ then a new weight vector, $(\frac{u^*}{h},\frac{v^*}{h})$ satisfies all constraints of the model (14) and its objective function's value is

$$\sum_{r=1}^{s}\frac{u_r^*}{h}y_{r0}-\sum_{i=1}^{m}\frac{v_i^*}{h}\left(\theta_0^* x_{i0}\right)=\frac{1}{h}\left(\sum_{r=1}^{s}u_r^* y_{r0}-\sum_{i=1}^{m}v_i^*\left(\theta_0^* x_{i0}\right)\right)=\frac{1}{h}\left(\theta_0^*-\theta_0^*\cdot 1\right)=0$$

So $(\frac{u^*}{h},\frac{v^*}{h})$ becomes an optimal solution of model (14). The efficiency calculated by model (14) is

$$\theta_0=\sum_{j=1}^{s}\frac{u_j^*}{h}y_{jo}\Big/\sum_{i=1}^{r}\frac{v_i^*}{h}x_{io}=\sum_{j=1}^{s}u_j^* y_{jo}\Big/\sum_{i=1}^{r}v_i^* x_{io}=\theta_0^*$$

, and is the same as one calculated by model (3). Conversely, the solution of model (3) can be obtained from the solution of model (14) and both efficiencies are the same. Thus, the efficiency calculated in model (3) and the one calculated in the model (14) are the same. In addition, two weight vectors of model (3) and model (14) for each DMU are parallel. So, model (2), model (3), and model (14) give the same results.

Now, let us adopt this transform method from model (2) to model (14) to our problem. Since $\tilde{\theta}_j^*=\left(\theta_j^{L*},\theta_j^{M*},\theta_j^{H*}\right)$ is the ideal fuzzy efficiency of DMU_j, virtual DMUs $(\theta_j^{L*}x_j^U, y_j^L)$, $(\theta_j^{M*}x_j^M, y_j^M)$, and $(\theta_j^{U*}x_j^L, y_j^U)$ become the efficient ones in the crisp model (8)-(10), separately. The following model (15)-(17) can be derived from model (8)-(10), and the results from the two groups of the models are the same.

$$\begin{aligned} &\max \sum_{r=1}^{s} u_r y_{r0}^{L} - \sum_{i=1}^{m} v_i \theta_0^{L*} x_{i0}^{U} \\ &s.t.\ \sum_{r=1}^{s} u_r y_{rj}^{U} - \sum_{i=1}^{m} v_i \theta_j^{U*} x_{ij}^{L} \le 0,\ \ j=1,\cdots,n \\ &\quad \sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i = 1 \\ &\quad u_r, v_i \ge 0,\ \ i=1,\cdots,m \ \ r=1,\cdots,s \end{aligned} \tag{15}$$

$$\begin{aligned} &\max \sum_{r=1}^{s} u_r y_{r0}^{M} - \sum_{i=1}^{m} v_i \theta_0^{M*} x_{i0}^{M} \\ &s.t.\ \sum_{r=1}^{s} u_r y_{rj}^{U} - \sum_{i=1}^{m} v_i \theta_j^{U*} x_{ij}^{L} \le 0,\ \ j=1,\cdots,n \\ &\quad \sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i = 1 \\ &\quad u_r, v_i \ge 0,\ \ i=1,\cdots,m \ \ r=1,\cdots,s \end{aligned} \tag{16}$$

$$\begin{aligned} &\max \sum_{r=1}^{s} u_r y_{r0}^{U} - \sum_{i=1}^{m} v_i \theta_0^{U*} x_{i0}^{L} \\ &s.t.\ \sum_{r=1}^{s} u_r y_{rj}^{U} - \sum_{i=1}^{m} v_i \theta_j^{U*} x_{ij}^{L} \le 0,\ \ j=1,\cdots,n \\ &\quad \sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i = 1 \\ &\quad u_r, v_i \ge 0,\ \ i=1,\cdots,m \ \ r=1,\cdots,s \end{aligned} \tag{17}$$

The constraints of the above three models are the same, but the objects are different. The aim is simultaneous optimization, not individual optimization. For constructing MODM from the above three models, the following model (18) can be established.

$$\begin{aligned} &\max \left\{ \sum_{r=1}^{s} u_r y_{r0}^{L} - \sum_{i=1}^{m} v_i \theta_0^{L*} x_{i0}^{U},\ \sum_{r=1}^{s} u_r y_{r0}^{M} - \sum_{i=1}^{m} v_i \theta_0^{M*} x_{i0}^{M},\ \sum_{r=1}^{s} u_r y_{r0}^{U} - \sum_{i=1}^{m} v_i \theta_0^{U*} x_{i0}^{L} \right\} \\ &s.t.\ \sum_{r=1}^{s} u_r y_{rj}^{U} - \sum_{i=1}^{m} v_i \theta_j^{U*} x_{ij}^{L} \le 0,\ \ j=1,\cdots,n \\ &\quad \sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i = 1 \\ &\quad u_r, v_i \ge 0,\ \ i=1,\cdots,m \ \ r=1,\cdots,s \end{aligned} \tag{18}$$

The constraints did not change from the original and the three objectives are grouped. Compared with the original model (11), the model (18) is a multiple objective linear program whereas the model (11) is a multiple objective fractional program.

Now, the ideal value of each objective in the model (18) can be regarded as the value obtained from (15)-(17), respectively. Considering the optimal value of each object in the model (15)-(17) is 0, we can transform the objective of the above model (18) into that for producing the nearest distance between the objective vector of the model (18) and the 3-dimensional 0 vector. For the concept of distance, we use the equation (12). Therefore, we can set a new objective function instead of that of the model (18) as follows:

$$\min\ d_p^3\left((0,0,0), \left(\sum_{r=1}^{s} u_r y_{r0}^{L} - \sum_{i=1}^{m} v_i \theta_0^{L*} x_{i0}^{U}, \sum_{r=1}^{s} u_r y_{r0}^{M} - \sum_{i=1}^{m} v_i \theta_0^{M*} x_{i0}^{M}, \sum_{r=1}^{s} u_r y_{r0}^{U} - \sum_{i=1}^{m} v_i \theta_0^{U*} x_{i0}^{L}\right)\right)$$

This objective function makes us get the compromise solution. When $p=1$, the above model becomes a linear program.

$$\begin{aligned} &\min \left\{ -\sum_{r=1}^{s} u_r \left(y_{r0}^{L} + y_{r0}^{M} + y_{r0}^{U} \right) + \sum_{i=1}^{m} v_i \left(\theta_0^{L*} x_{i0}^{U} + \theta_0^{M*} x_{i0}^{M} + \theta_0^{U*} x_{i0}^{L} \right) \right\} \\ &s.t.\ \sum_{r=1}^{s} u_r y_{rj}^{U} - \sum_{i=1}^{m} v_i \theta_j^{U*} x_{ij}^{L} \le 0,\ \ j=1,\cdots,n \\ &\quad \sum_{r=1}^{s} u_r + \sum_{i=1}^{m} v_i = 1 \\ &\quad u_r, v_i \ge 0,\ \ i=1,\cdots,m \ \ r=1,\cdots,s \end{aligned} \tag{19}$$

We call model (19) the DSW-FDEA simply. By the weight vector $\left(u^*, v^*\right)$ of DMU_0, obtained from model (18), its fuzzy efficiency is calculated as follows:

$$\left(\theta_0^L, \theta_0^M, \theta_0^U\right) = \left(\frac{\sum_{r=1}^{s} u_r^* y_{r0}^L}{\sum_{i=1}^{m} v_i^* x_{i0}^U}, \frac{\sum_{r=1}^{s} u_r^* y_{r0}^M}{\sum_{i=1}^{m} v_i^* x_{i0}^M}, \frac{\sum_{r=1}^{s} u_r^* y_{r0}^U}{\sum_{i=1}^{m} v_i^* x_{i0}^L} \right) \tag{20}$$

3.2. Fuzzy DEA based on CSW (CSW-FDEA)

In the evaluation by common weights, all DMUs are evaluated on the same basis so fair evaluation can be performed. From this, we try to extend the above study of fuzzy DEA to the common weights.

The fuzzy DEA using common weights can be viewed as the following MODM problem.

$$\begin{aligned} &\max \ \left\{\tilde{\theta}_1, \tilde{\theta}_2, \cdots, \tilde{\theta}_n\right\} \\ &s.t. \quad \tilde{\theta}_j = \frac{\sum_{r=1}^{s} u_r \tilde{y}_{rj}}{\sum_{i=1}^{m} v_i \tilde{x}_{ij}} \le 1, \quad j = 1, \cdots, n \\ &\qquad u_r, v_i \ge 0, \quad i = 1, \cdots, m \quad r = 1, \cdots, s \end{aligned} \tag{21}$$

We can derive the following model (22).

$$\begin{aligned} &\max \ \left\{\tilde{\theta}_1, \cdots, \tilde{\theta}_n\right\} \approx \left\{\left(\theta_1^L, \theta_1^M, \theta_1^U\right), \cdots, \left(\theta_n^L, \theta_n^M, \theta_n^U\right)\right\} = \\ &\left\{ \left(\frac{\sum_{r=1}^{s} u_r y_{r1}^L}{\sum_{i=1}^{m} v_i x_{i1}^U}, \frac{\sum_{r=1}^{s} u_r y_{r1}^M}{\sum_{i=1}^{m} v_i x_{i1}^M}, \frac{\sum_{r=1}^{s} u_r y_{r1}^U}{\sum_{i=1}^{m} v_i x_{i1}^L} \right), \cdots, \left(\frac{\sum_{r=1}^{s} u_r y_{rn}^L}{\sum_{i=1}^{m} v_i x_{in}^U}, \frac{\sum_{r=1}^{s} u_r y_{rn}^M}{\sum_{i=1}^{m} v_i x_{in}^M}, \frac{\sum_{r=1}^{s} u_r y_{rn}^U}{\sum_{i=1}^{m} v_i x_{in}^L} \right) \right\} \\ &s.t. \quad \theta_j^U = \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i x_{ij}^L} \le 1, \quad j = 1, \cdots, n \\ &\qquad u_r, v_i \ge 0, \quad i = 1, \cdots, m \quad r = 1, \cdots, s \end{aligned} \tag{22}$$

Model (22) is a non-linear program with multiple objectives. This model can be derived in the following form that simultaneously optimizes every mark value of fuzzy efficiencies.

$$\begin{aligned} &\max \ \left\{\theta_1^L, \theta_1^M, \theta_1^U, \cdots, \theta_n^L, \theta_n^M, \theta_n^U\right\} = \\ &\left\{ \frac{\sum_{r=1}^{s} u_r y_{r1}^L}{\sum_{i=1}^{m} v_i x_{i1}^U}, \frac{\sum_{r=1}^{s} u_r y_{r1}^M}{\sum_{i=1}^{m} v_i x_{i1}^M}, \frac{\sum_{r=1}^{s} u_r y_{r1}^U}{\sum_{i=1}^{m} v_i x_{i1}^L}, \cdots, \frac{\sum_{r=1}^{s} u_r y_{rn}^L}{\sum_{i=1}^{m} v_i x_{in}^U}, \frac{\sum_{r=1}^{s} u_r y_{rn}^M}{\sum_{i=1}^{m} v_i x_{in}^M}, \frac{\sum_{r=1}^{s} u_r y_{rn}^U}{\sum_{i=1}^{m} v_i x_{in}^L} \right\} \\ &s.t. \quad \theta_j^U = \frac{\sum_{r=1}^{s} u_r y_{rj}^U}{\sum_{i=1}^{m} v_i x_{ij}^L} \le 1, \quad j = 1, \cdots, n \\ &\qquad u_r, v_i \ge 0, \quad i = 1, \cdots, m \quad r = 1, \cdots, s \end{aligned} \tag{23}$$

We can also consider $\tilde{\theta}_j^* = \left(\theta_j^{L*}, \theta_j^{M*}, \theta_j^{H*}\right)$, the efficiency of DMU_j obtained in the model (8)-(10), as the ideal solution of DMU_j in model (23).

Like the model with dynamic weights in section 3.1, using the transformation in Remark 1, we can obtain compromise resolution from the following model (24).

$$\begin{aligned}&\max\Big\{\sum_{r=1}^{s}u_r y_{r1}^L-\sum_{i=1}^{m}v_i\theta_1^{L*}x_{i1}^U,\ \sum_{r=1}^{s}u_r y_{r1}^M-\sum_{i=1}^{m}v_i\theta_1^{M*}x_{i1}^M,\ \sum_{r=1}^{s}u_r y_{r1}^U-\sum_{i=1}^{m}v_i\theta_1^{U*}x_{i1}^L,\\&\quad\cdots,\\&\quad\sum_{r=1}^{s}u_r y_{rn}^L-\sum_{i=1}^{m}v_i\theta_n^{L*}x_{in}^U,\ \sum_{r=1}^{s}u_r y_{rn}^M-\sum_{i=1}^{m}v_i\theta_n^{M*}x_{in}^M,\ \sum_{r=1}^{s}u_r y_{rn}^U-\sum_{i=1}^{m}v_i\theta_n^{U*}x_{in}^L\Big\}\\&s.t.\ \sum_{r=1}^{s}u_r y_{rj}^U-\sum_{i=1}^{m}v_i\theta_j^{U*}x_{ij}^L\le 0,\ \ j=1,\cdots,n\\&\quad\sum_{r=1}^{s}u_r+\sum_{i=1}^{m}v_i=1\\&\quad u_r,v_i\ge 0,\ \ i=1,\cdots,m\ \ r=1,\cdots,s\end{aligned}\tag{24}$$

There are $3n$ items in the objective. Regarding the ideal value of each item in the objective is 0, this model can be considered the compromise one to obtain the common weights which produce the nearest distance from $3n$ dimensional 0 vector.

As in the above 3-dimensional space, a generalized family of distance measures in $3n$ dimensional space can be described as follows:

$$d_p^{3n}\left(\left(x_1,x_2,\cdots,x_{3n}\right),\left(y_1,y_2,\cdots,y_{3n}\right)\right)=\left(\sum_{i=1}^{3n}\left(x_i-y_i\right)^p\right)^{\frac{1}{p}}$$

If $p=1$, the above model (24) can be derived as following compromise model (25).

$$\begin{aligned}&\min\left\{\sum_{j=1}^{n}\left(-\sum_{r=1}^{s}u_r\left(y_{rj}^L+y_{rj}^M+y_{rj}^U\right)+\sum_{i=1}^{m}v_i\left(\theta_j^{L*}x_{ij}^U+\theta_j^{M*}x_{ij}^M+\theta_j^{U*}x_{ij}^L\right)\right)\right\}\\&s.t.\ \sum_{r=1}^{s}u_r y_{rj}^U-\sum_{i=1}^{m}v_i\theta_j^{U*}x_{ij}^L\le 0,\ \ j=1,\cdots,n\\&\quad\sum_{r=1}^{s}u_r+\sum_{i=1}^{m}v_i=1\\&\quad u_r,v_i\ge 0,\ \ i=1,\cdots,m\ \ r=1,\cdots,s\end{aligned}\tag{25}$$

This model is a linear program. We call the model (25) the CSW-FDEA simply. By the common weight vector $\left(u^*,v^*\right)$ obtained from model (25), the efficiency of DMU_j is calculated by equation (20).

4. Numerical Experiment

To illustrate the two models of fuzzy DEA proposed, we consider the example presented in Wang et al. [41]. For comparison, models of Wang et al. [41], Azar et al. [3] and Bhardwaj et al. [5] are applied to the example. The numerical example of eight DMUs with two inputs $\left(X_1,X_2\right)$ and two outputs $\left(Y_1,Y_2\right)$ is presented in Table 1.

Table 1. The numerical example of Wang, Luo [41]

Enterprises (DMUs)	Inputs		Outputs	
	MC	NOE	GOV	PQ
A	(2120, 2170, 2210)	1870	(14500, 14790, 14860)	(3.1, 4.1, 4.9)
B	(1420, 1460, 1500)	1340	(12470, 12720, 12790)	(1.2, 2.1, 3.0)
C	(2510, 2570, 2610)	2360	(17900, 18260, 18400)	(3.3, 4.3, 5.0)
D	(2300, 2350, 2400)	2020	(14970, 15270, 15400)	(2.7, 3.7, 4.6)
E	(1480, 1520, 1560)	1550	(13980, 14260, 14330)	(1.0, 1.8, 2.7)
F	(1990, 2030, 2100)	1760	(14030, 14310, 14400)	(1.6, 2.6, 3.6)
G	(2200, 2260, 2300)	1980	(16540, 16870, 17000)	(2.4, 3.4, 4.4)
H	(2400, 2460, 2520)	2250	(17600, 17960, 18100)	(2.6, 3.6, 4.6)

Before applying the proposed models, it is necessary to scale-transform the numerical data. Units invariance is satisfied in not only the original CCR model but the original fuzzy DEA model (4). Of course, Wang et al.'s model also satisfies "units invariance". In the case of this proposal, we can say model (18) satisfies "units invariance" but model (19) doesn't. Because of the difference in the value range of each indicator, the compromise optimization of the model (19) is affected by these different value ranges, and model (25). To solve this problem, it is necessary to discuss the values of all indicators in the same value range. So, we transform the original data to proportionally scaled values between [0,1], with the maximum value of each indicator being 1. The data after scale transformation are shown in Table 2.

Table 2. The scale-transformed data of Table 1.

Enterprises (DMUs)	Inputs		Outputs	
	MC	NOE	GOV	PQ
A	(0.813, 0.832, 0.847)	0.793	(0.789, 0.804, 0.808)	(0.62, 0.82, 0.98)
B	(0.545, 0.56, 0.575)	0.568	(0.678, 0.692, 0.696)	(0.24, 0.42, 0.6)
C	(0.962, 0.985, 1)	1	(0.973, 0.993, 1)	(0.66, 0.86, 1)
D	(0.882, 0.901, 0.92)	0.856	(0.814, 0.83, 0.837)	(0.54, 0.74, 0.92)
E	(0.568, 0.583, 0.598)	0.657	(0.76, 0.775, 0.779)	(0.2, 0.36, 0.54)
F	(0.763, 0.778, 0.805)	0.746	(0.763, 0.778, 0.783)	(0.32, 0.52, 0.72)
G	(0.843, 0.866, 0.882)	0.839	(0.899, 0.917, 0.924)	(0.48, 0.68, 0.88)
H	(0.92, 0.943, 0.966)	0.954	(0.957, 0.977, 0.984)	(0.52, 0.72, 0.92)

For DMUs of Table 2, the dynamic weights and common weights are calculated separately by the model (19) and model (25). Using of obtained DSW and CSW, the fuzzy efficiency for each DMU is calculated by equation (20) and presented in the last two columns of Table 3. We have also listed the results by Wang et al. [41], Azar et al. [3] and Bhardwaj et al. [5] in this table.

Table 3. The comparison between results from proposed methods and the results from different methods.

DMUs	Wang et al.'s Model 1.	Bhardwaj et al.'s model	Azar et al.'s model	DSW-FDEA	CSW-FDEA
A	(0.813, 0.904, 1)	(0.812, 0.829, 0.833)	(0.803, 0.821, 0.828)	(0.777, 0.904, 1)	(0.813, 0.829, 0.833)
B	(0.975, 0.995, 1)	(0.975, 1, 1)	(0.969, 0.992, 1)	(0.975, 0.995, 1)	(0.975, 0.995, 1)
C	(0.795, 0.813, 0.905)	(0.797, 0.815, 0.825)	(0.791, 0.809, 0.817)	(0.795, 0.811, 0.817)	(0.795, 0.811, 0.817)
D	(0.777, 0.805, 0.907)	(0.776, 0.792, 0.799)	(0.767, 0.785, 0.794)	(0.688, 0.805, 0.907)	(0.777, 0.792, 0.799)
E	(0.961, 0.988, 1)	(0.973, 1, 1)	(0.949, 0.971, 0.979)	(0.961, 0.988, 1)	(0.945, 0.964, 0.969)
F	(0.836, 0.852, 0.886)	(0.835, 0.852, 0.857)	(0.824, 0.844, 0.852)	(0.836, 0.852, 0.858)	(0.836, 0.852, 0.858)
G	(0.876, 0.893, 0.946)	(0.875, 0.893, 0.9)	(0.866, 0.886, 0.896)	(0.876, 0.893, 0.9)	(0.876, 0.893, 0.9)
H	(0.82, 0.837, 0.887)	(0.82, 0.836, 0.843)	(0.815, 0.834, 0.843)	(0.82, 0.837, 0.843)	(0.82, 0.837, 0.843)

We found that there is a problem with DMU B, C, and E in the result of Bhardwaj et al.'s model. The modal value of the fuzzy efficiency for DMU B, C, and E is larger in Bhardwaj et al.'s model than in Wang et al.'s model. Considering each value of fuzzy efficiency obtained by Wang et al.'s model is the ideal maximum, Bhardwaj et al.'s model has poor reliability of the results in B, C, and E. From the table, it can be confirmed that each mark value of the efficiency obtained by the proposed DSW-FDEA and CSW-FDEA is less than or equal to one by Wang et al.'s model. In addition, the mark values of the efficiency obtained from DSW-FDEA are larger than the corresponding mark values of CSW-FDEA and Azar et al.'s model. From this, it can be confirmed that the efficiency obtained from the fuzzy DEA by dynamic weights is larger than the efficiency obtained from the fuzzy DEA by common weights.

To provide a full ranking of all DMUs, Azar et al. [3] recommended the following defuzzification function:

$$\theta = \frac{\theta_L + 4\theta_M + \theta_U}{6} \tag{26}$$

Azar et al. [3] stated that other types of defuzzification could also be used and the defuzzification would not have much impact on the results. The ranking results by ranking methods recommended by Wang et al. [41] and Azar et al. [3] are the same as the experiment results of three models. Hence, we recommend the above defuzzification as the ranking method. The ranking result of each method is shown in Table 4.

Table 4. **The full rankings of DMUs by different methods.**

DMUs	Wang et al.'s Model 1		Bhardwaj et al.'s model		Azar et al.'s model		DSW-FDEA		CSW-FDEA	
	efficiency	rank	efficiency	rank	efficiency	rank	efficiency	rank	efficiency	rank
A	0.905	3	0.827	6	0.819	6	0.899	3	0.827	6
B	0.993	1	0.996	1	0.989	1	0.993	1	0.993	1
C	0.825	7	0.814	7	0.807	7	0.81	7	0.81	7
D	0.818	8	0.791	8	0.783	8	0.803	8	0.791	8
E	0.985	2	0.996	2	0.969	2	0.985	2	0.962	2
F	0.855	5	0.85	4	0.842	4	0.85	5	0.85	4
G	0.899	4	0.892	3	0.884	3	0.891	4	0.891	3
H	0.842	6	0.835	5	0.832	5	0.835	6	0.835	5

From this table, we can notice that the efficiency of each DMU in the DSW-FDEA is smaller than the one corresponding to Wang et al.'s model, and is larger than the one corresponding to Azar et al.'s model. Except for DMUs B, C, and E, which have problems in Bhardwaj et al.'s model, when comparing the results, the efficiency of each DMU is larger in the proposed method. The ranking result of the proposed model with dynamic weights is completely consistent with the one by Wang et al.'s model. In addition, the efficiency of each DMU in the CSW-FDEA is larger than that of Azar et al.'s model, except for DMU E, and is smaller than that of the DSW-FDEA. The ranking result of the DSW-FDEA is consistent with Azar et al.'s model. We don't consider the ranking in Bhardwaj et al.'s model because of the problem above mentioned.

From these experimental results, the results of the proposed models are reliable. Although the ranking results of the proposed DSW-FDEA and Wang et al.'s model are consistent and ones of the proposed CSW-FDEA and Azar et al.'s model are consistent, we can notice that the efficiency evaluation of proposed models is more accurate than others. The ranking results may differ from other methods if the proposed models are applied to other applications.

5. Empirical Study

This section reports on the empirical results and the analysis of the proposed models. We describe the data and variables for evaluating the energy efficiency of China's iron and steel industry in 5.1 and explain the framework of efficiency prediction in 5.2. Finally, we predict the energy efficiency of 40 iron and steel enterprises in 2018 in 5.3.

5.1. Data and variables

We estimate the subsequent energy efficiency of 40 iron and steel enterprises in China from the data of past years. Due to data access rights, the data of Chinese iron and steel enterprises only in the period 2014-2017 were collected, so the purpose of this study is to estimate the energy efficiency of these enterprises in 2018 using those data. In selecting indicators for estimating energy efficiency, we considered the views of previous studies. He et al. [14] selected three inputs, i.e. net fixed assets, the number of employees, and energy consumption, and one desirable output, i.e. industrial value-added, for evaluating the energy efficiency of iron and steel enterprises. Here, the industrial value-added is affected by various macroeconomic factors. The prediction of industrial value-added in the next period has difficulties due to the influence of several external and indeterminate conditions. In addition, some studies such as Kuosmanen [23] argued that using variables measured in terms of money, e.g. "net profit", is not appropriate for measuring (pure) "technical" efficiency.

If we discuss only the production efficiency with no change in value, we can select the enterprise's crude steel production as the output indicator. This indicator is relatively less affected by external conditions such as market price fluctuations; it is possible to predict the production of the next period due to its certain regularity.

The selected input indicators are as follows:

- Net fixed assets (X_1) – Net fixed assets of the enterprises are the sum of their net investments made each year, which are valued at the current price for that year.
- The number of employees (X_2) - Since data on working hours are not available to obtain, the number of employed workers is used as the labor force.
- Energy consumption (X_3) – The energy used in iron and steel production, such as coal, electricity, fuel, and natural gas, is converted into the corresponding amount of coal needed.

The selected output indicator is as follows:

- Crude steel production (Y_1) – The qualified output that has completed the whole process of the steelmaking production process.

All data were collected from China Iron and Steel Industry Association, and Table 5 shows the statistical summary of the data. These collected data are all crisp values.

Table 5. Statistical summary of data.

	Variables	Unit	Mean	SD [a]	Median	Min	Max
Inputs	Net fixed assets	billion Yuan	27.22	20.66	18.26	2.33	91.37
	Energy	million tons	5.38	3.57	4.36	0.9	20.09
	Employee	thousand workers	21.96	21.34	16.51	5.73	143.11
Output	Crude steel production	million tons	8.74	5.1	7.66	2.3	25.66

[a] Standard deviation

5.2. The framework of efficiency prediction

The efficiency prediction process consists of the following four steps;

First, the input/output data of each enterprise in the past is used to estimate the input/output in the next year.

Second, the input/output values estimated in the first step are expressed as triangular fuzzy numbers.

Third, by applying the fuzzy DEA, the fuzzy efficiency of each enterprise in the next year is calculated.

Finally, the defuzzification of fuzzy efficiency is carried out to analyze the ranking.

The data for the past four years is used to predict the efficiency for the next year. We assume that managers' efforts to improve the efficiency do not change. In other words, it implicitly assumes that the efforts between 2014 and 2017 will be continued in 2018. Linear estimation based on ordinary least squares (OLS) is introduced to predict the next value from four-time series data. The linear estimation is simple to use and is advantageous in short time series analysis. OLS is the most fundamental form of linear estimation and requires the least conditions of the model. OLS finds a regression line that the sum of the distances from all observations to the regression line is the smallest. We estimate the fifth value from the past four data. However, the prediction always exists error, so we express the predicted value as a triangular fuzzy number. In detail, the crisp value obtained by the linear estimation becomes the modal value of the triangular fuzzy number, and the variance of the linear estimation is used as the width of the ambiguity, as follows:

p_M = Estimation value

p_L = Estimation value – Variance of estimation

p_U = Estimation value + Variance of estimation

Then, the prediction value is expressed as the following fuzzy number,

$$\tilde{P} = (p_L, p_M, p_U)$$

If we assume the error term of the estimation follows a normal distribution, the prediction range represented by the above fuzzy number occupies 68.2% of the possible value range. Therefore, we get the prediction value with about 70% reliability. As result, 40 DMUs with three fuzzy inputs and one fuzzy output are created. The scale transforming to [0,1] is carried out to reduce the effectiveness of the value range of each indicator in resolving the optimal problem. Then, the fuzzy DEA is applied. DSW-FDEA and CSW-FDEA are separately applied to evaluate energy efficiency from different viewpoints.

To provide a full ranking of all DMUs, the defuzzification function (26) is used.

5.3. Energy efficiency prediction

We use data during 2014-2017 to predict the efficiency in 2018. Linear estimation is carried out to predict the input and output data for 2018; we expressed it as triangular fuzzy numbers; the result is shown in Table A1 (Appendix A).

For 40 virtual DMUs, fuzzy efficiency is calculated from DSW-FDEA and CSW-FDEA, respectively. The result is shown in Table A2 (Appendix A).

In the next step, we proceed with the defuzzification of fuzzy efficiency for ranking DMUs, and we get two kinds of efficiencies. The final crisp efficiencies predicted are shown in Table 6. For analysis, we have also listed the efficiencies and the ranking results of 2017, which were respectively calculated by DSW and CSW, in this table.

Table 6. The full rankings of 40 iron and steel enterprises in China by different method.

DMU	2017 (DSW)		2018 (DSW-FDEA)		2017 (CSW)		2018 (CSW-FDEA)	
	efficiency	rank	efficiency	rank	efficiency	rank	efficiency	rank
1	0.588	37	0.443	37	0.584	32	0.351	38
2	0.487	38	0.383	39	0.308	40	0.369	37
3	0.672	29	0.633	27	0.641	26	0.633	18
4	0.763	19	0.917	5	0.701	22	0.553	31
5	0.673	28	0.611	29	0.604	30	0.607	24
6	0.828	10	0.857	9	0.821	9	0.674	12
7	0.648	32	0.577	32	0.493	35	0.575	28
8	0.657	31	0.464	36	0.638	27	0.464	35
9	0.667	30	0.738	14	0.436	36	0.736	6
10	0.826	11	0.728	16	0.733	19	0.691	10
11	0.75	22	0.637	25	0.75	17	0.619	20
12	0.701	26	0.583	31	0.7	23	0.583	27
13	0.357	40	0.28	40	0.357	39	0.261	40
14	0.709	25	0.615	28	0.575	34	0.605	25
15	0.739	23	0.679	21	0.739	18	0.679	11
16	0.39	39	0.738	14	0.376	38	0.302	39
17	0.816	14	0.704	20	0.815	12	0.7	9
18	0.697	27	0.635	26	0.692	24	0.611	23
19	1	1	0.978	1	0.989	3	0.89	2
20	0.643	33	0.389	38	0.417	37	0.389	36
21	0.725	24	0.65	24	0.704	21	0.646	17
22	0.841	9	0.705	19	0.836	7	0.656	15
23	0.764	18	0.785	11	0.763	16	0.66	14
24	0.941	5	0.724	17	0.941	4	0.575	28
25	0.623	35	0.54	34	0.622	28	0.54	32
26	0.896	6	0.944	4	0.883	6	0.936	1
27	1	1	0.958	2	1	1	0.703	8
28	0.822	12	0.743	13	0.822	8	0.743	5
29	1	1	0.818	10	1	1	0.815	3
30	0.821	13	0.719	18	0.82	10	0.628	19
31	0.763	19	0.653	23	0.665	25	0.649	16
32	0.635	34	0.557	33	0.594	31	0.557	30
33	0.775	17	0.671	22	0.774	15	0.671	13
34	1	1	0.948	3	0.904	5	0.717	7
35	0.786	16	0.884	7	0.786	14	0.617	21
36	0.858	8	0.883	8	0.82	10	0.599	26

37	0.811	15	0.591	30	0.811	13	0.508	34
38	0.864	7	0.772	12	0.718	20	0.772	4
39	0.756	21	0.915	6	0.581	33	0.616	22
40	0.609	36	0.524	35	0.609	29	0.524	33

In the case of DSW-FDEA, DMU 19, 27, and 34 took the lead in 1st, 2nd, and 3rd places, respectively. Interestingly, all of these units had CCR efficiencies of 1 in 2017. DMU 13 took the last place in evaluation of both CCR and DSW-FDEA. In CSW-FDEA, DMU 26, 19, and 29 took the lead in 1st, 2nd, and 3rd, respectively, and DMU 13 occupies the last place. DMU 13 took the last place in the evaluation of both DSW and CSW.

For intuitive comparison, both the efficiency in 2017 and the efficiency predicted in 2018 according to the DMU are graphed. Fig. 1 and 2 show the efficiency results for DSW and CSW, respectively. We can see that the prediction efficiency has a pattern similar to the real efficiency in 2017 in the overall phase, regardless of whether in DSW or CSW.

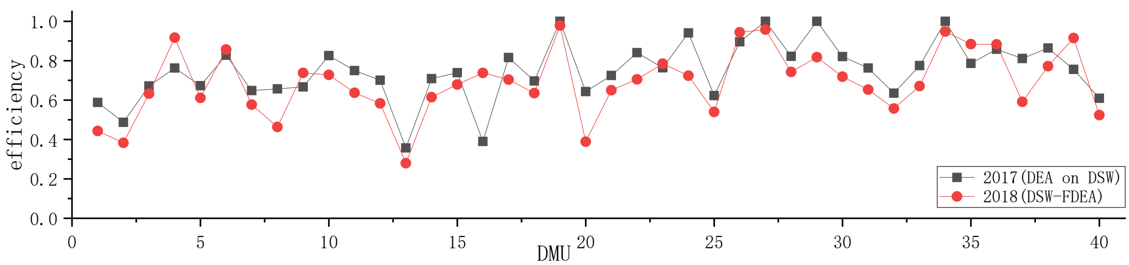

Fig. 1. Efficiencies by enterprise (DSW).

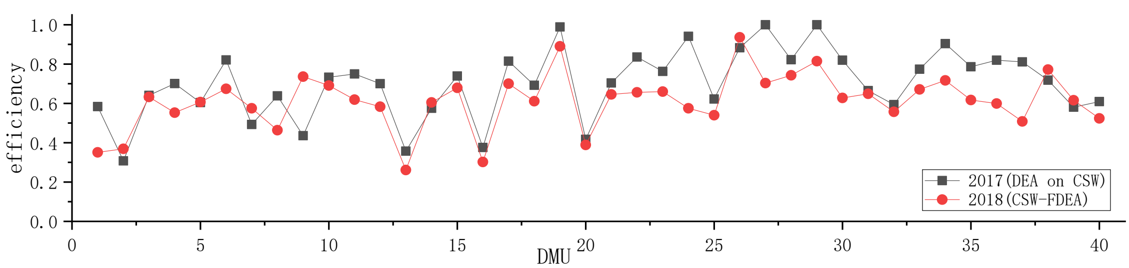

Fig. 2. Efficiencies by enterprise (CSW).

Discussing the ranking change in DSW-FDEA from the final year, the largest change in ranking occurred in DMU 16. The ranking of this rose from the original 39th to 14th. Except for this DMU, the rankings differed by more than 10 in the other 6 DMUs.

For the CSW method, the largest change in ranking occurred in DMU 9, 24, and 37. The first rose from the original 36th to 6th; the second fell from the original 4th to 28th; the third fell from 13th to 34th. Except for those three DMUs, the rankings differed by more than 10 in the other 3 DMUs.

Comparing the ranking results of DSW-FDEA and CSW-FDEA, there are large rank differences of more than 10 in DMU 38 and 39. This means that if DMU 38 and 39 are evaluated by the common weights, their rank will be evaluated lower than the ones by the dynamic weights.

The ranking was not clear in the method by CCR because there are 4 efficient DMUs, whereas, in the case of the two models proposed, the ranking was determined for all DMUs.

6. Conclusions

Although traditional DEA is a non-parametric method that does not require any prior information to evaluate the relative performance of DMUs, it is applicable only to crisp data. Recently, the application of fuzzy DEA proceeding with imprecise data is expanding. The fuzzy arithmetic approach, a kind of fuzzy DEA approach, has been performed in some literature, but they gave inappropriate solutions.

This paper proposed two fuzzy DEA models, obtaining the compromise solution by formulating them as MODM. First, we transformed the previous fuzzy DEA model with dynamic weights as a MODM problem to simultaneously maximize three-mark efficiency values given as triangular fuzzy numbers and derived a linear program. Considering various studies for evaluating DMUs in the same scale are also active, the first proposed fuzzy DEA model has been expanded to common weights.

The two models proposed in this paper, called DSW-FDEA and CSW-FDEA, were applied to the experimental data commonly used in the previous studies, and the results were compared with others. As a result of the numerical experiment, it was found that the models proposed derived reliable results and were better than the result of previous studies.

Finally, the two models proposed were applied to predict the energy efficiency of 40 iron and steel enterprises in China. We selected three input indicators and one output indicator for energy efficiency evaluation and suggested a new framework for efficiency prediction. Using data during 2014-2017, the energy efficiency in 2018 was predicted from two aspects: DSW-FDEA and CSW-FDEA.

Two fuzzy DEA models proposed in this paper will be effective in evaluating and ranking the efficiency of DMUs with imprecise data due to data measurement error, data dropout, etc., as well as prediction. Since two models with the dynamic weights and the common weights are proposed separately, the model corresponding to the application can be selected and applied. The efficiency prediction framework suggested in this paper will be applicable in various fields to need the prediction of efficiency. Although the paper predicted for one year, 2018, due to the condition of the data, we can use the framework to continuously predict for not only one year but also multiple years.

However, the method proposed has some defects. In the paper, fuzzy numbers only expressed with triangular membership functions are used, and ones with other types of membership functions are not applicable. And the proposed models have complex calculation stages. In addition, the indicator values of the next year are predicted by simple linear estimation and expressed by triangular fuzzy numbers in empirical studies, but there is a lack of consideration for the characteristics of each indicator. The future studies could discuss the fuzzy DEA based on fuzzy arithmetic in the consideration of various forms of fuzzy membership functions. And it could be discussed of introducing various estimation methods such as grey model – GM (1,1), instead of the linear estimation method, in the efficiency prediction framework.

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Appendix A

Table A1. 40 virtual DMUs scale-transformed (predict in 2018).

DMU	Net fixed assets	Energy	Employee	Crude steel production
1	(0.135, 0.138, 0.14)	(0.448, 0.468, 0.488)	(0.088, 0.09, 0.091)	(0.29, 0.291, 0.291)
2	(0.63, 0.636, 0.643)	(0.945, 0.972, 1)	(0.962, 0.981, 1)	(0.663, 0.669, 0.674)
3	(0.898, 0.918, 0.939)	(0.538, 0.543, 0.547)	(0.216, 0.223, 0.231)	(0.608, 0.618, 0.628)
4	(0.179, 0.222, 0.266)	(0.891, 0.904, 0.918)	(0.063, 0.078, 0.092)	(0.879, 0.882, 0.885)
5	(0.075, 0.083, 0.091)	(0.128, 0.142, 0.156)	(0.056, 0.062, 0.069)	(0.14, 0.154, 0.169)
6	(0.057, 0.06, 0.063)	(0.21, 0.218, 0.225)	(0.04, 0.042, 0.044)	(0.251, 0.26, 0.269)
7	(0.683, 0.686, 0.689)	(0.53, 0.543, 0.556)	(0.422, 0.422, 0.422)	(0.561, 0.573, 0.585)
8	(0.132, 0.152, 0.171)	(0.132, 0.137, 0.142)	(0.038, 0.041, 0.043)	(0.11, 0.114, 0.118)
9	(0.179, 0.183, 0.187)	(0.045, 0.057, 0.07)	(0.099, 0.104, 0.109)	(0.071, 0.08, 0.089)
10	(0.121, 0.122, 0.123)	(0.326, 0.33, 0.334)	(0.142, 0.143, 0.143)	(0.404, 0.41, 0.416)
11	(0.186, 0.188, 0.189)	(0.263, 0.268, 0.274)	(0.068, 0.07, 0.072)	(0.292, 0.296, 0.299)
12	(0.517, 0.526, 0.536)	(0.387, 0.397, 0.407)	(0.137, 0.137, 0.137)	(0.405, 0.414, 0.422)
13	(0.588, 0.589, 0.589)	(0.463, 0.473, 0.482)	(0.111, 0.112, 0.113)	(0.21, 0.219, 0.228)
14	(0.095, 0.095, 0.096)	(0.198, 0.202, 0.206)	(0.116, 0.117, 0.117)	(0.217, 0.222, 0.227)
15	(0.126, 0.126, 0.126)	(0.113, 0.123, 0.133)	(0.035, 0.036, 0.037)	(0.147, 0.148, 0.15)
16	(0.014, 0.023, 0.032)	(0.214, 0.218, 0.222)	(0.048, 0.048, 0.048)	(0.115, 0.117, 0.119)
17	(0.101, 0.103, 0.106)	(0.225, 0.228, 0.232)	(0.066, 0.066, 0.067)	(0.28, 0.285, 0.29)
18	(0.096, 0.102, 0.107)	(0.265, 0.267, 0.269)	(0.071, 0.077, 0.083)	(0.289, 0.291, 0.293)
19	(0.081, 0.081, 0.082)	(0.276, 0.283, 0.29)	(0.077, 0.078, 0.078)	(0.443, 0.449, 0.456)
20	(0.424, 0.431, 0.437)	(0.124, 0.128, 0.131)	(0.153, 0.158, 0.164)	(0.084, 0.093, 0.103)
21	(0.113, 0.115, 0.117)	(0.16, 0.163, 0.166)	(0.079, 0.082, 0.085)	(0.187, 0.19, 0.193)
22	(0.157, 0.158, 0.158)	(0.396, 0.401, 0.405)	(0.101, 0.102, 0.103)	(0.461, 0.468, 0.476)
23	(0.357, 0.379, 0.4)	(0.516, 0.518, 0.52)	(0.087, 0.094, 0.101)	(0.603, 0.606, 0.609)
24	(0.235, 0.248, 0.261)	(0.374, 0.393, 0.412)	(0.059, 0.06, 0.06)	(0.397, 0.399, 0.401)
25	(0.527, 0.555, 0.583)	(0.199, 0.206, 0.214)	(0.066, 0.071, 0.077)	(0.196, 0.2, 0.203)
26	(0.066, 0.068, 0.07)	(0.142, 0.15, 0.159)	(0.04, 0.042, 0.044)	(0.248, 0.251, 0.253)
27	(0.374, 0.376, 0.378)	(0.767, 0.784, 0.802)	(0.09, 0.091, 0.092)	(0.946, 0.973, 1)
28	(0.409, 0.436, 0.463)	(0.337, 0.344, 0.35)	(0.103, 0.104, 0.105)	(0.446, 0.456, 0.466)
29	(0.126, 0.135, 0.145)	(0.263, 0.263, 0.264)	(0.096, 0.096, 0.096)	(0.368, 0.385, 0.401)
30	(0.924, 0.962, 1)	(0.617, 0.62, 0.622)	(0.121, 0.125, 0.129)	(0.687, 0.691, 0.695)
31	(0.051, 0.051, 0.051)	(0.096, 0.099, 0.103)	(0.037, 0.038, 0.04)	(0.11, 0.116, 0.121)
32	(0.189, 0.192, 0.195)	(0.088, 0.094, 0.099)	(0.049, 0.049, 0.049)	(0.089, 0.094, 0.1)
33	(0.384, 0.402, 0.421)	(0.328, 0.33, 0.331)	(0.121, 0.123, 0.124)	(0.394, 0.397, 0.4)
34	(0.04, 0.042, 0.043)	(0.184, 0.188, 0.192)	(0.042, 0.045, 0.048)	(0.234, 0.24, 0.245)
35	(0.031, 0.036, 0.042)	(0.156, 0.158, 0.16)	(0.023, 0.025, 0.027)	(0.171, 0.172, 0.174)
36	(0.128, 0.144, 0.159)	(0.547, 0.552, 0.556)	(0.065, 0.071, 0.076)	(0.573, 0.584, 0.594)
37	(0.17, 0.176, 0.182)	(0.239, 0.248, 0.257)	(0.047, 0.048, 0.049)	(0.221, 0.223, 0.226)
38	(0.131, 0.134, 0.138)	(0.068, 0.069, 0.071)	(0.043, 0.043, 0.044)	(0.094, 0.097, 0.1)
39	(0.053, 0.059, 0.066)	(0.302, 0.303, 0.304)	(0.177, 0.178, 0.18)	(0.337, 0.339, 0.341)
40	(0.124, 0.137, 0.151)	(0.101, 0.105, 0.108)	(0.03, 0.032, 0.034)	(0.096, 0.098, 0.1)

Table A2. **Fuzzy efficiencies calculated by proposed DSW-FDEA and CSW-FDEA (predicted in 2018).**

DMU	DSW-FDEA	CSW-FDEA
1	(0.429, 0.443, 0.457)	(0.335, 0.351, 0.367)
2	(0.369, 0.383, 0.397)	(0.356, 0.369, 0.383)
3	(0.618, 0.633, 0.649)	(0.618, 0.633, 0.649)
4	(0.843, 0.915, 1)	(0.543, 0.553, 0.564)
5	(0.499, 0.607, 0.738)	(0.497, 0.604, 0.734)
6	(0.792, 0.856, 0.927)	(0.627, 0.673, 0.723)
7	(0.552, 0.577, 0.602)	(0.55, 0.574, 0.6)
8	(0.431, 0.464, 0.498)	(0.431, 0.464, 0.498)
9	(0.536, 0.722, 1)	(0.534, 0.72, 1)
10	(0.709, 0.728, 0.748)	(0.673, 0.691, 0.71)
11	(0.616, 0.637, 0.659)	(0.6, 0.618, 0.638)
12	(0.556, 0.582, 0.61)	(0.556, 0.582, 0.61)
13	(0.265, 0.28, 0.296)	(0.245, 0.26, 0.276)
14	(0.59, 0.615, 0.642)	(0.58, 0.604, 0.63)
15	(0.619, 0.677, 0.746)	(0.619, 0.677, 0.746)
16	(0.555, 0.718, 1)	(0.292, 0.302, 0.312)
17	(0.679, 0.703, 0.729)	(0.677, 0.699, 0.723)
18	(0.621, 0.635, 0.649)	(0.602, 0.611, 0.621)
19	(0.956, 0.978, 1)	(0.858, 0.89, 0.924)
20	(0.34, 0.388, 0.44)	(0.34, 0.388, 0.44)
21	(0.627, 0.65, 0.674)	(0.623, 0.646, 0.67)
22	(0.687, 0.705, 0.723)	(0.638, 0.656, 0.675)
23	(0.759, 0.785, 0.811)	(0.654, 0.66, 0.667)
24	(0.696, 0.723, 0.753)	(0.546, 0.574, 0.606)
25	(0.51, 0.54, 0.571)	(0.51, 0.54, 0.571)
26	(0.892, 0.944, 1)	(0.877, 0.935, 1)
27	(0.917, 0.958, 1)	(0.668, 0.702, 0.738)
28	(0.714, 0.743, 0.773)	(0.714, 0.743, 0.773)
29	(0.78, 0.818, 0.857)	(0.778, 0.815, 0.853)
30	(0.705, 0.719, 0.734)	(0.622, 0.628, 0.635)
31	(0.601, 0.652, 0.706)	(0.598, 0.649, 0.703)
32	(0.497, 0.556, 0.621)	(0.497, 0.556, 0.621)
33	(0.664, 0.671, 0.679)	(0.664, 0.671, 0.679)
34	(0.898, 0.948, 1)	(0.685, 0.716, 0.75)
35	(0.787, 0.88, 0.997)	(0.603, 0.617, 0.631)
36	(0.829, 0.882, 0.94)	(0.583, 0.599, 0.614)
37	(0.566, 0.591, 0.617)	(0.484, 0.508, 0.533)
38	(0.733, 0.772, 0.811)	(0.733, 0.772, 0.811)
39	(0.839, 0.913, 1)	(0.61, 0.616, 0.622)
40	(0.499, 0.524, 0.551)	(0.499, 0.524, 0.551)